Braid Monodromy of Algebraic Curves

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Preface

These are the notes from a one-week course on Braid Monodromy of Algebraic Curves given at the Université de Pau et des Pays de l'Adour during the Première Ecole Franco-Espagnole: Groupes de tresses et topologie en petite dimension in October 2009.

This is intended to be an introductory survey through which we hope we can briefly outline the power of the concept *monodromy* as a common area for group theory, algebraic geometry, and topology of projective curves.

The main classical results are stated in §2, where the Zariski-Van Kampen method to compute a presentation for the fundamental group of the complement to projective plane curves is presented. In §1 these results are prefaced with a review of basic concepts like fundamental groups, locally trivial fibrations, branched and unbranched coverings and a first peek at monodromy. Descriptions of the main motivations that have lead mathematicians to study these objects are included throughout this first chapter. Finally, additional tools and further results that are direct applications of braid monodromy will be considered in §3.

While not all proofs are included, we do provide either originals or simplified versions of those that are relevant in the sense that they exhibit the techniques that are most used in this context and lead to a better understanding of the main concepts discussed in this survey.

Nothing here is hence original, other than an attempt to bring together different results and points of view.

It goes without saying that this is not the first, and hopefully not the last, survey on the topic. For other approaches to braid monodromy we refer to the following beautifully-written papers [**71, 20, 6**].

We finally wish to thank the organizers and the referee for their patience and understanding in the process of writing and correcting these notes.

CHAPTER 1

Settings and Motivations

For the sake of completeness we will define the main objects and will state the problems that motivate the study of braid monodromy in connection with algebraic curves.

1. Fundamental Groupoids

Consider X a topological space and $\Gamma(X, x_0, y_0)$ the set of continuous paths from x_0 to y_0, that is,

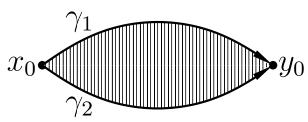
$$\Gamma(X, x_0, y_0) := \{\gamma : [0,1] \to X \mid \gamma \text{ continuous}, \gamma(0) = x_0, \gamma(1) = y_0\}.$$

The set of equivalence classes of $\Gamma(X, x_0, y_0)$ under *homotopy relative to* x_0 and y_0 will be denoted by $\pi_1(X, x_0, y_0)$. In other words:

$$\pi_1(X, x_0, y_0) := \Gamma(X, x_0, y_0)/\sim$$

where $\gamma_1 \sim \gamma_2 \Leftrightarrow \exists h : [0,1] \times [0,1] \to X$ continuous such that:

- $h(\lambda, 0) = \gamma_1(\lambda)$,
- $h(\lambda, 1) = \gamma_2(\lambda)$,
- $h(0, \mu) = x_0$,
- $h(1, \mu) = y_0$.

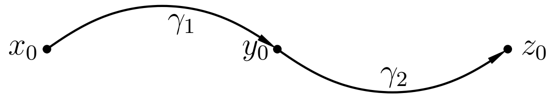


The category $(X, \{\pi_1(X, x_0, y_0)\}_{x_0, y_0 \in X})$, where X is the set of objects and $\pi_1(X, x_0, y_0)$ is the family of morphisms between x_0 and y_0, has a *groupoid* structure, that is, it satisfies the following properties:

- Associative composition law of morphisms:
  if $\gamma_1 \in \pi_1(X, x_0, y_0)$ and $\gamma_2 \in \pi_1(X, y_0, z_0)$, then $\gamma_1\gamma_2 \in \pi_1(X, x_0, z_0)$, where

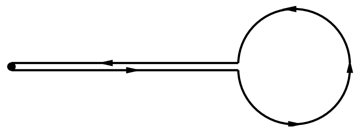
$$\gamma_1\gamma_2(\lambda) = \begin{cases} \gamma_1(2\lambda) & \lambda \in [0, \frac{1}{2}] \\ \gamma_2(2\lambda - 1) & \lambda \in [\frac{1}{2}, 1] \end{cases}$$

  Moreover $(\gamma_1\gamma_2)\gamma_3 = \gamma_1(\gamma_2\gamma_3)$ for any three paths $\gamma_1 \in \pi_1(X, x_0, y_0)$, $\gamma_2 \in \pi_1(X, y_0, z_0)$, and $\gamma_3 \in \pi_1(X, z_0, w_0)$.



- $\pi_1(X, x_0) := \pi_1(X, x_0, x_0)$ has a group structure (with the composition law): where $1_{x_0} \equiv x_0 \in \pi_1(X, x_0, x_0)$ and $\gamma^{-1}(\lambda) = \gamma(1-\lambda) \in \pi_1(X, y_0, x_0)$.

REMARK 1.1. In our paper, X will always have a complex manifold structure and thus any class of paths has a Piecewise Smooth representative. From now on, all the paths γ will be considered Piecewise Smooth.



REMARK 1.2. Also note that if x_0 and y_0 are in the same path-connected component of X, then the groups $\pi_1(X, x_0)$ and $\pi_1(X, y_0)$ are naturally isomorphic by an inner automorphism. In case X is path connected, such groups are denoted by $\pi_1(X)$ and called the *fundamental group of* X.

EXAMPLE 1.3. $\pi_1(\mathbb{S}^1) = \mathbb{Z}$ (see the comment after Theorem 2.3).

EXAMPLE 1.4 (Ordered Configuration Spaces). Let $X_n := \{(z_1, ..., z_n) \in \mathbb{C}^n \mid z_i \neq z_j, i \neq j\}$. A path in X_n from $x = (x_1, ..., x_n)$ to $y = (y_1, ..., y_n)$ is nothing but a collection of n paths γ_i, $i = 1, ..., n$ from x_i to y_i such that $\gamma_i(\lambda) \neq \gamma_j(\lambda)$ if $i \neq j$. Then $\pi_1(X_n) = \mathbb{P}_n$, the *pure braid group on n strings (on $\mathbb{C}$).*

EXAMPLE 1.5 (Non-ordered Configuration Spaces). Let $\mathcal{P}_n := \{f(z) \in \mathbb{C}[z] \mid \deg(f) = n\}$, $Y_n := \mathbb{P}(\mathcal{P}_n \setminus \Delta_n)$, where $\Delta_n := \{f \in \mathcal{P}_n \mid f$ has multiple roots$\}$. Note that $Y_n \underset{\varphi}{\cong} X_n/\Sigma_n$, where Σ_n represents the action of the symmetric group of n elements on X_n by permuting the coordinates, that is, if $\sigma \in \Sigma_n$, then $\sigma(z_1, ..., z_n) = (z_{\sigma(1)}, ..., z_{\sigma(n)})$ (note that the elements of X_n/Σ_n are simply sets of n distinct complex numbers). The homeomorphism φ is given as follows: any polynomial $f(z) \in Y_n$ can be normalized as $f(z) = (z - z_1)\cdots(z - z_n)$ where $z_i \neq z_j$. Thus $\varphi(f) := \{z_1, ..., z_n\} \in X_n/\Sigma_n$. Conversely, given a set of n distinct complex numbers $\{z_1, ..., z_n\} \in X_n/\Sigma_n$ one can obtain $f(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ as $a_i = \sigma_{n-i}(z_1, ..., z_n) =$ the symmetric polynomial of degree $n - i$ on $z_1, ..., z_n$. Therefore, if γ is a path in Y_n from $f_1 = (z - x_1)\cdots(z - x_n)$ to $f_2 = (z - y_1)\cdots(z - y_n)$, then $\varphi\gamma$ can be seen as a collection of n disjoint paths γ_i, $i = 1, ..., n$ from x_i to $y_{\sigma(i)}$ for a certain $\sigma \in \Sigma_n$. Then $\pi_1(Y_n) = \mathbb{B}_n$, the *(geometric) braid group on n strings (on $\mathbb{C}$).*

Analogously, if we consider $\bar{\mathcal{P}}_n := \{f(s,t) \in \mathbb{C}[s,t] \mid f$ homogeneous $\deg(f) = n\}$, $\bar{Y}_n := \mathbb{P}(\mathcal{P}_n \setminus \bar{\Delta}_n)$, where $\bar{\Delta}_n := \{f \in \bar{\mathcal{P}}_n \mid f$ has multiple roots$\}$. Note that $\pi_1(\bar{Y}_n) = \mathbb{B}_n(\mathbb{P}^1)$, the *braid group on n strings on* $\mathbb{P}^1 \cong \mathbb{S}^2$.

In the previous examples fundamental groups are either computed directly or by finding suitable homomorphisms to other spaces whose fundamental group was easier to compute. The idea behind it is that the fundamental group is a *topological invariant*, that is, if $X \overset{\varphi}{\cong} Y$ are two homeomorphic spaces, then the map $\pi_1(X; x_0, x_1) \xrightarrow{\varphi_*} \pi_1(Y; \varphi(x_0), \varphi(x_1))$ given by the set-theoretical image by φ of paths in X is well defined, it is a bijection for any choice of $x_0, x_1 \in X$, and it preserves the products, hence it is an isomorphism in the category of groupoids. In particular, φ_* defines isomorphisms of fundamental groups.

However, homeomorphisms are not the only continuous maps that induce isomorphisms of fundamental groups. The following result generalizes the map φ_* referred to in the previous paragraph and it serves as a way to introduce notation. Its proof is straightforward from the definitions and it is left as a useful exercise for the beginners.

LEMMA 1.6. *Any continuous map $\varphi : X \to Y$ between two topological spaces induces morphisms $\pi_1(X; x_0, x_1) \xrightarrow{\varphi_{x_0,x_1}} \pi_1(Y; \varphi(x_0), \varphi(x_1))$ for any choice of $x_0, x_1 \in X$.*

Moreover, if $\varphi : X \to Y$ and $\psi : Y \to Z$, then $(\psi \circ \varphi)_{x_0,x_1} = \psi_{x_0,x_1} \circ \varphi_{x_0,x_1}$.

To simplify notation, and whenever there is no likely ambiguity, we will simply refer to φ_{x_0,x_1} as φ_*.

EXAMPLE 1.7. Assume that $Y \subset X$ and that there is a surjective continuous map $\varphi : X \to Y$ such that $Y \xrightarrow{i} X \xrightarrow{\varphi} Y$ is the identity on Y. Then φ_* is an epimorphism, since $\varphi_* \circ i_* = (\mathrm{Id}_Y)_*$ (see 1.6) which is an isomorphism. Such a map is called a *retraction* of X onto Y.

The following is a very common way to find maps that induce equivalent morphisms of fundamental groups.

DEFINITION 1.8. Let $f, g : X \to Y$ two continuous maps. We say that f and g are two *homotopic* maps if there exists $H : X \times [0,1] \to Y$ a continuous map such that, if $x \in X$ then $H(x,0) = f(x)$ and $H(x,1) = g(x)$. The map H is called a *homotopy* from f to g and it is denoted as $f \overset{H}{\sim} g$.

Two topological spaces X and Y are called *homotopy equivalent* if there exist maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \sim \mathrm{Id}_Y$ and $g \circ f \sim \mathrm{Id}_X$

EXAMPLE 1.9. If two topological spaces X and Y are homotopy equivalent, then their groupoid fundamental groups are isomorphic.

Note, in particular, that a homotopy equivalence $\varphi : X \to Y$ induces isomorphisms of fundamental groups $\pi_1(X; x) \xrightarrow{\varphi_*} \pi_1(Y; \varphi(x))$. Moreover, if both spaces are connected, then one can simply say that $\pi_1(X) \xrightarrow{\varphi_*} \pi_1(Y)$ is an isomorphism.

EXAMPLE 1.10. Assume the hypothesis of Example 1.7 and also assume that the retraction φ is homotopic to the identity in X. Then X and Y are homotopy equivalent and the retraction φ is an equivalence of homotopies. Such retractions are called *deformation retract*.

EXAMPLE 1.11. $\pi_1(\mathbb{C} \setminus \{0\}) = \mathbb{Z}$ is a consequence of Examples 1.3 and 1.10, since the normalization map $\mathbb{C} \setminus \{0\} \to \mathbb{S}^1$ given by $z \mapsto \frac{z}{|z|}$ is a deformation retract.

2. The Seifert-Van Kampen Theorem

One of the basic tools to compute fundamental groups (and fundamental groupoids) is the Seifert-Van Kampen Theorem. This was first proved by H. Seifert [**68**] and later on, and independently, by E.R. Van Kampen [**73**]. Originally Van Kampen wrote this paper in an attempt to prove that the construction O. Zariski [**76**] built in order to compute the fundamental group (aka Poincaré group) of the complement of a plane curve in $\mathbb{P}^2$ was correct.

In order to state this result we will need to define the amalgamated product of two groups.

DEFINITION 2.1. Let $G_{12}\xrightarrow{i_1}G_1$ and $G_{12}\xrightarrow{i_2}G_2$ be two group homomorphisms. The *amalgamated free product of* G_1 *and* G_2 *w.r.t.* G_{12} is a group G that fits in a commutative diagram

$$
\begin{array}{ccc}
G_{12} & \xrightarrow{i_1} & G_1 \\
\downarrow i_2 & & \downarrow j_{1G} \\
G_2 & \xrightarrow{j_{2G}} & G
\end{array}
\tag{1}
$$

and has the following universal property: for any other such G' there exists a homomorphism $G\xrightarrow{\varphi}G'$ that commutes with both diagrams, that is, $\varphi j_{1G} = j_{1G'}$ and $\varphi j_{2G} = j_{2G'}$.

This can also be described by saying that the diagram (1) is a *pushout (in the category of groups)*.

In more *down-to-earth* terms, if G_{12}, G_1, and G_2 are groups is as in the previous definition with morphisms i_1, i_2 respectively, then the amalgamated free product of G_1 and G_2 w.r.t. G_{12}, commonly denoted by $G_1 *_{G_{12}} G_2$, can be described as the quotient

$$(G_1 * G_2)/N,$$

where N is the smallest normal subgroup of the free product $G_1 * G_2$ generated by $i_1(\gamma)i_2(\gamma)^{-1}$ for all $\gamma \in G_{12}$.

EXAMPLE 2.2. For instance, if G_i, G_{12} admit presentations $\langle \bar{x}_i : \bar{R}_i(\bar{x}_i)\rangle$ and $\langle \bar{y} : \bar{R}_{12}(\bar{y})\rangle$, then

$$G_1 *_{G_{12}} G_2 = \langle \bar{x}_1, \bar{x}_2 : \bar{R}_1(\bar{x}_1), \bar{R}_2(\bar{x}_2), i_1(y) = i_2(y), y \in \bar{y}\rangle.$$

Therefore, if G_i, are finitely presented, and G_{12} is finitely generated, then $G_1 *_{G_{12}} G_2$ is finitely presented.

We will give the following version of the main theorem.

THEOREM 2.3 (Seifert-Van Kampen Theorem). *Let* U_1 *and* U_2 *path-connected open subsets of* X *such that:*

- $U_1 \cup U_2 = X$ *and*
- $U_{12} := U_1 \cap U_2$ *is also path-connected.*

Then

$$\pi_1(X) = \pi_1(U_1) *_{\pi_1(U_{12})} \pi_1(U_2).$$

In other words, the commutative diagram given by the inclusions:

$$\begin{array}{ccc} \pi_1(U_{12}) & \xrightarrow{i_1} & \pi_1(U_1) \\ \downarrow i_2 & & \downarrow j_1 \\ \pi_1(U_2) & \xrightarrow{j_2} & \pi_1(X) \end{array}$$

is a pushout.

Originally Van Kampen considered the general case scenario, where the open sets U_1, U_2 and U_{12} are not necessarily path-connected. In this case, the result above generalizes claiming that $\pi_1(X, x_0, y_0)$ is a pushout of $\pi_1(U_1, x_0, y_0)$ and $\pi_1(U_2, x_0, y_0)$ in the category of groupoids (see [**13**, 6.7.2]).

This theorem gives a very simple proof of Example 1.3 (see [**13**, 6.7.5]).

EXAMPLE 2.4. Consider X and Y two path connected topological spaces, and $x \in X$, $y \in Y$ points on them. One can define $X \vee Y$, the *bouquet* of X and Y as the quotient space $X \sqcup Y/\{x, y\}$ of the disjoint union of X and Y by $\{x, y\}$. Note that, since X and Y are path connected, the homotopy type of the space $X \vee Y$ does not depend on the choice of x and y.

In order to compute $\mathbb{S}^1 \vee \mathbb{S}^1$ one can consider $U_1 := (\mathbb{S}^1 \setminus \{x\}) \vee \mathbb{S}^1$ and $U_2 := \mathbb{S}^1 \vee (\mathbb{S}^1 \setminus \{y\})$. If $\{x, y\}$ is not the set of points chosen to quotient by, then U_1 and U_2 are open subsets of $\mathbb{S}^1 \vee \mathbb{S}^1$. Moreover, $U_{12} = (\mathbb{S}^1 \setminus \{x\}) \vee (\mathbb{S}^1 \setminus \{y\})$ is contractible and hence $\pi_1(\mathbb{S}^1 \vee \mathbb{S}^1) = \mathbb{F}_2$ the free product of rank 2.

By induction, if $\bigvee^n \mathbb{S}^1 := \mathbb{S}^1 \vee ... \vee \mathbb{S}^1$ is the bouquet of n spheres, then $\pi_1(\bigvee^n \mathbb{S}^1) = \mathbb{F}_n$.

EXAMPLE 2.5. Let $z_1, ..., z_n \in \mathbb{C}$, $Z_n := \{z_1, ..., z_n\}$. Then $\pi_1(\mathbb{C} \setminus Z_n) = \mathbb{F}_n$. The case $n = 1$ is shown in Example 1.11. The case $n = 2$ is given in Figure 1 by describing $\mathbb{S}^1 \vee \mathbb{S}^1$ as a deformation retract of $\mathbb{C} \setminus \{\pm 1\}$. In general, one can describe $\bigvee^n \mathbb{S}^1$ as a deformation retract of $\mathbb{C} \setminus Z_n$, and hence the result will follow from Examples 1.10 and 2.4.

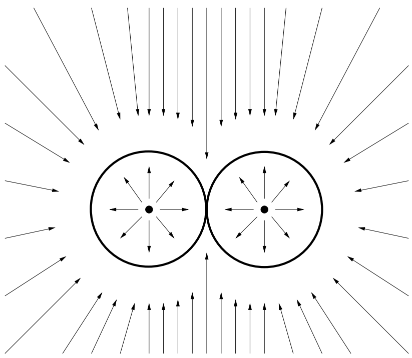


FIGURE 1. Deformation retract from $\mathbb{C} \setminus \{\pm 1\}$ to $\mathbb{S}^1 \vee \mathbb{S}^1$.

EXAMPLE 2.6. Let $z_1, ..., z_n \in \mathbb{P}^1$, $Z_n := \{z_1, ..., z_n\}$. Then $\pi_1(\mathbb{P}^1 \setminus Z_n) = \langle \gamma_1, \ldots, \gamma_n : \gamma_n \cdots \gamma_1 = 1 \rangle = \mathbb{F}_{n-1}$. Since $\mathbb{P}^1 \setminus \{z_1\} \cong \mathbb{C}$ and applying Example 4.1.

3. Locally Trivial Fibrations

DEFINITION 3.1. A surjective smooth map $\pi : X \to M$ of smooth manifolds is a *locally trivial fibration* if there is an open cover $\mathcal{U}$ of M and diffeomorphisms $\varphi_U : \pi^{-1}(U) \to U \times \pi^{-1}(p_U)$, with $p_U \in U$, such that φ_U is fiber-preserving, that is $pr_1 \circ \varphi_U = \pi$. The diffeomorphisms φ_U are called *trivializations* of π. The submanifold $\pi^{-1}(p) \subset X$ is called the *fiber* of π at p and usually denoted by F_p.

Two fibrations $\pi : X \to M$, $\pi' : X' \to M$ are said to be *equivalent* if there exists a diffeomorphism $\varphi : X \to X'$ such that

$$\begin{tikzcd} X \arrow[d, "\pi"] \arrow[dr, "\varphi"] & \\ & X' \arrow[dl, "\pi'"] \\ M & \end{tikzcd}$$

is a commutative diagram.

REMARK 3.2. Note that, if U is a trivialization open set, then $\pi^{-1}(p_1) \sim \pi^{-1}(p_2)$ for any two points $p_1, p_2 \in U$, simply considering $\varphi_U|^{-1}_{\pi^{-1}(p_1)} \circ \varphi_U|_{\pi^{-1}(p_1)}$. Therefore, the existence of the points $p_U \in U$ in Definition 3.1, might be replaced by the same property at any point of U. Hence all the fibers of a locally trivial fibration are all diffeomorphic to F_p as long as X is connected.

EXAMPLE 3.3. Any product $X := M \times F$ produces a locally trivial fibration just by projecting onto a component, say $\pi : X = M \times F \to M$, where $\pi(x, y) = x$. The open cover of M that trivializes the fibration is given simply by the total space M. The fiber of this fibration at any point is isomorphic to F. Such a fibration is called a *trivial fibration*.

One of the main properties of locally trivial fibrations, which will be extensively used here, is the fact that homotopies on the base can be *lifted*. A precise statement is the following (cf. [**75**, p. 45]):

THEOREM 3.4 (Homotopy Lifting Property). *Let $\pi : X \to M$ be a locally trivial fibration, consider:*

(1) $\gamma : [0,1] \to M$ *a continuous map,*
(2) $\tilde{\gamma} : [0,1] \to X$ *a lifting of γ (that is, a continuous map such that $\gamma = \pi \circ \tilde{\gamma}$), and*
(3) $h : [0,1] \times [0,1] \to M$ *a homotopy from γ (that is $h(\lambda, 0) = \gamma(\lambda)$).*

Then h can be lifted to a homotopy $\tilde{h} : [0,1] \times [0,1] \to X$ from $\tilde{\gamma}$.

Moreover, if two paths $\omega_1, \omega_2 : [0,1] \to X$ are given such that $\pi \circ \omega_1(\mu) = h(0, \mu)$ and $\pi \circ \omega_2(\mu) = h(1, \mu)$, then $\tilde{h}$ can be found such that $\tilde{h}(0, \mu) = \omega_1(\mu)$ and $\tilde{h}(1, \mu) = \omega_2(\mu)$.

EXAMPLE 3.5. Note that any locally trivial fibration $\pi : X \to [0,1]$ has a section $s : [0,1] \to X$ such that $s(0) = x_0$ for any $x_0 \in \pi^{-1}(0) \subset X$. Consider the constant map $\gamma(\lambda) = 0$ and fix a lifting $\tilde{\gamma}(\lambda) = x_0$. The retraction $h : [0,1] \times [0,1] \to [0,1]$, $h(\lambda, \mu) = \lambda\mu$ can be lifted using the Homotopy Lifting Property 3.4. Then $s(\lambda) = \tilde{h}(\lambda, 1)$ is a section such that $s(0) = \tilde{h}(0,1) = \tilde{\gamma}(0) = x_0$.

EXAMPLE 3.6. Any locally trivial fibration $\pi : X \to [0,1]$ is in fact a trivial fibration. In order to prove this, note that one can *patch* trivializations as follows. Suppose that $\varphi_1 : \pi^{-1}([a,b]) \to [a,b] \times F$ and $\varphi_2 : \pi^{-1}([b,c]) \to [b,c] \times F$ are trivializations of π (resp.) on $U_1 \supset [a,b]$ and $U_2 \supset [b,c]$ restricted to $[a,b] \subset [0,1]$ and $[b,c] \subset [0,1]$ (resp.). Note that $\psi_a := \varphi_1 \circ \varphi_2^{-1}|_{\{b\}\times F}$ is an automorphism of F. One can build the following isomorphism $\varphi : \pi^{-1}([a,c]) \to [a,c] \times F$ such that:

$$\varphi(x) := \begin{cases} \varphi_1(x) & \text{if } x \in \pi^{-1}([a,b]) \\ (\varphi_{2,1}(x), \psi_a \circ \varphi_{2,2}(x)) & \text{if } x \in \pi^{-1}([b,c]). \end{cases}$$

Consider a finite covering $\mathcal{U} := \{U_1 = [\alpha_1 = 0, \beta_1), U_2 = (\alpha_2, \beta_2), ..., U_n = (\alpha_n, \beta_n = 1]\}$, where the fibration trivializes and define $a_0 = 0$, $\beta_i < a_i < \alpha_{i+1}$, $(i = 2, \ldots, n-1)$, $a_n = 1$. Using the paragraph above, one can patch the trivializations to obtain the trivialization $\varphi : \pi^{-1}([0,1]) = X \to [0,1] \times F$.

Even though it is true that every locally trivial fibration has homeomorphic fibers, the converse is not true, as we will see later. In general, proving that a certain map with homeomorphic fibers is a locally trivial fibration is not an easy task. The main tool in our context is the following fundamental result (cf. [**28, 47**]).

THEOREM 3.7 (Ehresmann's Fibration Theorem). *Any proper submersion $\pi : X \to M$ is a locally trivial fibration. Moreover, if $B \subset X$ is a closed submanifold such that $\pi|_B$ is still a proper submersion, then $\pi|_{X\setminus B}$ is also a locally trivial fibration.*

4. Unbranched Coverings, Branched Coverings, and Monodromy

We will briefly discuss the notion of unbranched and branched coverings as both, a motivation and a first approximation to braid monodromy. Conditions for the existence of branched coverings of smooth lines and surfaces ramified along a given locus has been a classical problem that becomes a common place for (low dimensional) Topology, Algebraic Geometry, (inverse) Galois Theory, and Geometry. The main results of this section can be found in much more detailed in [**57**].

4.1. Unbranched Coverings.

DEFINITION 4.2. An *unbranched covering* is a locally trivial fibration whose fiber is a discrete subset.

EXAMPLE 4.3. The map $\pi : \mathbb{B}^* \to \mathbb{B}^*$, defined as $\pi(z) = z^e$ from the punctured disc to itself is a finite unbranched covering whose fiber is a finite set of e elements. In particular, $\pi^{-1}(z^e) = \{\xi_e^i z \mid i = 0, ..., e-1\}$, where $\xi := \exp\left(\frac{2\pi\sqrt{-1}}{e}\right)$.

Analogously, the map $\pi : \mathbb{B}^{n-1} \times \mathbb{B}^* \to \mathbb{B}^{n-1} \times \mathbb{B}^*$ defined as $\pi(z_1, ..., z_{n-1}, z_n) = (z_1, ..., z_{n-1}, z_n^e)$ is also an unbranched covering whose fiber is a finite set of e elements.

The following result classifies unbranched coverings.

THEOREM 4.4. *Let M be a locally contractible topological space. Then the following holds:*

(1) *For any unbranched covering $\pi : X \to M$, the induced morphism $\pi_* : \pi_1(X, x_0) \to \pi_1(M, \pi(x_0))$ is a monomorphism.*
(2) *Conversely, for any subgroup $G < \pi_1(M)$ there exists a covering $\pi : X \to M$ such that $G = \pi_*(\pi_1(X))$.*
(3) *Two coverings $\pi : X \to M$, $\pi' : X' \to M$ are equivalent if and only if $\pi_*(\pi_1(X, x_0)) < \pi_1(M, q_0)$ and $\pi'_*(\pi_1(X', x'_0)) < \pi_1(M, q_0)$ are conjugate of each other (for some $x_0 \in X$ and $x'_0 \in X'$ such that $q_0 = \pi(x_0) = \pi'(x'_0)$).*

(for a proof of Theorem 4.4 see any basic textbook on Algebraic Topology, for instance see [**75**]).

EXAMPLE 4.5. Note that Example 4.3 induces the following:

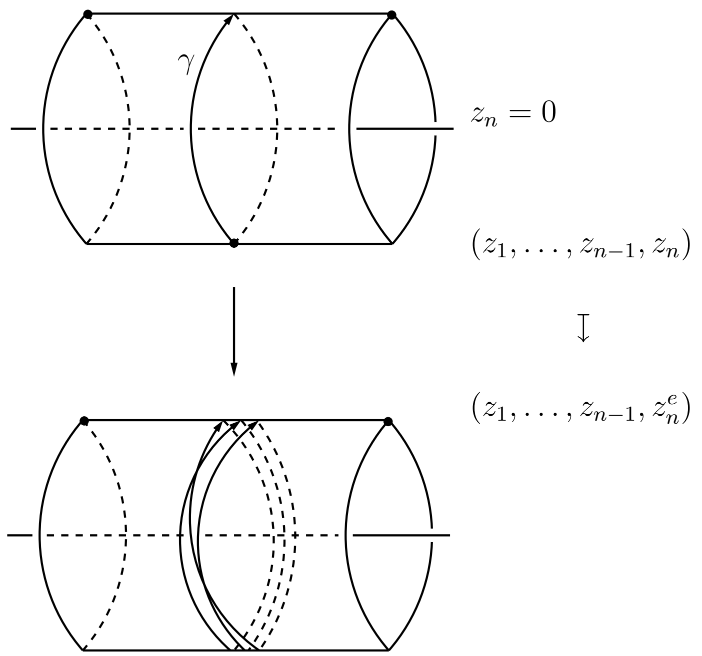
$$\pi_1(\mathbb{B} \setminus \{0\}) = \mathbb{Z}\gamma \overset{\pi_*}{\hookrightarrow} = \pi_1(\mathbb{B} \setminus \{0\}) = \mathbb{Z}\gamma,$$

where the map is given by $\pi_*(\gamma) = e\gamma$. This corresponds to the inclusion $\mathbb{Z}e < \mathbb{Z}$.

Analogously, the map $\pi : \mathbb{B}^{n-1} \times \mathbb{B}^* \to \mathbb{B}^{n-1} \times \mathbb{B}^*$ defined as $\pi(z_1, \dots, z_n) = (z_1, \dots, z_{n-1}, z_n^e)$ induces the following:

$$\pi_1(\mathbb{B}^{n-1} \times \mathbb{B}^*) = \mathbb{Z}\gamma \overset{\pi_*}{\hookrightarrow} = \pi_1(\mathbb{B}^{n-1} \times \mathbb{B}^*) = \mathbb{Z}\gamma,$$

where the map is also given by $\pi_*(\gamma) = e\gamma$. This corresponds to the inclusion $\mathbb{Z}e < \mathbb{Z}$ as shown below.



4.6. Monodromy of Unbranched Coverings. Any unbranched covering $\pi : X \to M$ is, by definition, a locally trivial fibration whose fiber is a discrete set $S := \pi^{-1}(q_0)$. There is a monodromy action of $\pi_1(M, q_0)$ on S as follows.

Let $\gamma : [0,1] \to M$ be a closed path in $\pi_1(M, q_0)$. One has the following diagram

$$\begin{array}{rccc} \pi^{-1}(\gamma) = & \tilde{X} & \hookrightarrow & X \\ & \downarrow \tilde{\pi} & & \downarrow \pi \\ & [0,1] & \xrightarrow{\gamma} & M \end{array}$$

According to Example 3.6, $\tilde{\pi}$ is a trivial fibration. The trivialization of $\tilde{\pi}$ defines a bijection $\gamma : S \to S$. In other words, for any given $x_0 \in S$ one can construct a section $s_{x_0} : [0,1] \to \tilde{X}$ such that $s_{x_0}(0) = x_0$ (see Example 3.5), then $\tilde{\pi}(x_0) = s_{x_0}(1) \in S$.

EXAMPLE 4.7. Finally, in order to understand the monodromy of the map given in Examples 4.3 and 4.5 consider the path $\gamma(\lambda) := z^e \exp(2\pi\sqrt{-1}\lambda) \in \pi(\mathbb{B}^*, z^e)$ which generates this fundamental group. Note that $s_z(\lambda) = z \exp\left(\frac{2\pi\sqrt{-1}\lambda}{e}\right)$ is the section constructed above and hence $s_z(1) = z \exp\left(\frac{2\pi\sqrt{-1}}{e}\right) = \xi z$ is the image of z by the monodromy of γ. Analogously, note that $s_{\xi^i z}(\lambda) = \xi^i z \exp\left(\frac{2\pi\sqrt{-1}\lambda}{e}\right)$ and hence

$$\begin{array}{rrcl} \gamma : & S = \{\xi^i z \mid i = 0, ..., e-1\} & \to & S \\ & \xi^i z & \mapsto & \xi^{i+1} z \end{array}$$

defines the monodromy of γ on S, which is just a cyclic transformation of order e.

4.8. Branched Coverings. In this section we will focus on the study of branched coverings of complex manifolds.

DEFINITION 4.9. Let M be an m-dimensional (connected) complex manifold. A *branched covering* of M is an m-dimensional irreducible normal complex space X together with a surjective holomorphic map $\pi : X \to M$ such that:

- every fiber of π is discrete in X,
- the set $R_\pi := \{x \in X \mid \pi^* : \mathcal{O}_{\pi(x),M} \to \mathcal{O}_{x,X}$ is not an isomorphism$\}$ called the *ramification locus*, and $B_\pi = \pi(R_\pi)$ called the *branched locus*, are hypersurfaces of X and M, respectively,
- the map $\pi| : X \setminus \pi^{-1}(B_\pi) \to M \setminus B_\pi$ is an unramified (topological) covering, and
- for any $q \in M$ there is a connected open neighborhood $W^q \subset M$ such that for every connected component U of $\pi^{-1}(W)$:
  (1) $\pi^{-1}(q) \cap U$ has only one element, and
  (2) $\pi|_U : U \to W$ is surjective and proper.

A branched cover $\pi : X \to M$ will be called *Galois*, (resp. *finite*) if $\pi_*(\pi_1(X \setminus \pi^{-1}(B_\pi))$ is a normal (resp. *finite index*) subgroup of $\pi_1(M \setminus B_\pi)$.

EXAMPLE 4.10. The map $\pi : \mathbb{B} \to \mathbb{B}$ defined by $\pi(z) = z^e$ is a branched covering ramified at $B_\pi = \{0\}$.

Analogously, the map $\pi : \mathbb{B}^n \to \mathbb{B}^n$ defined by $\pi(z_1, \ldots, z_n) = (z_1, \ldots, z_{n-1}, z_n^e)$ is a branched covering ramified at $B_\pi = \{z_n = 0\}$.

REMARK 4.11. In the context of complex manifolds, Example 4.10 is the only local situation that one can encounter (cf. [**57**, Theorem 1.1.8]).

The purpose of this section is to study Theorem 4.4(2) for branched coverings, that is, what are the conditions, in terms of the monodromy or in terms of fundamental groups for the existence of branched coverings ramified along a given divisor. In order to do so, let us develop the key concept of meridian.

4.12. Meridians. Let M be a complex manifold, B' an irreducible component of a hypersurface $B \subset M$, and $b \in B'$ a smooth point on B. By definition, this means that there exists an open neighborhood U of b in M and a holomorphic function f on U such that $B \cap U = \{z \in U \mid f(z) = 0\}$. As a simple application of the Implicit Function Theorem on f and U, there exists a change of coordinates such that U can be chosen to be $V \times \mathbb{B}$ where V is a polydisk and $B \cap U = V \times \{0\}$. Hence the point $b \in B \cap U$ will have coordinates $b = (b_0, 0)$. Let $\gamma_b = \{b_0\} \times \{\exp(2\pi\sqrt{-1}\lambda)\}$ be a closed path centered at $\tilde{b} = (b_0, 1)$.

DEFINITION 4.13. Under the above conditions, a closed path in $\pi_1(M \setminus B, q_0)$ is called a *meridian* of B' if there is a representative γ in its homotopy class that can be written as $\gamma = \alpha \cdot \gamma_b \cdot \alpha^{-1}$ where $\alpha \in \pi_1(M \setminus B, q_0, \tilde{b})$ for a certain $b \in B'$ as above (see Figure 2).

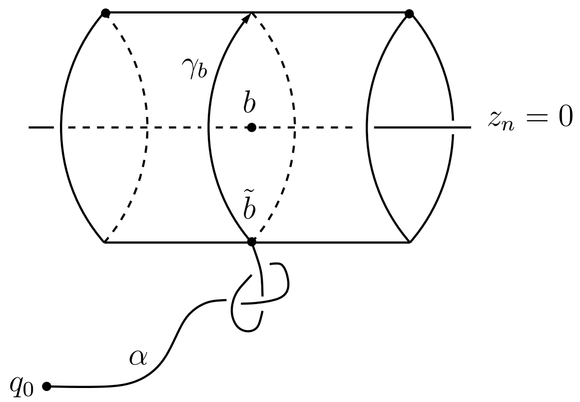


FIGURE 2. Meridian

PROPOSITION 4.14. *Any two meridians, say $\gamma_1, \gamma_2 \in \pi_1(M \setminus B, q_0)$, of the same irreducible component B' are conjugated, that is, $\gamma_2 = \omega\gamma_1\omega^{-1}$ for a certain $\omega \in \pi_1(M \setminus B, q_0)$.*

Moreover, the conjugacy class of a meridian coincides with the set of homotopy classes of meridians around the same irreducible component.

PROOF. The main ingredient of this proof is that $B' \setminus \mathrm{Sing}(B)$ is a path connected space as long as B' is irreducible since $\mathrm{Sing}(B)$ has real codimension 2 in B. Therefore consider δ

a path in B' from $\tilde{b}_2$ to $\tilde{b}_1$, where $\gamma_i = \alpha_i \cdot \gamma_{b_i} \cdot \alpha_i^{-1}$, $i = 1, 2$ and γ_i are paths around $b_i \in B$. One can deform δ along the normal bundle so that δ connects $\tilde{b}_2$ and $\tilde{b}_1$. This way, note that $\gamma_2 = \omega\gamma_1\omega^{-1}$ where $\omega = \alpha_2 \cdot \delta \cdot \alpha_1^{-1}$ (see Figure 3).

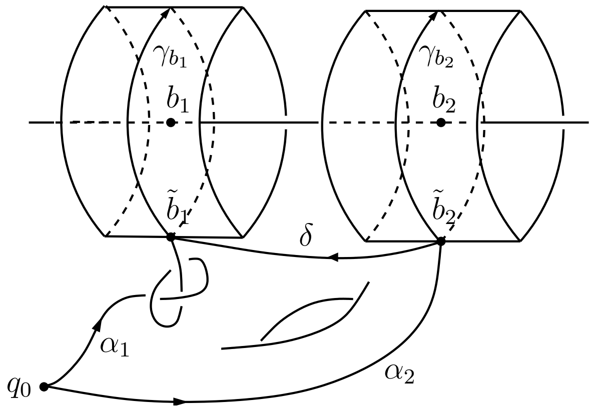


FIGURE 3. Conjugate meridians

The *moreover* part is obvious by definition of meridian. If $\gamma = \alpha \cdot \gamma_b \cdot \alpha^{-1}$ is a meridian decomposed as in Definition 4.13, and $\omega \in \pi_1(M \setminus B, q_0)$ then $(\omega \cdot \alpha) \cdot \gamma_b \cdot (\omega \cdot \alpha)^{-1}$ also satisfies the conditions of Definition 4.13, and hence it is a meridian of M around B'. $\square$

4.15. Existence and construction of branched coverings: smooth case. Consider B a non-singular hypersurface, $B = D_1 \cup ... \cup D_r$ its decomposition in irreducible components, choose $\bar{e} := (e_1, ..., e_r) \in \mathbb{N}^r$, $e_i > 1$ and denote $D = \sum n_i D_i$ a divisor on M. Let $q_0 \in M \setminus B$ base point.

Let $\gamma_1, \ldots, \gamma_r \in \pi_1(M \setminus B, q_0)$ be meridians of the irreducible components of B. The elements $\gamma_1^{e_1}, ..., \gamma_r^{e_r} \in \pi_1(M \setminus B, q_0)$ normally generate a subgroup

$$J_{\bar{e}} := N(\gamma_1^{e_1}, ..., \gamma_r^{e_r}) \triangleleft \pi_1(M \setminus B, q_0).$$

According to Proposition 4.14, $J_{\bar{e}}$ does not depend on the choice of the meridians.

DEFINITION 4.16. Under the above notation, π is said to *ramify* (resp. *ramify at most*) *along* D if B is the ramification locus of π and e_i coincides with (resp. is a multiple of) the ramification index of π at D_i.

A branched cover π ramified along D is said to be *maximal* if it factors through any other branched cover π' ramified at most along D, that is, there exists a holomorphic map $\varphi : X \to X'$ such that:

$$\begin{tikzcd} X \arrow[dd, "\pi"] \arrow[dr, "\varphi"] & \\ & X' \arrow[dl, "\pi'"] \\ M & \end{tikzcd}$$

is a commutative diagram.

REMARK 4.17. Note that if $\pi : X \to M$ is a branched covering ramified along D, then $\gamma_i^{e_i}$ can be lifted to a meridian of $\pi^{-1}(D_i)$ (see Remark 4.11 and Example 4.5). Therefore $J_{\bar{e}} \triangleleft \pi_1(X \setminus \pi^{-1}(B), q_0)$.

CONDITION 4.18. We say that $K < \pi_1(M \setminus B)$ satisfies this condition if, given any meridian γ_i of D_i, one has that $\gamma_i^d \in K$ implies $d \equiv 0 \ (mod\ e_i)\ \forall\ 1 \le i \le r$.

The following result can be found in [**57**, Theorem 1.2.7]. It characterizes the branched covers of a complex manifold ramified along a smooth hypersurface with prescribed ramification indices and it is a partial equivalent of Theorem 4.42.

THEOREM 4.19. *There is a natural one-to-one correspondence between*

$$\left\{ K \overset{f.i}{\triangleleft} \pi_1(M \setminus B) \mid \begin{array}{c} K \supset J_{\bar{e}} \\ \textit{satisfying } (4.18) \end{array} \right\} \leftrightarrow \left\{ \pi : X \to M \quad \begin{array}{c} \textit{Galois, finite,} \\ \textit{ramified along } D \end{array} \right\} \Big/ \sim . \tag{2}$$

Moreover, there is a maximal Galois covering X_D of M ramified along D iff

$$K_\pi = \bigcap_{K \textit{as in } (2)} K \overset{f.i}{\triangleleft} \pi_1(M \setminus B)$$

satisfies (4.18).

Note that we use $K \overset{f.i}{\triangleleft} \pi_1(M \setminus B)$ for *finite index normal subgroup*. As a consequence of Theorem 4.19 one has the following classical result, for compact complex manifolds of dimension 1, part of which is known as the Riemann Existence Theorem. Consider M^g a compact complex manifold of dimension 1, that is, a *Riemann surface* and $Z_n \subset M^g$ a finite set of n points in M^g.

THEOREM 4.20. *Any monodromy action $\pi_1(M^g \setminus Z_n) \to \Sigma_s$ can be realized by a branched covering of the Riemann surface M^g.*

PROOF. Let $K = \ker(\pi_1(M^g \setminus Z_n) \xrightarrow{\mu} \Sigma_s)$. For any meridian γ_z of an element of $z \in Z_n$, consider $\mu(\gamma_z) \in \Sigma_s$. Since Σ_s is finite, the order of $\mu(\gamma_z)$, say e_z, is also finite. Define $B = \{z \in Z_n \mid e_z > 1\}$ and $D = \sum_{z \in B} e_z z$. Note that $K \overset{f.i}{\triangleleft} \pi_1(M \setminus B)$ and $K \supset J_{\bar{e}}$ by construction. All one needs to check is condition (4.18), but this is also immediate. If $\gamma_z^d \in K$, then $\mu(\gamma_z)^d = 1$. Therefore $e_z | d$, since e_z is the order of $\mu(\gamma_z)$. Finally, one can apply Theorem 4.19, since B is a smooth hypersurface. $\square$

4.21. Existence and construction of branched coverings: general case. We will follow the notation introduced in the previous item. Consider B a (possibly singular) hypersurface, $B = D_1 \cup ... \cup D_r$ its decomposition in irreducible components, choose $\bar{e} := (e_1, ..., e_r) \in \mathbb{N}^r$, $e_i > 1$ and denote $D = \sum n_i D_i$ a divisor on M. Let $q_0 \in M \setminus B$ base point.

Let $\gamma_1, \ldots, \gamma_r \in \pi_1(M \setminus B)$ be meridians of the irreducible components of B and define $J_{\bar{e}}$ as above.

In addition, for any $q \in \operatorname{Sing} B$ one can consider the inclusion of a local neighborhood of q in B, say $i_q : W^q \setminus B \hookrightarrow M \setminus B$. By the special structure of analytic singularities (see [**53**, Theorem 2.10]), it turns out that i_q does not depend on W^q for a small enough neighborhood. Therefore, given any subgroup $K < \pi_1(M \setminus B)$ one can define $K_q := i_q^{-1}(K)$.

CONDITION 4.22. We say $K \triangleleft \pi_1(M \setminus B)$ satisfies this condition if, for any point $q \in \operatorname{Sing} B$, $K_q \overset{f.i}{\triangleleft} \pi_1(W \setminus B)$.

It is reasonable, but not so obvious anymore, that given a branched cover $\pi : X \to M$ ramified along D, then $K = \pi_*(\pi_1(X \setminus \pi^{-1}(B)))$ satisfies 4.22 (see [**57**, Theorem 1.3.8] or [**37**, p.340] for a proof).

THEOREM 4.23. *There is a one-to-one correspondence:*

(3)
$$\left\{ K \overset{f.i}{\triangleleft} \pi_1(M \setminus B) \mid \begin{array}{c} K \supset J_{\bar{e}} \textit{ satisfying} \\ (4.18) \textit{ and } (4.22) \end{array} \right\} \leftrightarrow \left\{ \pi : X \to M \quad \begin{array}{c} \textit{Galois, finite,} \\ \textit{ramified along } D \end{array} \right\} \Big/ \sim .$$

Moreover, there is a maximal Galois covering X_D of M ramified along D iff

$$K_\pi = \bigcap_{K \textit{as in } (3)} K \overset{f.i}{\triangleleft} \pi_1(M \setminus B)$$

satisfies (4.18) *and* (4.22).

This will allow for a general study of branched covers of $\mathbb{P}^2$ ramified along plane curves, which is the classical problem, already stated by Enriques [**29**], Zariski [**76, 77**], and many others, known as the *multiple plane* problem. The original problem was stated as follows:

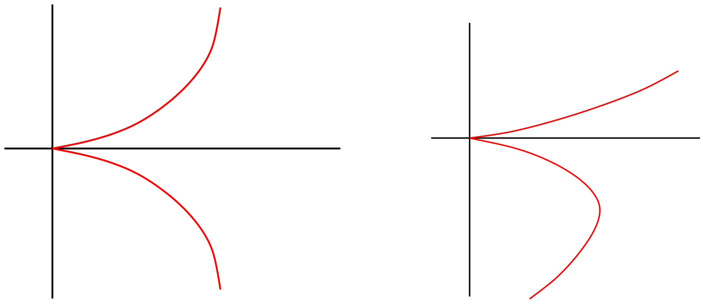
PROBLEM 4.24. **Enriques-Zariski Problem** [**76**] *Does an algebraic function z of x and y exist, possessing a preassigned curve f as branched curve?*

EXAMPLE 4.25. Consider $M = \mathbb{P}^2$, $D_1 = \{ZY^2 = X^3\}$, $D_2 = \{Z = 0\}$. Let us study the possible Galois covers of $\mathbb{P}^2$ ramified along $D = e_1 D_1 + e_2 D_2$.

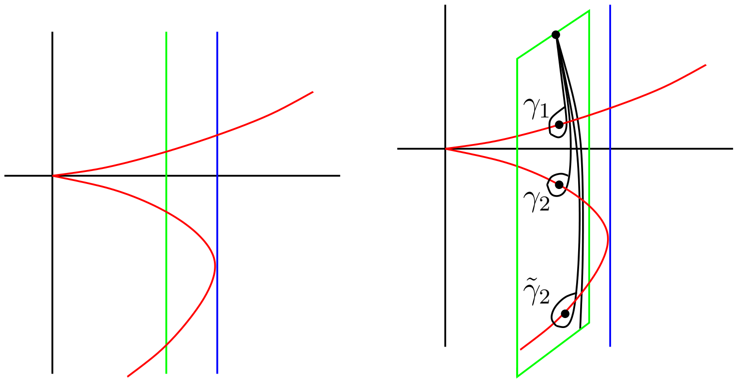
In order to do so, one needs to compute the fundamental group $\pi_1(\mathbb{P}^2 \setminus (D_1 \cup D_2))$. This will be presented in a more systematic way in Chapter §2. You can go ahead read it and come back, or just bare with me a couple of calculations and hopefully everything will be understood later.

The space $\mathbb{P}^2 \setminus (D_1 \cup D_2)$ is nothing but $\mathbb{C}^2 \setminus \{y^2 = x^3\}$, where $\mathbb{C}^2 = \mathbb{P}^2 \setminus D_2$ is one of the standard affine charts of $\mathbb{P}^2$. The identification is given as $(x, y) \mapsto [X : Y : 1]$, whose inverse is $[X : Y : Z] \mapsto \left(\frac{X}{Z}, \frac{Y}{Z}\right)$.

Avoiding tangencies at infinity will make our life easier in this case, so one can change the affine coordinate system and simply work with the curve $\mathcal{C} := \{27y^2 = 4(x - y)^3\}$. Since this transformation is continuous. The fundamental group is not affected by that. First of all note that $\mathcal{C}$ has only one singular point at $(0, 0)$. Consider the projection $(x, y) \mapsto x$, and note that, when restricted to $\mathcal{C}$, it produces a cover of $\mathbb{C}$ branched along $x = 0$ (the projection of the singular point) and $x = 1$ (the tangency shown by the blue line). Precisely the non existence of vertical asymptotes will allow us to take big disks $\mathbb{D}_x$, $\mathbb{D}_y$ such that $(\mathbb{C}^2, D_1)$ is a deformation retract of $(\mathbb{D}_x \times \mathbb{D}_y, D \cap (\mathbb{D}_x \times \mathbb{D}_y))$. On the other hand, consider the disk $\mathbb{D} := \{\frac{1}{2}\} \times \mathbb{D}_y$ shown below.



$y^2 = x^3$ $27y^2 = 4(x-y)^3$



Note that $D_1 \cap \mathbb{D} = \{p_1 := (\frac{1}{2}, -\frac{5}{2} + \frac{3\sqrt{3}}{2}), p_2 := (\frac{1}{2}, -\frac{5}{2} - \frac{3\sqrt{3}}{2}), \tilde{p}_2 := (\frac{1}{2}, -\frac{1}{4})\}$. Consider γ_1, γ_2, $\tilde{\gamma}_2$ meridians around p_1, p_2, and $\tilde{p}_2$ respectively. One can check that these meridians satisfy the following relations as closed paths in the total space $\mathbb{C}^2 \setminus D_1$:

$$\tilde{\gamma}_2 = \gamma_2$$
$$\gamma_2\gamma_1\gamma_2 = \gamma_1\gamma_2\gamma_1$$

Moreover,

$$\pi_1(\mathbb{P}^2 \setminus (D_1 \cup D_2)) = \langle \gamma_1, \gamma_2, \gamma_\infty : \gamma_2\gamma_1\gamma_2 = \gamma_1\gamma_2\gamma_1 = \gamma_\infty \rangle,$$

where γ_∞ is a meridian of D_2, the line at infinity of $\mathbb{P}^2$.

According to Theorem 4.23 we need to study subgroups $J_{\bar{e}}$ normally generated by $\gamma_1^{e_1}$, $\gamma_2^{e_1}$, and $\gamma_\infty^{e_2}$ for $\bar{e} = (e_1, e_2) \in \mathbb{N}^2$. Equivalently, one can study quotients of $\pi_1(\mathbb{P}^2 \setminus (D_1 \cup D_2))$ of the form

$$G_{\bar{e}} = \langle \gamma_1, \gamma_2, \gamma_\infty : \gamma_2\gamma_1\gamma_2 = \gamma_1\gamma_2\gamma_1 = \gamma_\infty, \gamma_1^{e_1} = \gamma_2^{e_1} = \gamma_\infty^{e_2} = 1 \rangle.$$

Such subgroups are well known (c.f. [**23**]) and $G_{\bar{e}}$ is finite if and only if $\bar{e} = (2,2)$, $(3,4)$, $(4,8)$, $(5,20)$ or $(6,2)$. In which cases one has the following result (c.f. [**57**, Propositions 1.3.27 and 1.3.29]):

THEOREM 4.26. *In the following cases there is a maximal Galois covering of $\mathbb{P}^2$ ramified along D:*

(e_1, e_2)	$G = \pi_1(\mathbb{P}^2 \setminus D)/J_{\bar{e}}$	$\lvert G \rvert$
$(2,2)$	Σ_3	6
$(3,4)$	$SL(2,\mathbb{Z}/3\mathbb{Z})$	24
$(4,8)$	$\Sigma_4 \ltimes \mathbb{Z}/4\mathbb{Z}$	96
$(5,20)$	$SL(2,\mathbb{Z}/5\mathbb{Z}) \times \mathbb{Z}/5\mathbb{Z}$	600

However, there is no maximal Galois cover of $\mathbb{P}^2$ ramified along $D = 6D_1 + 2D_2$.

Analogously to the Riemann Existence Theorem 4.20, one has the following result on the existence of branched covers ramified along divisors with prescribed ramification index.

THEOREM 4.27. *Let $B = D_1 \cup ... \cup D_n$ be a projective manifold. Then any representation of $\pi_1(M \setminus B)$ on a linear group $GL(r, \mathbb{C})$ such that the image of a meridian γ_i has order e_i, gives rise to a Galois cover of M branched along $D = e_1 D_1 + ... + e_n D_n$.*

The proof of this result is similar to the one presented here for Theorem 4.20 and it relies on the fact that $\pi_1(M \setminus B)$ is finitely generated, which is a consequence of the Zariski Theorems of Lefschetz Type (see §1.7) and §2.

4.28. Chisini Problem. In this context, another interesting motivation is the following problem:

PROBLEM 4.29. **Chisini Problem** [**19**] *Let S be a non-singular compact complex surface, let $\pi : S \to \mathbb{P}^2$ be a finite morphism having simple branching, and let B be the branch curve; then "to what extent does the pair $(\mathbb{P}^2, B)$ determine π"?*

Partial results have been given to this problem for *generic* coverings [**55, 44, 43, 58**], or special types of singularities [**42, 52**], but a global answer to this is yet to be determined. Certain restrictions, like the fact that the degree of the covering has to be ≥ 5, are also known [**54, 15**].

5. Monodromy Action on Fundamental Groups

Probably the first appearance in the literature of this fact is due to O.Chisini [**18**], and has been implicitly used by V.Kampen [**74**] and O.Zariski [**76**] in the context of computing the fundamental group of plane projective curve complements. The first systematic approach for the case of plane curves is given by B.Moishezon [**55**] with the purpose of studying the Chisini Conjecture.

In order to give a general definition in our setting let us recall the notion of section.

DEFINITION 5.1. Let $\pi : X \to M$ be a locally trivial fibration. We say that a morphism $s : M \to X$ is a *section* if $\pi \circ s = \mathbf{1}_M$.

Associated with a locally trivial fibration $\pi : X \to M$ and a section $s : M \to X$ there is a right action of the groupoid $\{\pi_1(M, p_1, p_2)\}$ on the groups $\{\pi_1(F, q_0)\}$, called *monodromy action of* M on F. More specifically, given a path $\gamma \in \pi_1(M, p_1, p_2)$ ($s(p_1) = (p_1, q_1), s(p_2) = (p_2, q_2)$) and a closed path $\alpha \in \pi_1(F, q_1)$, one obtains another closed path $\alpha^\gamma \in \pi_1(F, q_2)$.

In addition, if $\gamma_1 \in \pi_1(M, p_1, p_2)$ and $\gamma_2 \in \pi_1(M, p_2, p_3)$, with $s(p_i) = (p_i, q_i)$, then

$$\alpha^{(\gamma_1\gamma_2)} = (\alpha^{\gamma_1})^{\gamma_2}.$$

5.2. Construction of the monodromy. Consider γ an open path representing an element in $\pi_1(M, p_1, p_2)$. The following diagram comes from restriction:

$$\begin{array}{rccc} \pi^{-1}(\gamma) = & \tilde{X} & \hookrightarrow & X \\ & \downarrow \tilde{\pi} & & \downarrow \pi \\ & [0,1] & \xrightarrow{\gamma} & M \end{array}$$

Note the following:

(1) The map $\tilde{\pi}$ is a fibration which, by Example 3.6, is trivial, and hence consider a trivialization $[0,1] \times F \xrightarrow{\varphi} \tilde{X}$ and a section $\tilde{s} := \varphi^{-1} \circ s \circ \gamma : [0,1] \to [0,1] \times F$ (see (4)).
(2) Any path $\alpha \in \pi_1(F, q_1)$ can be regarded as a path $\alpha : [0,1] \to \{0\} \times F$, based at $s(p_1) = (p_1, q_1)$, and it is a lifting of $\mathbf{0} : [0,1] \to [0,1]$ the 0 constant path (see (4)).
(3) By the Homotopy Lifting Property 3.4, the homotopy $h : [0,1] \times [0,1] \to [0,1]$ given by $h(\lambda, \mu) = \mu$, which takes the constant zero path $\mathbf{0}$ to the constant path $\mathbf{1}$ can be lifted to $\tilde{h} : [0,1] \times [0,1] \to [0,1] \times F$ such that $\tilde{h}(\lambda, 0) = \alpha(\lambda)$ and $\tilde{h}(0, \mu) = \tilde{h}(1, \mu) = \tilde{s}(\mu)$.

$$\begin{array}{ccccccc} & & [0,1] \times F & \xrightarrow{\varphi} & \tilde{X} & \hookrightarrow & X \\ & {}^{\alpha}\nearrow & \downarrow pr_1 & & \downarrow \tilde{\pi} & & \downarrow \pi \\ [0,1] & \xrightarrow{\mathbf{0}} & [0,1] & = & [0,1] & \xrightarrow{\gamma} & M \end{array} \tag{4}$$

DEFINITION 5.3. The closed path $\alpha^\gamma(\lambda) := \varphi \circ \tilde{h}(\lambda, 1) \in \pi_1(F, q_2)$ constructed above is called the *monodromy action* of γ over α.

REMARK 5.4. Intuitively, α is being *pushed* fiberwise along γ and keeping the base point along the section s.

One needs to check that the previous construction is independent of φ, the choice of representative of γ and α. This is all a consequence of the Homotopy Lifting Property.

Note that, according to our discussion, $\alpha^\gamma = s(\gamma)^{-1} \alpha s(\gamma)$ (see Figure 4)

The following example will clarify the previous construction.

EXAMPLE 5.5. The trivial case occurs when $q_0 := q_1 = q_2$, the fibration π is trivial, and the section $s : M \to M \times F$ is given by $s(p) = (p, q_0)$. In this case, the monodromy action is trivial.

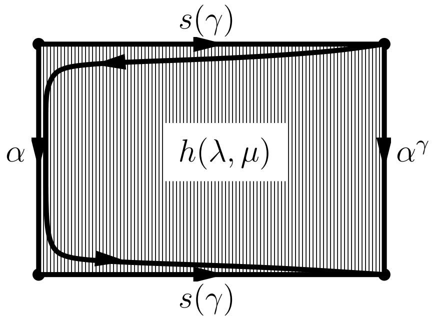


FIGURE 4.

EXAMPLE 5.6. The simplest non-trivial case arises when $q_0 := q_1 = q_2$, the fibration π is trivial, but the section is not constant on the second coordinate. For instance, consider $[0,1] \times F \overset{pr_1}{\longrightarrow} [0,1]$, where $F = \mathbb{D} \setminus \{(0, \frac{1}{2}), (0, -\frac{1}{2})\}$, (see Figure 5)

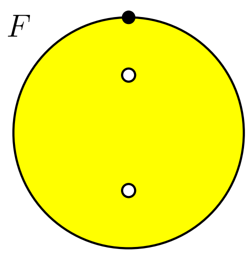


FIGURE 5. Fiber F

If γ represents the identity on the base, $p_1 = 0$, $p_2 = 1$, $q_0 = (0,1)$, and $s(\lambda) = \exp(2\pi\sqrt{-1}\lambda)$ is a section, then Figure 6 describes this monodromy action on two closed paths α_1, α_2.

EXAMPLE 5.7. Consider F as before, define $X = \mathbb{S}^1 \times F$ with a non-trivial section $s : \mathbb{S}^1 \to X$ given by $s(\lambda) := (\lambda, \lambda)$ (note that $\mathbb{S}^1 \subset F$). Note that φ can be given as the exponential map. In particular, the trivialization φ along γ is not trivial (see Figure 7).

where $\tilde{s}$ is just the section $\varphi^{-1} \circ s \circ \gamma : I \to I \times F$.

In this case, the closed paths α_1, α_2 shown in the previous example are transformed as shown in Figure 8,

that is, $\alpha_1^\gamma = \alpha_2$, $\alpha_2^\gamma = \alpha_2\alpha_1\alpha_2^{-1}$.

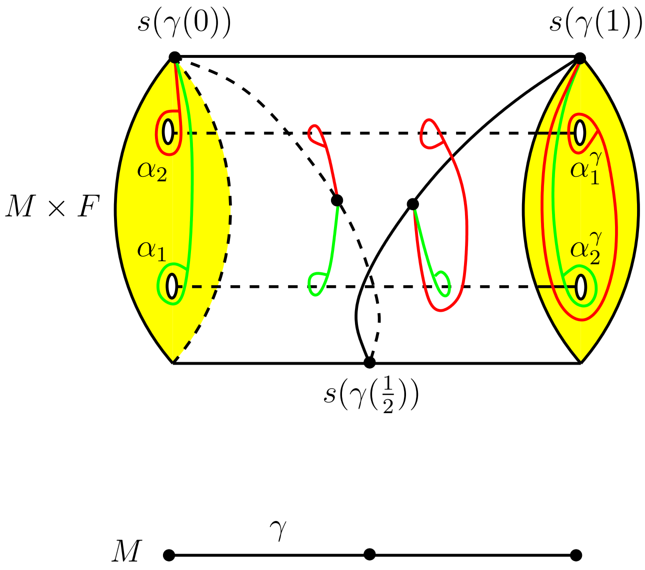


FIGURE 6. Monodromy Action

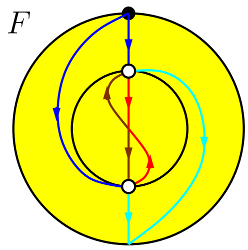


FIGURE 7. Trivialization

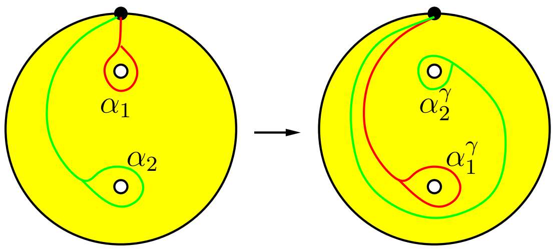


FIGURE 8. Monodromy Action

6. Mapping Class Groups and Braid Action

The group of oriented isomorphisms of a compact orientable surface S of genus g fixing a set of n points up to homotopy relative to its boundary is called the *mapping class group* of S_n^g, and will be denoted by $\mathfrak{M}(S_n^g)$.

A classical interpretation of the geometric braid group on n-strings (see Example 1.5) is the following.

THEOREM 6.1 ([**12**]). *There is an isomorphism between the* geometric *group of braids on n-strings and the mapping class group of the disk $\mathbb{D}$ fixing a set of n points, that is,*

$$\mathfrak{M}(\mathbb{D}_n) = \pi_0(\mathit{Diff}^+(\mathbb{D}_n, \partial\mathbb{D})) \cong \pi_1(Y_n) = \mathbb{B}_n.$$

This allows one to interpret the action of the braid group on free groups as a monodromy action.

REMARK 6.2. The proof of Theorem 6.1 usually involves proving another interesting result: $\mathrm{Diff}^+(\mathbb{D}_n, \partial\mathbb{D})$ is contractible, namely, any diffeomorphism in $\mathrm{Diff}^+(\mathbb{D}_n, \partial\mathbb{D})$ is isotopic to the identity map $\mathbb{1}_{\mathbb{D}}$.

The previous Remark implies the following.

PROPOSITION 6.3. *The set $\mathit{Diff}^+(\mathbb{D}_n, \partial\mathbb{D})$ is naturally in bijection with the set of trivializations along $[0,1]$ of locally trivial fibrations of fiber $\mathbb{D} \setminus Z_n$.*

Note that the trivializations are nothing but the isotopy that joins a diffeomorphism and the identity.

Using Proposition 6.3 and Theorem 6.1 one can consider the action, via monodromy, of a braid in $\mathbb{B}_n$ on $\pi_1(\mathbb{D} \setminus Z_n) = \mathbb{F}_n = \mathbb{Z}g_1 * ... * \mathbb{Z}g_n$.

It is an interesting exercise to convince oneself that the monodromy action of a standard basis $\sigma_1, \ldots, \sigma_{n-1}$ on $g_1 \ldots, g_n$ is given as follows:

$$g_j^{\sigma_i} = \begin{cases} g_{i+1} & j = i \\ g_{i+1} g_i g_{i+1}^{-1} & j = i+1 \\ g_i & \text{otherwise.} \end{cases} \tag{5}$$

This is basically a consequence of Example 5.7 and Figure 8.

REMARK 6.4. Since $(g_n \cdot ... \cdot g_1) = \partial\mathbb{D}$, note that one obtains $(g_n \cdot ... \cdot g_1)^\sigma = (g_n \cdot ... \cdot g_1)$.

EXAMPLE 6.5. Consider $\pi : X = \mathbb{D}^* \times \mathbb{D} \setminus \{y^2 - x^k = 0\} \to M = \mathbb{D}^*$, where $\mathbb{D}$ is the disk centered at 0 of radius 2, defined by $(x, y) \mapsto x$. Note that π is a proper submersion, and hence a locally trivial fibration by the Ehresmann Fibration Theorem 3.7.

Since $\pi_1(M) = \mathbb{Z}$ (Example 2.5), in order to calculate the monodromy action of the base, it is enough to compute the braid produced by the path $\gamma(\lambda) = \exp(2\pi\sqrt{-1}\lambda)$, which generates $\pi_1(M, 1)$. Note that $\pi^{-1}(\gamma(\lambda)) = \{(\exp(2\pi\sqrt{-1}\lambda), \exp(\pi\sqrt{-1}\lambda k))\}$. The braid $(\exp(\pi\sqrt{-1}\lambda k), \lambda)$ is depicted in Figure 9 and it can be described as σ_1^k.

Therefore,

$$g_1^\gamma = g_1^{\sigma_1^k} = \begin{cases} (g_2 g_1)^{\frac{k}{2}} g_1 (g_2 g_1)^{-\frac{k}{2}} & \text{if } k \text{ even} \\ (g_2 g_1)^{\frac{k-1}{2}} g_2 (g_2 g_1)^{-\frac{k-1}{2}} & \text{if } k \text{ odd,} \end{cases}$$

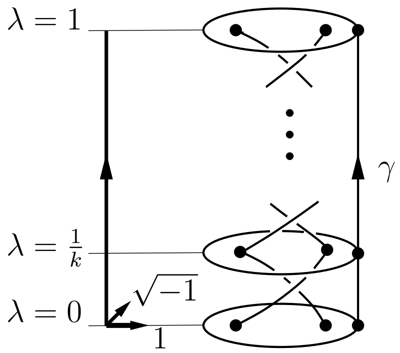
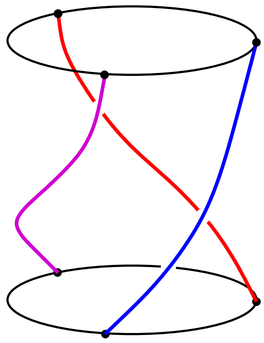


FIGURE 9. Braid monodromy of $\{y^2 - x^k = 0\}$

and

$$g_2^\gamma = g_2^{\sigma_1^k} = g_1^{\sigma_1^{k+1}}.$$

EXAMPLE 6.6. Another interesting example is the monodromy of the fibration $\pi : X = \mathbb{D}^* \times \mathbb{D} \setminus \{y^k = x\} \to M = \mathbb{D}^*$. Following Example 6.5 one obtains



which corresponds to the braid $\sigma := \sigma_1 \sigma_2 \cdots \sigma_{k-1}$. Note that

$$g_i = g_i^\sigma = \begin{cases} g_k & i = 1 \\ g_k^{-1} g_{i-1} g_k & i \neq 1. \end{cases} \tag{6}$$

EXAMPLE 6.7. Based on Example 6.6 one can generalize this construction to study the monodromy of the fibration $\pi : X = \mathbb{D}^* \times \mathbb{D} \setminus \{y^q = x^p\} \to M = \mathbb{D}^*$. It is easy to see that such monodromy is nothing but p times the monodromy of $\pi : X = \mathbb{D}^* \times \mathbb{D} \setminus \{y^q = x\} \to M = \mathbb{D}^*$, which corresponds to the braid $(\sigma_1 \sigma_2 \cdots \sigma_{q-1})^p$. In particular, one can recuperate the result given in Example 6.5 for $q = 2$, $p = k$.

7. Zariski Theorem of Lefschetz Type

From the previous sections one fact seems to be worth stressing:

> *In order to understand coverings of M ramified along D one needs to study $\pi_1(M \setminus B)$.*

How to compute the fundamental group $\pi_1(M \setminus B)$ of a quasi-projective variety? The following crucial result, known as the *Zariski Theorem of Lefschetz Type* (cf. [**38, 33**]) states that it is enough to understand complements of curves on surfaces.

THEOREM 7.1 (Hamm, Goreski-MacPherson). *Let $M \subset \mathbb{P}^n$ be a closed subvariety which is locally a complete intersection of dimension m. Let $\mathcal{A}$ be a* Whitney stratification *of M and consider $B \subset \mathbb{P}^n$ another subvariety such that $B \cap M$ is a union of strata of $\mathcal{A}$. Consider H a hyperplane transversal to $\mathcal{A}$ in $M \setminus B$, then the inclusion*

$$(M \setminus B) \cap H \hookrightarrow M \setminus B$$

is an $(m-1)$-homotopy equivalence.

For this reason, we will be mostly concerned about complements of projective curves in the complex plane $\mathbb{P}^2$. However, it is important to stress that the general problem of computing homotopy groups of complements to singular varieties and relating them to other invariants of the complement is a very interesting question in and of its own (see [**64, 49, 50**]).

CHAPTER 2

Zariski-Van Kampen Method

Originally sketched by O. Zariski [**76**] and later completed by E.R.Van Kampen [**74**]. Later on, D. Chéniot [**16**] gave a modern approach to this method. The Zariski-Van Kampen method allow one to give a finite presentation for the fundamental group of the complement to a projective plane curve. It is hence a constructive method and in some cases it is even effective, i.e. it has been implemented in the case of line arrangements, curves with *easy singularities* and equations on the Gaussian integers $\mathbb{Z}[\sqrt{-1}]$ (see [**14, 11**]). A very nice approach to this method can be found in the unpublished notes written by I.Shimada in [**71**].

We will put together several ingredients, among which, the *Van Kampen Theorem* is key.

1. Fundamental Group of the Total Space of a Locally Trivial Fibration

Let $\pi : X \to M$ be a locally trivial fibration with section $s : M \to X$. Consider $p \in M$ and $x_0 \in F_p$.

THEOREM 1.1. $\pi_1(X, x_0) = \pi_1(F_p, x_0) \rtimes \pi_1(M, p)$, *where the action of* $\pi_1(M, p)$ *on* $\pi_1(F_p, x_0)$ *is given by the monodromy of* π.

PROOF. First of all note that the existence of a section implies that

$$\pi_* \circ s_* : \pi_i(M) \xrightarrow{s_*} \pi_i(X) \xrightarrow{\pi_*} \pi_i(M)$$

is the identity, and hence $\pi_i(M) \xrightarrow{s_*} \pi_i(X)$ is surjective. Therefore, the homotopy exact sequence of the fibration becomes:

$$1 \to \pi_i(F_p) \xrightarrow{i_*} \pi_i(X) \underset{\pi_*}{\overset{s_*}{\rightleftarrows}} \pi_i(M) \to 1$$

for any $i \in \mathbb{N}$. In particular, we are interested in $i = 1$. Since $\pi_i(X) \xrightarrow{\pi_*} \pi_i(M)$ splits, $\pi_i(X)$ endows a semi-direct product structure, that is, $\pi_1(M) = \{(\gamma, \alpha) \mid \gamma \in \pi_i(M), \alpha \in \pi_i(F_p)\}$ as a set (where (γ, α) is nothing but $s_*(\gamma)i_*(\alpha)$) and the product structure is given by

$$\begin{aligned}(\gamma_1, \alpha_1) \cdot (\gamma_2, \alpha_2) &= s_*(\gamma_1)i_*(\alpha_1)s_*(\gamma_2)i_*(\alpha_2) = \\ &= s_*(\gamma_1)s_*(\gamma_2)s_*(\gamma_2)^{-1}i_*(\alpha_1)s_*(\gamma_2)i_*(\alpha_2) = \\ &= (\gamma_1\gamma_2, s_*(\gamma_2)^{-1}\alpha_1 s_*(\gamma_2)\alpha_2).\end{aligned}$$

By Remark 5.4, $s_*(\gamma_2)^{-1}\alpha_1 s_*(\gamma_2) = \alpha_1^{\gamma_2}$, thus

$$(\gamma_1, \alpha_1) \cdot (\gamma_2, \alpha_2) = (\gamma_1\gamma_2, \alpha_1^{\gamma_2}\alpha_2)$$

is given by the monodromy action of $\pi_1(M)$ on $\pi_1(F_p)$. □

Besides Proposition 4.14, we need another basic result on meridians.

PROPOSITION 1.2. *Let $B \subset M$ be an irreducible hypersurface in M, then the inclusion $M \setminus B \hookrightarrow M$ induces a surjective morphism $\pi_1(M \setminus B) \to \pi_1(M)$, whose kernel is $\langle \gamma \rangle$, the normal subgroup of $\pi_1(M \setminus B)$ generated by a meridian of B.*

PROOF. Basically, if $\alpha \in \pi_1(M \setminus B)$ is such that $i_*(\alpha) = 1$, then α is the boundary of a disk, say $\mathbb{D}$, in M. Since B is a hypersurface and $\mathbb{D}$ is compact, then (after pushing $\mathbb{D}$ in general position) the intersection $\mathbb{D} \cap B$ is a finite number of points $b_1, \dots, b_n$ (see Figure 1).

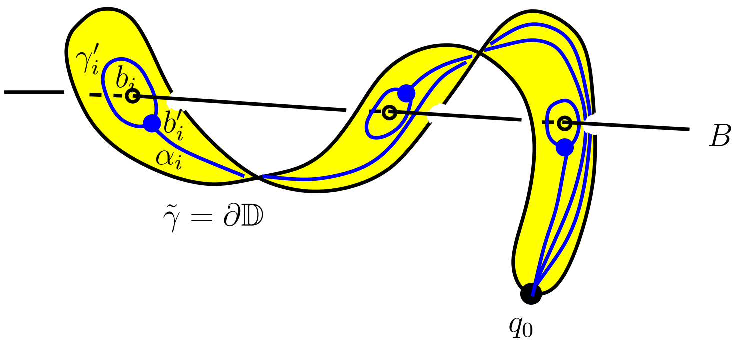


FIGURE 1. Kernel

Note that $n \geq 1$ or else $\mathbb{D} \subset M \setminus B$ and hence $\tilde{\gamma} = 1$ in $\pi_1(M \setminus B)$. Consider disks $\mathbb{D}_1, \dots, \mathbb{D}_n$ on $\mathbb{D}$ such that $\mathbb{D}_i \cap B = \{b_i\}$ and paths α_i from q_0 to $b_i' \in \partial\mathbb{D} = \gamma_i$ $(i = 1, \dots, n)$ such that $\gamma = \prod_{i=1}^n \alpha_i \cdot \gamma_i^{\varepsilon_i} \cdot \alpha^{-1}$, where $\varepsilon_i = \pm 1$. Note that $\gamma_i' = \alpha_i \cdot \gamma_i \cdot \alpha^{-1}$ is a meridian around B, and hence, by Proposition 4.14 $\gamma_i' \in \langle \gamma \rangle$, which implies $\tilde{\gamma} \in \langle \gamma \rangle$. □

2. Zariski-Van Kampen Theorem

Let $\mathcal{C} \subset \mathbb{P}^2$ be a projective plane curve given as the zeroes of a reduced homogeneous polynomial $f \in \mathbb{C}[X, Y, Z]$ of degree d. After a suitable change of coordinates one can assume $P = [0 : 1 : 0] \in \mathbb{P}^2 \setminus \mathcal{C}$ and thus one can consider the projection $\pi : \mathbb{P}^2 \setminus \{P\} \to \mathbb{P}^1$ from P. Note that, for any point $z = [x_0 : z_0]$ the preimage $\pi|_{\mathcal{C}}$ consists of a finite number of points, precisely the roots of the one-variable polynomial $f(x_0, t, z_0)$.

LEMMA 2.1. *If $P \notin \mathcal{C}$, then $f(x_0, t, z_0) \in \mathbb{C}[t]$ has degree exactly d.*

PROOF. One can write $f(X, Y, Z) = aY^d + Xf_1(X, Y, Z) + Zf_2(X, Y, Z)$, where $\deg_Y f_i < d$, $i = 1, 2$. By hypothesis, $f(P) = f(0, 1, 0) = a \neq 0$, therefore $f(x_0, t, z_0) = at^d + x_0 f_1(x_0, t, z_0) + z_0 f_2(x_0, t, z_0)$ has degree d as a polynomial in $\mathbb{C}[t]$. □

This implies that $\pi|_{\mathcal{C}}$ is a branched cover of $\mathbb{P}^1$ of degree d ramified on $\Delta := \{[x_0 : z_0] \in \mathbb{P}^1 \mid \partial_t f(x_0, t, z_0) = f(x_0, t, z_0) = 0\}$, that is, $\Delta = \{\mathrm{Discrim}_Y(f) = 0\}$. In other words, $\pi|_{\mathcal{C}}$ ramifies along those points of $\mathbb{P}^1$ whose *vertical lines* above them intersect $\mathcal{C}$ in less than d distinct points (see Figure 2).

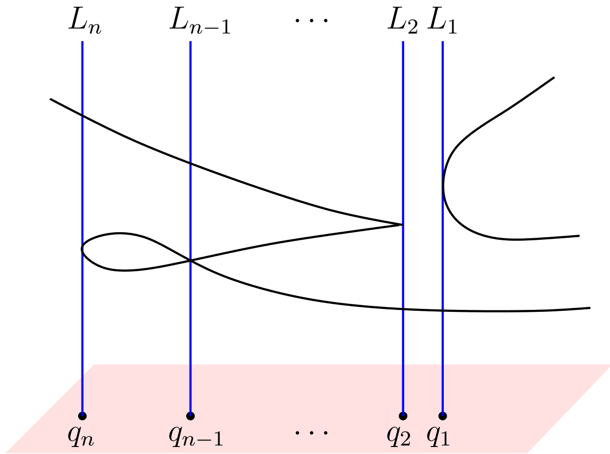


FIGURE 2. The projection from P

Let $\mathcal{L} := L_1 \cup \cdots \cup L_n$ be the union of the non-generic vertical lines, that is, $\mathcal{L} := \pi^{-1}(\Delta)$. Even though π is a locally trivial fibration, there are two problems: first of all it is not so obvious since all the fibers are *very close* to P and second of all, the fiber is NOT $\pi^{-1}([x_0 : z_0]) \cong \mathbb{P}^1 \setminus \{P\}$. We would like to *separate* the fibers. In order to do so one can construct another complex space X from $\mathbb{P}^2$ by replacing P by the $\mathbb{P}^1$ of lines passing through P. In other words, each line $L_q := \pi^{-1}([x_0 : z_0])$ will be compactified not by adding P, but by adding a point P_q. Algebraically this can be done as follows. Consider $U_Y = \{[X : Y : Z] \mid Y \neq 0\}$ an affine chart of $\mathbb{P}^2$ containing P and define the following map $\varepsilon : \tilde{U}_Y \to U_Y$, where $\tilde{U}_Y = \{[X : Y : Z] \times [u : v] \in U_Y \times \mathbb{P}^1 \mid uZ = vX\}$, given by the projection onto the first component $\varepsilon([X : Y : Z], [u : v]) = [X : Y : Z]$. Note that

$$\varepsilon^{-1}([x_0 : 1 : z_0]) = \begin{cases} ([x_0 : 1 : z_0], [z_0 : x_0]) & \text{if } x_0 z_0 \neq 0 \\ ([0 : 1 : 0], [u : v]) =: E \cong \mathbb{P}^1 & \text{if } x_0 = z_0 = 0. \end{cases} \tag{7}$$

and hence $\tilde{U}_Y \setminus E \cong U_Y \setminus \{P\}$. Since the other standard affine charts of $\mathbb{P}^2$ do not contain P, namely $P \notin U_X := \{[X : Y : Z] \mid X \neq 0\}$, $P \notin U_Z := \{[X : Y : Z] \mid Z \neq 0\}$, one can glue the charts $\tilde{U}_Y$, U_X, and U_Z using the same transition functions as for U_Y, U_X, and U_Z. This way one defines the manifold X.

Now $\tilde{\pi} = \pi \circ \varepsilon$ can be extended to X as follows

$$\tilde{\pi}(\tilde{P}) = \begin{cases} [v : u] & \text{if } \tilde{P} = ([X : Y : Z], [u : v]) \in \tilde{U}_Y \\ [x_0 : z_0] & \text{if } P = [x_0 : 0 : z_0]. \end{cases}$$

According to (7) one can check that $\tilde{\pi}|_{X \setminus E} = \pi|_{\mathbb{P}^2 \setminus \{P\}}$. Moreover, if $\tilde{L}_q := \tilde{\pi}^{-1}([x_0 : z_0])$, $q = [x_0, z_0]$, then

$$\tilde{L}_q \cap \tilde{U}_Y = \{([tx_0 : s : tz_0], [z_0 : x_0]) \mid [t : s] \in \mathbb{P}^1, s \neq 0\}$$
$$\tilde{L}_q \cap U_X = \tilde{L}_q \cap U_Z = \{[tx_0 : s : tz_0] \mid [t : s] \in \mathbb{P}^1, t \neq 0\}$$

Hence $L_q \cup \{\tilde{P}_q := ([0:1:0],[z_0:x_0])\}$ and $\tilde{L}_q \cong \mathbb{P}^1$.

Define $\tilde{\mathcal{C}} := \varepsilon^{-1}(\mathcal{C})$, $\tilde{\mathcal{L}} := \varepsilon^{-1}(\mathcal{L})$ the preimages of $\mathcal{C}$ and $\mathcal{L}$, respectively, by the blow-up. Note that $\tilde{\mathcal{C}} \cong \mathcal{C}$ by the above discussion, since $P \notin \mathcal{C}$. Also, note that $\tilde{\pi}|_{\tilde{\mathcal{C}}}$ is a branched cover and $\tilde{\pi}|_{(\tilde{\mathcal{C}}\cup\tilde{\mathcal{L}})}$ is an unbranched cover.

Finally, $\tilde{\pi}$ is a proper submersion, and hence (by the Ehresmann's Fibration Theorem 3.7) a locally trivial fibration of fiber $\tilde{L}_q \cong \mathbb{P}^1$. Moreover, since $\tilde{\mathcal{C}}$ is compact, using once again Theorem 3.7, $\tilde{\pi}|_{X\setminus(\tilde{\mathcal{C}}\cup\tilde{\mathcal{L}})}$ is also a locally trivial fibration of fiber $\mathbb{P}^1 \setminus Z_d$, where Z_d is a union of d distinct points.

Summarizing:

PROPOSITION 2.2. *The map $\tilde{\pi} : X \setminus (\tilde{\mathcal{C}} \cup \tilde{\mathcal{L}}) \to \mathbb{P}^1 \setminus \Delta$ is a locally trivial fibration of fiber $F := \mathbb{P}^1 \setminus Z_d$. Moreover, $\pi_1(X \setminus (\tilde{\mathcal{C}} \cup \tilde{\mathcal{L}})) = \pi_1(\mathbb{P}^2 \setminus (\mathcal{C} \cup \mathcal{L}))$.*

PROOF. The first part is a consequence of the discussion above. For the second part, note that the map ε induces a morphism $\pi_1(X \setminus (\tilde{\mathcal{C}} \cup \tilde{\mathcal{L}})) \xrightarrow{\varepsilon_*} \pi_1(\mathbb{P}^2 \setminus (\mathcal{C} \cup \mathcal{L}))$. Let us show that ε_* is an isomorphism. Note that any class in $\pi_1(\mathbb{P}^2 \setminus (\mathcal{C} \cup \mathcal{L}))$ can be described by a closed path γ avoiding P (since we can restrict ourselves to piecewise smooth representatives as mentioned in Remark 1.1) as shown in Figure 3.

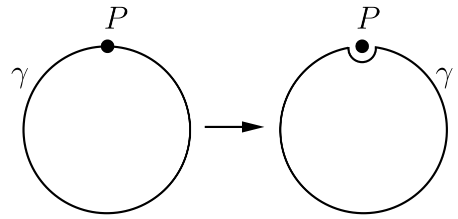


FIGURE 3. Avoiding a zero dimensional subset

Since $X \setminus (E \cup \tilde{\mathcal{C}} \cup \tilde{\mathcal{L}}) \overset{\varepsilon}{\cong} \mathbb{P}^2 \setminus (\{P\} \cup \mathcal{C} \cup \mathcal{L})$, there exists $\tilde{\gamma} \in \pi_1(X \setminus (E \cup \tilde{\mathcal{C}} \cup \tilde{\mathcal{L}}))$ such that $\varepsilon_*(\tilde{\gamma}) = \gamma$, which shows that ε_* is surjective. Analogously, $\varepsilon_*(\tilde{\gamma})$ is trivial in $\pi_1(\mathbb{P}^2 \setminus (\mathcal{C} \cup \mathcal{L}))$ for some $\tilde{\gamma} \in \pi_1(X \setminus (\tilde{\mathcal{C}} \cup \tilde{\mathcal{L}}))$ if $\varepsilon_*(\tilde{\gamma}) = \partial\mathbb{D}$ for a certain disk in $\mathbb{P}^2 \setminus (\mathcal{C} \cup \mathcal{L})$. As above, if $\mathbb{D}$ intersects P, one can find a homotopic representative avoiding P, that is, $\mathbb{D} \subset \mathbb{P}^2 \setminus (\{P\} \cup \mathcal{C} \cup \mathcal{L})$. Again, since $X \setminus (E \cup \tilde{\mathcal{C}} \cup \tilde{\mathcal{L}}) \overset{\varepsilon}{\cong} \mathbb{P}^2 \setminus (\{P\} \cup \mathcal{C} \cup \mathcal{L})$, there exists $\tilde{\mathbb{D}} \subset X \setminus (E \cup \tilde{\mathcal{C}} \cup \tilde{\mathcal{L}})$ such that $\varepsilon(\tilde{\mathbb{D}}) = \mathbb{D}$ and $\partial\tilde{\mathbb{D}} = \tilde{\gamma}$, which shows that $\tilde{\gamma}$ is trivial in $\pi_1(X \setminus (E \cup \tilde{\mathcal{C}} \cup \tilde{\mathcal{L}}))$ and hence in $\pi_1(X \setminus (\tilde{\mathcal{C}} \cup \tilde{\mathcal{L}}))$. This shows that ε_* is injective. $\square$

As shown in Figure 4, one can choose $g_1, \ldots, g_d \in \pi_1(F)$ meridians around Z_d such that $g_d \cdots g_1 = 1$, that is,

$$\pi_1(F) = \langle g_1, \ldots, g_d : g_d \cdots g_1 = 1 \rangle$$

(see Example 2.6). Analogously, let us denote by $\gamma_1, \ldots, \gamma_n \in \pi_1(\mathbb{P}^1 \setminus \Delta)$ meridians around Δ such that $\gamma_n \cdots \gamma_1 = 1$, that is,

$$\pi_1(\mathbb{P}^1 \setminus \Delta) = \langle \gamma_1, \ldots, \gamma_n : \gamma_n \cdots \gamma_1 = 1 \rangle = \langle \gamma_1, \ldots, \gamma_{n-1} \rangle.$$

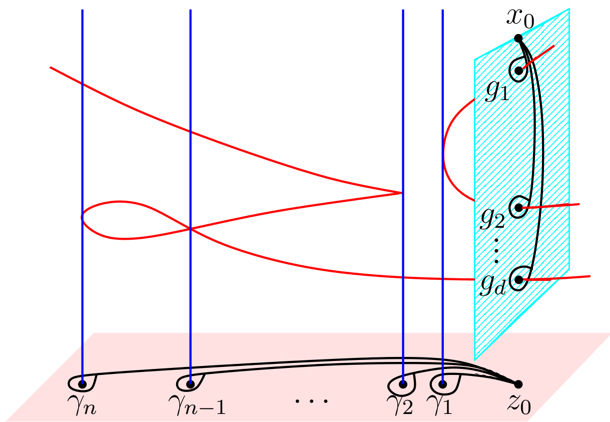


FIGURE 4. Choices of meridians

Under these conditions one has the following.

PROPOSITION 2.3.

$$\langle g_1, \ldots, g_d, \gamma_1, \ldots, \gamma_n : \gamma_n \cdots \gamma_1 = 1,\ g_d \cdots g_1 = 1,\ g_i^{\gamma_j} = \gamma_j^{-1} g_i \gamma_j,\ j = 1, \ldots, n-1\rangle$$

is a finite presentation of $\pi_1(\mathbb{P}^2 \setminus (\mathcal{C} \cup \mathcal{L}))$.

PROOF. It is an immediate consequence of Theorem 1.1 and Proposition 2.2. For the sake of simplicity, note that we have replaced $s_*(\gamma_j)$ simply by γ_j. The relations coming from the monodromy should read $g_i^{\gamma_j} = s_*(\gamma_j)^{-1} g_i s_*(\gamma_j)$ to be precise. □

Finally, one can give a presentation of $\pi_1(\mathbb{P}^2 \setminus \mathcal{C})$ as follows.

THEOREM 2.4 (Zariski-Van Kampen Theorem). *Let* $\mathcal{C} \subset \mathbb{P}^2$ *be a curve and* g_i, γ_j *meridians as constructed above. Then*

$$\langle g_1, \ldots, g_d : g_d \cdots g_1 = 1, g_i^{\gamma_j} = g_i,\ j = 1, \ldots, n-1\rangle$$

is a finite presentation of $\pi_1(\mathbb{P}^2 \setminus \mathcal{C})$.

PROOF. After using Proposition 1.2 for each irreducible component of $\mathcal{L}$, one obtains the following

$$\pi_1(\mathbb{P}^2 \setminus \mathcal{C}) = \langle g_1, \ldots, g_d, \gamma_1, \ldots, \gamma_{n-1} : g_d \cdots g_1 = 1, g_i^{\gamma_j} = \gamma_j^{-1} g_i \gamma_j\rangle / \langle \gamma_1, \ldots, \gamma_{n-1}\rangle =$$

$$= \langle g_1, \ldots, g_d : g_d \cdots g_1 = 1, g_i^{\gamma_j} = g_i\rangle.$$

□

REMARK 2.5. This is a first approach to a very fruitful combination of methods, which appears in several instances in singularity theory. For instance, a similar idea can be applied to higher homotopy groups as nicely presented by Chéniot-Libgober in [**17**].

As an immediate application of this Theorem one has the following.

COROLLARY 2.6. *Let $\mathcal{C} = \mathcal{C}_1 \cup \ldots \cup \mathcal{C}_r$ the decomposition of $\mathcal{C}$ in its irreducible components, then*

$$H_1(\mathbb{P}^2 \setminus \mathcal{C}) = \mathbb{Z}^{r-1} \oplus \mathbb{Z}/\tau,$$

where $d_i := \deg \mathcal{C}$ and τ is the greatest common divisor of $d_1, \ldots, d_r$.

PROOF. We will use the fact that $H_1(X) = \pi_1(X)/[\pi_1(X), \pi_1(X)]$, that is, the first homology group $H_1(X)$ of a topological space X is the abelianization of $\pi_1(X)$ its fundamental group (see for instance [**56**, Lemma 11.69.3]).

First of all, by Proposition 4.14, we know that $H_1(\mathbb{P}^2 \setminus \mathcal{C})$ is a quotient of $\mathbb{Z}^r$, since $\pi_1(\mathbb{P}^2 \setminus \mathcal{C})$ is generated by meridians of the irreducible components of $\mathcal{C}$ and any two meridians of the same irreducible component are conjugated.

Finally, and this is the key here, Theorem 2.4 specifies that the quotient of $\mathbb{Z}^r$ mentioned above comes from *abelianizing* the relations $g_d \cdots g_1 = 1$, $g_i^{\gamma_j} = g_i$, $j = 1, \ldots, n-1$. By construction of the monodromy, the element $g_i^{\gamma_j}$ is a meridian around the same irreducible component as g_i, hence these relations are trivial in H_1. The only relation left is $g_d \cdots g_1 = 1$. Note that in the set $\{g_1, \ldots, g_d\}$ there are exactly d_i meridians of the component $\mathcal{C}_i$, hence, after abelianizing, $g_d \cdots g_1 = 1$ becomes

$$d_1 m_1 + \cdots + d_r m_r = 0, \tag{8}$$

where $m_1, \ldots, m_r$ represent cycles around the component $\mathcal{C}_1, \ldots, \mathcal{C}_r$ respectively. Therefore

$$H_1(\mathbb{P}^2 \setminus \mathcal{C}) = \frac{\mathbb{Z}m_1 \oplus \cdots \oplus \mathbb{Z}m_r}{d_1 m_1 + \cdots + d_r m_r},$$

which has rank $r-1$ and non-trivial torsion τ if and only if $\tau = (d_1, ..., d_r) > 1$. □

REMARK 2.7. The projection π used for the Zariski-Van Kampen method as presented here is only required to be performed from a point $P \notin \mathcal{C}$. Originally, π was asked to be *generic* in the following sense:

(1) Any line L through P contains at most one singular point of $\mathcal{C}$ or one tangency,
(2) no lines through P are higher order tangents at a smooth point of $\mathcal{C}$, and
(3) any line L through P that intersects $\mathcal{C}$ at a singular point Q satisfies that $\operatorname{mult}_Q(\mathcal{C}) = \operatorname{mult}_Q(L, \mathcal{C})$.

Geometrically, this means that the following cases are avoided in the locally trivial fibration π:

Obviously, one can always chose P so that π is generic, since the set of higher order tangencies at $\mathcal{C}$, lines containing more that one singular point, bitangencies, and lines in the tangent cone of a singularity of $\mathcal{C}$ is finite, so P can be chosen outside this set and $\mathcal{C}$.

The following, very natural, result assures that if two curves can be joint by a smooth path of equisingular curves, then their fundamental groups are isomorphic (for a proof see [**14**]).

PROPOSITION 2.8. *All curves in the same connected family of equisingular curves are isotopic.*

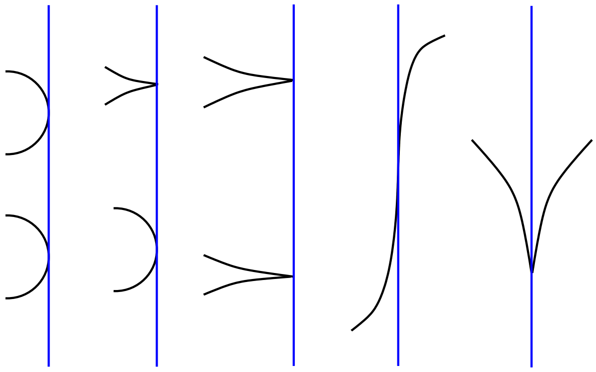


FIGURE 5. Non-generic projections

One also has results on how the fundamental group of a family of equisingular curves changes when degenerating onto other curves outside the equisingular locus.

PROPOSITION 2.9 ([**25**]). *Let $\{\mathcal{C}_t\}_{t\in(0,1]}$ be a continuous family of equisingular curves degenerating onto the reduced curve $\mathcal{C}_0$. Then there is a natural epimorphism*

$$\pi_1(\mathbb{P}^2 \setminus \mathcal{C}_0) \to \pi_1(\mathbb{P}^2 \setminus \mathcal{C}_t).$$

In particular,

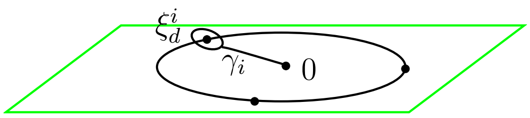
COROLLARY 2.10. *If $\mathcal{C}$ can be continuously degenerated onto a curve with abelian fundamental group, then $\pi_1(\mathbb{P}^2 \setminus \mathcal{C})$ is abelian as well.*

Other interesting degeneration results can be found in [**5**].

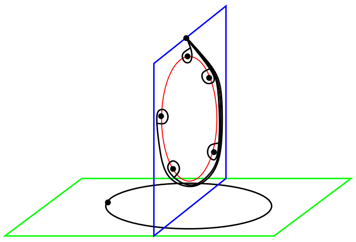
3. Basic examples

3.1. The fundamental group of smooth and nodal curves. First, we will compute the fundamental group of the curve $\mathcal{C} := \{F(X,Y,Z) = X^d + Y^d - Z^d = 0\}$ using the Zariski-Van Kampen method described above.

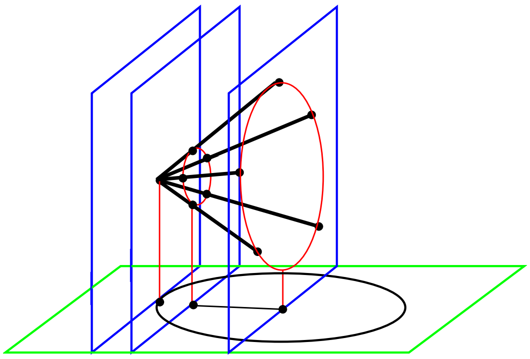
(1) Choose a point $P = [0 : 1 : 0] \notin \mathcal{C}$ and project from P.
(2) The projection $\pi : \mathbb{P}^2 \setminus \mathcal{C} \to \mathbb{P}^1$ ramifies along $\Delta := \{F = X^d + Y^d - Z^d = F_Y = dY^{d-1} = 0\} = \{[\xi_d^i : 0 : 1] \mid \xi = \exp\left(\frac{2\pi\sqrt{-1}}{d}\right)\}$. After blowing up, the projection $\tilde\pi|_{\tilde\pi^{-1}(\mathbb{P}^1\setminus\Delta)}$ is a locally trivial fibration of fiber $\mathbb{P}^1 \setminus Z_d$. Note that this projection is highly not generic, since each non-generic fiber, say $\tilde L_i = \tilde\pi^{-1}([\xi_d^i : 1])$, intersects $\tilde{\mathcal{C}}$ only at $[\xi_d^i : 0 : 1]$, that is, $\tilde L_i$ is a tangent of $\tilde{\mathcal{C}}$ of order d.
(3) Fix a base point on the base, say $[0 : 1]$
(4) $\pi_1(\mathbb{P}^1 \setminus \Delta) = \langle \gamma_1, \dots, \gamma_d : \gamma_d \cdots \gamma_1 = 1\rangle$, where γ_i is a meridian of ξ_d^i based at 0.



(5) Choose a basis on the fiber $L_0 := \tilde\pi^{-1}([0 : 1])$



(6) In order to compute the monodromy along γ_i, one can decompose $\gamma_i = \alpha_i \cdot \gamma'_i \cdot \alpha_i^{-1}$ where α_i is the straight path joining 0 and a point close to ξ_d^i and γ'_i is a loop around ξ_d^i. Note that the monodromy along α_i only produces a contraction of the points on the fiber.



(7) The monodromy along γ'_i is given in Example 6.6 as $\sigma_1 \cdots \sigma_{d-1}$ (independently of i).
(8) From (6) one obtains:

$$g_i = g_i^{(\sigma_1\sigma_2\cdots\sigma_{d-1})} = \begin{cases} g_d & i = 1 \\ g_d^{-1} g_{i-1} g_d & i \neq 1, \end{cases}$$

hence $g_2 = g_d^{-1} g_1 g_d = g_1$, and by induction $g_1 = \ldots = g_d = g$. Finally, $g_1 \cdots g_d = 1$ becomes $g^d = 1$. Therefore,

$$\pi_1(\mathbb{P}^2 \setminus \mathcal{C}) = \langle g : g^d = 1 \rangle = \mathbb{Z}/d\mathbb{Z}. \tag{9}$$

Note that all necessary relations are obtained by the monodromy action of any meridian γ_i. This can be further improved.

THEOREM 3.2. *If $\mathcal{C}$ is an irreducible curve with a maximal order tangent, that is, if there exists a line L such that $L \cap \mathcal{C} = \{Q\}$, then $\pi_1(\mathbb{P}^2 \setminus \mathcal{C})$ is abelian.*

PROOF. Consider $P \in L$, $P \neq Q$ and project from P. Since L becomes a non-generic fiber of the projection, one can fix a base point x_0 on $\mathbb{P}^1$ sufficiently close to the projection of L, say z_1. The monodromy around z_1 is given as in Example 6.6. The computation above shows that the relations obtained from this monodromy are enough to verify that $\pi_1(\mathbb{P}^2 \setminus \mathcal{C})$ is abelian. $\square$

Another application of the computations above.

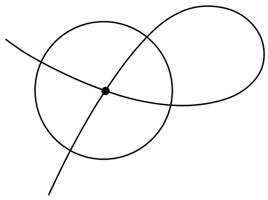
THEOREM 3.3. *If $\mathcal{C}$ is a smooth curve of degree d, then $\pi_1(\mathbb{P}^2 \setminus \mathcal{C}) = \mathbb{Z}/d\mathbb{Z}$.*

PROOF. The family of smooth curves of degree d is a quasi-projective variety in the projective space $\mathbb{P}^N$ of dimension $N = \binom{d-2}{2}$, where $N+1$ is the number of coefficients of a generic homogeneous polynomial of degree d in $\mathbb{C}[X, Y, Z]$. Therefore it is path connected, and, by Proposition 2.8, it is enough to compute the fundamental group of a particular smooth curve of degree d. The curve $\mathcal{C}$ defined above is smooth since

$$\begin{gathered}\operatorname{Sing}\mathcal{C} = \{[X:Y:Z] \in \mathbb{P}^2 \mid F_X = F_Y = F_Z = 0\} = \\ = \{[X:Y:Z] \in \mathbb{P}^2 \mid X^{d-1} = Y^{d-1} = Z^{d-1} = 0\} = \\ = \{[X:Y:Z] \in \mathbb{P}^2 \mid X = Y = Z = 0\} = \emptyset.\end{gathered}$$

Hence (9) gives the required fundamental group. □

The simplest singularities a curve can have are *nodes* (aka. *ordinary double points*), that is, singular points that admit local equations of the form $x^2 + y^2$, where x and y are generators of the local ring $\mathcal{O}_{\mathbb{P}^2,P}$. Note that $x^2 + y^2$ is equivalent to $x^2 - y^2 = (x-y)(x+y)$ by a *complex* change of coordinates. In other words, a node *locally looks* like a product of smooth transversal branches (locally meaning inside a neighborhood of the point, as shown below).



A more general result regarding nodal curves was already given by Zariski [**76**, Theorem 7].

THEOREM 3.4 (Zariski, Fulton, Deligne, Salvetti). *Any nodal curve has an abelian fundamental group.*

REMARK 3.5. As in our proof of Theorem 3.3 Zariski's proof of Theorem 3.4 depended on the irreducibility of the moduli spaces of nodal curves (there are different strata depending on the number and degrees of irreducible components). Such result had been claimed by Severi [**70**, Anhang F], and hence the proof given by Zariski was completed. However, later on, a gap was found in Severi's proof and hence Zariski's result was not complete anymore. Severi's assertion became *Severi's problem* and the original result by Zariski turned into the *Zariski conjecture on nodal curves* and they remained open until 1980, when Fulton [**32**] first and then Deligne [**24**] proved the Zariski conjecture on nodal curves (giving algebraic and topological proofs respectively) without using Severi's result. Finally, in 1986, J. Harris [**39**] solve the Severi problem. For a further study of such problems see [**61, 72, 34, 35, 36**] among others.

One can also find more recent proofs of this result by means of monodromy computations (see M.Salvetti [**66**]).

Also, generalizations of this result have been proved by M.V.Nori in [**59**].

The same ideas in Deligne's proof lead to the following result.

PROPOSITION 3.6. *If $\mathcal{C}_1$ and $\mathcal{C}_2$ are two curves intersecting transversally (only in ordinary double points), then*

$$\pi_1(\mathbb{P}^2 \setminus (L \cup \mathcal{C}_1 \cup \mathcal{C}_2)) = \pi_1(\mathbb{P}^2 \setminus (L \cup \mathcal{C}_1)) \oplus \pi_1(\mathbb{P}^2 \setminus (L \cup \mathcal{C}_2)),$$

where L is a line tranversal $\mathcal{C}_1 \cup \mathcal{C}_2$.

3.7. Further examples. By the previous sections we know how to compute fundamental groups of all curves of degrees one, two, and three:

(1) Degree one: $\pi_1(\mathbb{P}^2 \setminus L) = \{1\}$ (since L is smooth and of degree one)
(2) Degree two:
   (a) $\pi_1(\mathbb{P}^2 \setminus (L_1 \cup L_2)) = \mathbb{Z}$, where L_i is a line ($L_1 L_2$ is a nodal curve union of two smooth curves of degree one).
   (b) $\pi_1(\mathbb{P}^2 \setminus \mathcal{Q}) = \mathbb{Z}_2$, where $\mathcal{Q}$ is a conic (smooth curve of degree two).
(3) Degree three:
   (a) $\pi_1(\mathbb{P}^2 \setminus (L_1 \cup L_2 \cup L_3)) = \mathbb{Z}^2$, where L_i, $i = 1, 2, 3$, are lines in general position ($L_1 L_2 L_3$ is a nodal curve union of three smooth curves of degree one).
   (b) $\pi_1(\mathbb{P}^2 \setminus (L_1 \cup L_2 \cup L_3)) = \mathbb{Z} * \mathbb{Z}$, where L_i, $i = 1, 2, 3$, are concurrent lines.

   PROOF. Projecting from a point outside the lines one realizes that there is only one special fiber. Since Theorem 2.4 involves the monodromy action of all meridians but one, then there are no relations coming from monodromy, that is, $\pi_1(\mathbb{P}^2 \setminus (L_1 \cup L_2 \cup L_3)) = \langle g_1, g_2, g_3 : g_3 g_2 g_1 = 1 \rangle = \mathbb{Z} * \mathbb{Z}$. □

   (c) $\pi_1(\mathbb{P}^2 \setminus (\mathcal{Q} \cup L)) = \mathbb{Z}$, where $\mathcal{Q}$ is a conic and L is a line transversal to $\mathcal{Q}$.
   (d) $\pi_1(\mathbb{P}^2 \setminus (\mathcal{Q} \cup L)) = \mathbb{Z}$, where $\mathcal{Q}$ is a conic and L is a tangent line to $\mathcal{Q}$.

   PROOF. Projecting from a point P on L one realizes that there are two special fibers: L and L' both tangent lines to $\mathcal{Q}$ through P. Consider γ a meridian around the projection of L. Note that the monodromy induced by γ is the only necessary to obtain the required presentation. By Proposition 2.3 one obtains the following $\pi_1(\mathbb{P}^2 \setminus (\mathcal{Q} \cup L)) = \langle g_1, g_2, \gamma : g_2 g_1 = 1, g_1^\gamma = \gamma^{-1} g_1 \gamma, g_2^\gamma = \gamma^{-1} g_2 \gamma \rangle$. Since $g_1^\gamma = g_2$, $g_2 = g_2 g_1 g_2^{-1}$ by Example 6.6, one obtains the required result. □

   (e) $\pi_1(\mathbb{P}^2 \setminus \mathcal{C}_3) = \mathbb{Z}_3$, where $\mathcal{C}_3$ is a smooth, nodal, or cuspidal cubic.

   PROOF. Since $\mathcal{C}_3$ has an inflection point, one simply applies Theorem 3.2. □

Probably the easiest example of non-abelian fundamental group of an irreducible quartic (i.e. a curve of degree four) is the three-cuspidal quartic. Zariski [**78**] showed this in a more general setting using a brilliant argument. Let us sketch the proof.

THEOREM 3.8. *Let $\mathcal{C}$ be a (rational) curve of degree $2d$, with $2(d-1)(d-2)$ nodes and $3(d-1)$ cusps. Then $\pi_1(\mathbb{P}^2 \setminus \mathcal{C}) = \mathbb{B}_{d+1}(\mathbb{P}^1)$ (see Example 1.5).*

PROOF. Such curves are generic plane sections of the space $\bar{\Delta}_{d+1}$ of homogeneous polynomials of degree $d+1$ in two variables with multiple roots, described in Example 1.5. The reason is the following: a plane in the space $\bar{Y}_{d+1}$ of homogeneous polynomials of degree $d+1$ in two variables is nothing but a family of polynomials $E := \{\lambda_0 f_0 + \lambda_1 f_1 + \lambda_2 f_2 \mid [\lambda_0 : \lambda_1 : \lambda_2] \in \mathbb{P}^2\}$. Note that a polynomial $\lambda_0 f_0 + \lambda_1 f_1 + \lambda_2 f_2 \in E$ has multiple roots if the line $\lambda_0 X + \lambda_1 Y + \lambda_2 Z = 0$ intersects the parametrized curve $F := [f_0(s,t) : f_1(s,t) : f_2(s,t)] \subset \mathbb{P}^2$ at a tangent. Note that $\mathcal{F}$ has degree $d+1$. In fact it is a (rational) curve with $\frac{d(d-1)}{2}$ nodes.

Therefore $E \cap \bar{\Delta}_{d+1}$ is exactly the dual of F, say $\check{F}$, which has to have degree $2d$, $2(d-1)(d-2)$ nodes and $3(d-1)$ cusps.

Hence, using Zariski Theorem of Lefschetz Type 7.1, $\pi_1(E \setminus \check{F}) = \pi_1(\bar{Y}_n) = \mathbb{B}_{d+1}(\mathbb{P}^1)$. □

COROLLARY 3.9 (Zariski [**78**]). *Let $\mathcal{C}$ be an irreducible tricuspidal quartic. Then*

$$\pi_1(\mathbb{P}^2 \setminus \mathcal{C}) = \langle g_1, g_2 : g_1 g_2 g_1 = g_2 g_1 g_2, g_2 g_1^2 g_1 = 1 \rangle.$$

This type of result has been generalized to the study of complements of discriminant varieties by Dolgachev-Libgober in [**26**].

4. Braid monodromy of curves: local versus global

When computing the monodromy action of the locally trivial fibration $\pi : X \to \mathbb{P}^1 \setminus \Delta$ constructed in the Zariski-Van Kampen method one needs a collection of meridians around the points in Δ. We recall that a meridian γ around $z \in \Delta$ can be decomposed as $\gamma = \omega \cdot \gamma_z \cdot \omega^{-1}$, where ω is a path joining the base point z_0 and a point z' near z, and γ_z is the boundary of a disk centered at z (see Definition 4.13).

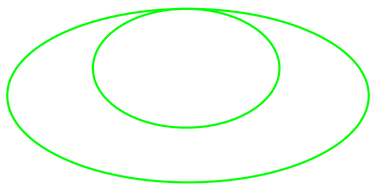
The action of γ on $\pi_1(F, s_*(x_0))$ will also be decomposed as the action of γ_z on $\pi_1(F, s_*(z'))$ and the action of ω on $\pi_1(F, s_*(x_0), s_*(z'))$. The first one will be called the *local monodromy at z* and the second one will be called the *global monodromy at z*.

The local monodromy is completely determined by the local topological type of the curve on the points on the fiber (see Examples 6.5, 6.7, and 6.6). For instance, the Puiseux expansion at each singular point on the fiber determines the local monodromy.

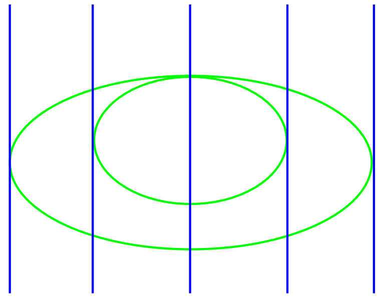
However, the global monodromy depends on the position of singularities and, in general, it depends on the global geometry of the curve. Whether or not there is a finite set of global data on the curve that determines the global monodromy is still unknown.

In the previous sections only examples were presented where the local monodromy information was enough to give the monodromy action, but this is far from being the case in general. The following example will hopefully depict the general situation.

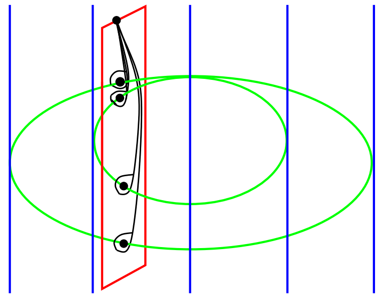
EXAMPLE 4.1. Consider the following quartic, which is a union of two smooth conics intersecting at one point:



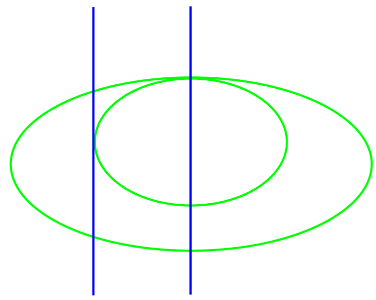
When projecting from $[0 : 1 : 0]$ there are five special fibers



After choosing a base point we can start computing the braid monodromy as follows:



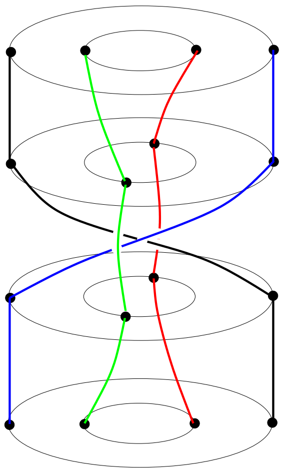
The tangency and the high order tacnode can basically be obtained directly from the local monodromy, since the global monodromy is trivial (the base point is *close enough* to both special fibers)



The tangency can be computed directly from Example 6.6 as σ_2 and the tacnode, whose local equation is $y^2 = x^8$, that is two smooth branches with multiplicity of intersection 4, can be obtained from Example 6.5 as σ_1^8.

However, the remaining braids depend on global monodromy for two different reasons:

- the left-most tangency depends on global monodromy basically due to the fact that the branches from the *small* conic become complex conjugated and intertwine with the branches of the *big* conic as one approaches the tangency obtaining the following braid.



that is, $\sigma_3^{-1}\sigma_1^{-1}\sigma_2\sigma_1\sigma_3$.

- the tangent immediately to the right of the tacnode also depends on global monodromy, even though all the branches remain real. The reason in this case is that the approaching path ω consists of *half a turn* around the tacnode. The braid becomes $\sigma_1^4 \cdot \sigma_2 \cdot \sigma_1^{-4}$.
- the right-most tangent also depends on global monodromy for both reasons, one has to avoid the branching values by performing half turns and also the real branches become complex conjugated at some point. However, since one only needs to compute all the monodromy actions but one, this one can be disregarded.

Finally, straightforward computations give the following:

(10)

$$\begin{array}{lll}
(r_1) & g_1 = g_1^{\sigma_1^8} = (g_2g_1)^4 g_1 (g_2g_1)^{-4} & \Rightarrow [(g_2g_1)^4, g_1] = 1 \\
 & g_2 = g_2^{\sigma_1^8} = (g_2g_1)^4 g_2 (g_2g_1)^{-4} & \Rightarrow [(g_2g_1)^4, g_2] = 1 \\
 & g_3 = g_3^{\sigma_1^8} = g_3 & \\
 & g_4 = g_4^{\sigma_1^8} = g_4 &
\end{array}$$

$$\begin{array}{lll}
 & g_1 = g_1^{\sigma_2} = g_1 & \\
(r_2) & g_2 = g_2^{\sigma_2} = g_3 & \Rightarrow g_2 = g_3 \\
 & g_3 = g_3^{\sigma_2} = g_3 g_2 g_3^{-1} & \Rightarrow g_2 = g_3 \\
 & g_4 = g_4^{\sigma_2} = g_4 &
\end{array}$$

$$\begin{array}{lll}
(r_3) & g_1 = g_1^{(\sigma_3^{-1}\sigma_1^{-1}\sigma_2\sigma_1\sigma_3)} = g_2^{-1} g_4 g_2 & \Rightarrow g_4 = g_2 g_1 g_2^{-1} \\
 & g_2 = g_2^{(\sigma_3^{-1}\sigma_1^{-1}\sigma_2\sigma_1\sigma_3)} = g_2 & \\
 & g_3 = g_3^{(\sigma_3^{-1}\sigma_1^{-1}\sigma_2\sigma_1\sigma_3)} = g_3 & \\
 & g_4 = g_4^{(\sigma_3^{-1}\sigma_1^{-1}\sigma_2\sigma_1\sigma_3)} = g_4 g_2 g_1 g_2^{-1} g_4^{-1} & \Rightarrow g_4 = g_2 g_1 g_2^{-1}
\end{array}$$

$$\begin{array}{ll}
(r_5) & g_1 = g_1^{(\sigma_1^4\sigma_2\sigma_1^{-4})} = (g_3(g_2g_1)^{-2}(g_1g_2)^2 g_1 g_3 (g_2g_1)^{-2}) * g_1 \\
(r_6) & g_2 = g_2^{(\sigma_1^4\sigma_2\sigma_1^{-4})} = (g_3(g_2g_1)^{-2}(g_1g_2)^2 g_1 g_3 (g_2g_1)^{-2}(g_1g_2)^2 g_1) * g_3 \\
(r_7) & g_3 = g_3^{(\sigma_1^4\sigma_2\sigma_1^{-4})} = g_1^{-1} g_2^{-1} g_1^{-1} g_2 g_1 g_2 g_1 \\
 & g_4 = g_4^{\sigma_1^4\sigma_2\sigma_1^{-4}} = g_4,
\end{array}$$

where $w * g_i = w g_i w^{-1}$. Also one needs to add the relation $g_4g_3g_2g_1 = 1$, which, after using (r_2) and (r_3) becomes $(r_8) \equiv (g_2g_1g_2^{-1})g_2g_2g_1 = (g_2g_1)^2 = 1$.

Finally, using (r_2), (r_3), and (r_8) one can easily check that (r_1), (r_5), (r_6), and (r_7) become trivial. Therefore, according to the Zariski-Van Kampen Theorem 2.4

$$\pi_1(\mathbb{P}^2 \setminus \mathcal{C}) = \langle g_1, g_2 : (g_2g_1)^2 = 1 \rangle = \mathbb{Z} * \mathbb{Z}_2,$$

which is the *biggest* group whose abelianized is $\mathbb{Z} \oplus \mathbb{Z}_2$. This result can also be obtained from the fact that both conics generate a very special pencil with a reduced member, the tangent line to both conics at the tacnode, but that would be another story and it is left to the interested reader.

CHAPTER 3

Braid Monodromy Tools

1. Definitions and First Properties

Let $\bar{\mathcal{C}} = \mathcal{C}_0 \cup \mathcal{C}_1 \cup ... \cup \mathcal{C}_r$, be the decomposition in irreducible components of a projective plane curve $\bar{\mathcal{C}}$. Let us denote by d_i the degree of $\mathcal{C}_i$ and assume $\mathcal{C}_0$ is a transversal line. An alternative construction similar to the Zariski-Van Kampen method occurs when studying $\mathbb{C}^2 := \mathbb{P}^2 \setminus \mathcal{C}_0$, $\mathcal{C} := \bar{\mathcal{C}} \cap \mathbb{C}^2$. The space $\mathbb{C}^2 \setminus \mathcal{C}$ retracts into a compact polydisk minus $\mathcal{C}$ as in the Figure 1.

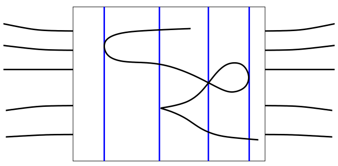
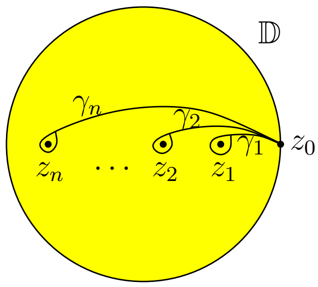


FIGURE 1. Affine projection

The projection onto the first coordinate outside the special fibers (see notation from §2.2) $\pi : \mathbb{D}_x \times \mathbb{D}_y \setminus (\mathcal{C} \cup \mathcal{L}) \to \mathbb{D}_x \setminus Z_n$ is a locally trivial fibration outside the fibers $\mathcal{L}$ whose intersection with $\mathcal{C}$ has less than d points.

DEFINITION 1.1. A set of meridians $\gamma_1, \gamma_2, \ldots, \gamma_n$ of a finite set on a disk $\mathbb{D}$ is called a *geometric basis* if $\gamma_n \gamma_{n-1} \cdots \gamma_1 = \partial\mathbb{D}$ with the positive (counterclockwise) orientation.



REMARK 1.2. A classical result by Artin **[7]** states that the set of geometric bases is in bijection with $\text{Diff}^+(\mathbb{D} \setminus Z_n, \partial\mathbb{D}) \cong \mathbb{B}_n$.

DEFINITION 1.3. Consider μ the braid monodromy action of the fundamental group of the base of π relative to the section $s_*(x) := (x, q_0)$ where $q_0 \in \partial\mathbb{D}_y$:

$$\mu : \pi_1(\mathbb{D}_x \setminus Z_n, z_0) \longrightarrow \text{Diff}^+(F_{z_0}) \cong \mathbb{B}_d$$

and fix $\Gamma := (\gamma_1, ..., \gamma_n)$ a geometric basis of $\pi_1(\mathbb{D}_x \backslash Z_n, z_0)$. The list of braids $(\mu(\gamma_1), ..., \mu(\gamma_n)) \in \mathbb{B}_d^n$ is called the *Braid Monodromy Representation* of $\mathcal{C}$ relative to (π, Γ, z_0, s_*).

REMARK 1.4. Due to the fact that any projective plane curve (outside its singular points) is an oriented Riemann surface, the braids obtained in any braid monodromy representation are *quasi-positive*, that is, they are conjugate of positive braids (braids that can be written as products of positive powers of the standard generators σ_i).

Moreover, the braids that appear in any braid monodromy representation are called *algebraic braids* because they can be realized as local monodromy of an algebraic function.

Our purpose is to construct an invariant of the projection π. In order to do so, we need to understand the different braid monodromy representations of $\mathcal{C}$ relative to (π, Γ, z_0, s_*) for the different choices of Γ, z_0, and s_*.

(1) Choice of geometric basis of $\mathbb{D}_x \setminus Z_n$. Let $\Gamma = (\gamma_1, \ldots, \gamma_n)$ and $\Gamma' = (\gamma'_1, \ldots, \gamma'_n)$ two geometric bases. By Remark 1.2, there exists a braid $\beta \in \mathbb{B}_n$ such that $\Gamma' = \Gamma^\beta$. This action is given as shown in (5), that is,

$$(\gamma_1, \ldots, \gamma_{i-1}, \gamma_i, \gamma_{i+1}, \ldots, \gamma_n)^{\sigma_i} = (\gamma_1, \ldots, \gamma_{i-1}, \gamma_i^{-1}\gamma_{i+1}\gamma_i, \gamma_i, \ldots, \gamma_n).$$

Therefore, the action of $\mathbb{B}_n$ on Γ naturally turns into an action on the monodromy representations associated with Γ.

(2) Choice of section, or analogously, choice of base point $q_0 \in \{z_0\} \times \mathbb{D}_y = F_{z_0}$. This produces, as mentioned in Remark 1.2, an inner automorphism, that is, a conjugation by a braid $\beta \in \mathbb{B}_d$. Hence, there is another action:

$$(\mu\gamma_1, \ldots, \mu\gamma_n)^\beta = (\beta^{-1}\mu\gamma_1\beta, \ldots, \beta^{-1}\mu\gamma_n\beta).$$

It is a mere exercise to check that the action of $\mathbb{B}_n$ and $\mathbb{B}_d$ on the set of geometric bases commute. This means that there is an right action of $\mathbb{B}_n \times \mathbb{B}_d$ on the set of monodromy representations, which takes care of all the possible choices of Γ, base points, and sections. Such an action is called the *Hurwitz moves* of a monodromy representation. Summarizing

THEOREM 1.5. *Given a monodromy representation μ of $\mathcal{C}$ with respect to (π, Γ, z_0). There is a one-to-one map between*

$$\{\textit{Monodromy representations of } \mathcal{C} \textit{ with respect to } \pi\} \leftrightarrow \{\textit{ Hurwitz class of } \mu\}$$

DEFINITION 1.6. Two monodromy representations of $\mathcal{C}$ are called *(Hurwitz) equivalent* if they belong to the same orbit by the Hurwitz moves described above. That is, if there exists $(\sigma, \beta) \in \mathbb{B}_n \times \mathbb{B}_d$ such that $\mu\Gamma' = \mu\Gamma^{(\sigma,\beta)}$.

The orbit of a braid monodromy representation by the action of Hurwitz moves will be called the *braid monodromy class* of a curve.

REMARK 1.7. Note that $\mu(\gamma_n)\mu(\gamma_{n-1})\cdots\mu(\gamma_2)\mu(\gamma_1) = \mu(\partial\mathbb{D}_x)$, Since $\partial\mathbb{D}$ can be seen as a meridian of the *point at infinity* of $\mathbb{C}$, that is, the projection of the line $\mathcal{C}_0$. The condition that $\mathcal{C}_0$ is transversal to $\mathcal{C}$ implies that $\mu\partial\mathbb{D} = \Delta_d^2 = (\sigma_1 \cdots \sigma_{d-1})^d$, the *Garside element* of $\mathbb{B}_d$, generator of its center. Thus, $\mu(\gamma_n)\mu(\gamma_{n-1})\cdots\mu(\gamma_2)\mu(\gamma_1) = \Delta_d^2 = (\sigma_1 \cdots \sigma_{d-1})^d$. This is another way to present a monodromy representation. This is usually known as a *Braid Monodromy Factorization*.

Many questions are still open regarding braid monodromy factorizations of algebraic curves. We mention just a few:

QUESTION 1.8. *Which (algebraic) factorizations are realizable in the algebraic category? This problem was solved in a bigger category called* Hurwitz category *by B. Moishezon* [**54**].

The braid monodromy factorization of a smooth curve is a product of conjugates of the standard generators σ_i. Any two such products (realizable or not) are Hurwitz equivalent [**10**], therefore, by Theorem 1.5 the realization problem is solved for the smooth case.

One can also find interesting local versions of the realization problem [**60**].

So far we have proved (Zariski-Van Kampen Theorem 2.4) that the braid monodromy representation of a curve determines the fundamental group of its complement. In fact it is much stronger than that, as shows the following result, which has been proved by Kulikov-Teicher in [**46**] for cuspidal curves and by Carmona [**14**] in full generality.

THEOREM 1.9. *The braid monodromy class of $\mathcal{C}$ fully determines the topology of the pair $(\mathbb{P}^2, \mathcal{C})$. In other words, if two curves $\mathcal{C}_1$ and $\mathcal{C}_2$ have the same braid monodromy class, then there is a homeomorphism $\varphi : \mathbb{P}^2 \to \mathbb{P}^2$ such that $\varphi(\mathcal{C}_1) = \mathcal{C}_2$.*

The converse is not known in general, basically because the homeomorphism $\varphi : \mathbb{P}^2 \to \mathbb{P}^2$ may not send lines to lines. Therefore, the pencil of lines through P, which determines the braid monodromy of $\mathcal{C}_1$ is not preserved by φ. There are some partial positive converses:

THEOREM 1.10 (Carmona [**14**]). *The pair $(\mathbb{P}^2, \mathcal{C})$ fully determines the braid monodromy class of $\mathcal{C}$ with respect to a projection.*

Another partial result in this direction is the following. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two curves and L_1, L_2 be lines such that the affine curves $\mathcal{C}_i \cup L_1 \subset \mathbb{C}^2 := \mathbb{P}^2 \setminus L_i$ have no vertical asymptotes. Consider $\mathcal{L}_1$ and $\mathcal{L}_2$ the union of vertical lines as described in §22 and $\varphi : (\mathbb{P}^2, \mathcal{C}_1, \mathcal{L}_1, L_1) \to (\mathbb{P}^2, \mathcal{C}_2, \mathcal{L}_2, L_2)$ a homeomorphism, then one has the following:

THEOREM 1.11 (Artal, Carmona,- [**2**]). *The braid monodromy factorization of $\mathcal{C}_1$ from a point $P \in L_1$ is Hurwitz equivalent to the braid monodromy representation of $\mathcal{C}_2$ from $\varphi(P) \in L_2$.*

In a different direction, there is also a negative converse to Theorem 1.9.

THEOREM 1.12 (Kharlamov-Kulikov [**41**]). *There are two sequences of plane irreducible cuspidal curves, $\mathcal{C}_{m,1}$ and $\mathcal{C}_{m,2}$, $m \geq 5$, such that the pairs $(\mathbb{C}^2, \mathcal{C}_{m,1})$ and $(\mathbb{C}^2, \mathcal{C}_{m,2})$ are diffeomorphic, but $\mathcal{C}_{m,1}$ and $\mathcal{C}_{m,2}$ are not isotopic and have different braid monodromy classes.*

Obviously, the diffeomorphisms cannot be extended to $\mathbb{P}^2$, otherwise the hypothesis in Theorem 1.11 would hold and the braid monodromy representations would be equivalent.

2. The Homotopy Type of $(\mathbb{C}^2, \mathcal{C})$

Let us consider the affine curve scenario as described at the beginning of §3.1, that is, $\bar{\mathcal{C}} = \mathcal{C}_0 \cup \mathcal{C}_1 \cup \ldots \cup \mathcal{C}_r$, where $\mathcal{C}_0$ is a transversal line, $d_i = \deg \mathcal{C}_i$, and $d := \deg \mathcal{C} - 1$. $\mathbb{C}^2 := \mathbb{P}^2 \setminus \mathcal{C}_0$, $\mathcal{C} := \bar{\mathcal{C}} \cap \mathbb{C}^2$. Consider $\pi : \mathbb{D}_x \times \mathbb{D}_y \setminus \mathcal{C} \to \mathbb{D}_x \setminus Z_n$ *generic*, where $\mathbb{D}_x$ is big enough to contain all the critical values Z_n of the projection from $\mathcal{C}$.

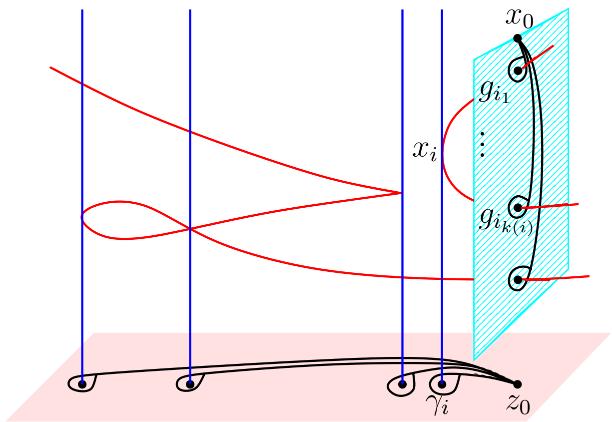


FIGURE 2.

Let us take a closer look at the relations (10) in Example 4.1. Note that, the relations derived from the tacnode σ_1^8 (involving branches 1 and 2), are trivial for the generators g_3 and g_4. Analogously, the braid σ_2 coming from the first tangency (involving branches 2 and 3) preserves generators g_1 and g_4. This is a general result. To be more precise, let $\gamma_i = \omega \cdot \gamma_i' \cdot \omega^{-1}$ be a meridian around $z_i \in Z_n$ and let $g_{i_1}, \ldots, g_{i_k}$ denote the meridians that approach the singular point x_i over z_i (see Figure 2) when running along ω. The following result is well known, see [**48, 55**].

PROPOSITION 2.1. *Under the above hypothesis $g_j^{\gamma_i} = g_j$ for any $j \notin \{i_1, \ldots, i_k\}$. Moreover, the relations $g_j^{\gamma_i} = g_j$, $j = i_1, \ldots, i_k - 1$ imply $g_{i_k}^{\gamma_i} = g_{i_k}$.*

Therefore, one has the following.

THEOREM 2.2 (Zariski-Van Kampen Theorem revisited).

$$\langle g_1, ..., g_d : g_i^{\gamma_j} = \gamma_j^{-1} g_i \gamma_j, i = 1, ..., d, j = 1, ..., i_{k(i)} - 1\rangle$$

is a presentation of $\pi_1(\mathbb{C}^2 \setminus \mathcal{C})$.

This presentation will be called *Zariski presentation*.

Associated with a (finite) group presentation

$$G = \langle g_1, \ldots, g_d : r_1(\bar{g}) = r_2(\bar{g}) = \cdots = r_n(\bar{g}) = \rangle,$$

one can construct a (finite) connected 2-dimensional CW-complex K as follows.

(1) the 0-dimensional skeleton of the complex will be given by only one 0-cell, $K_0 = \{e^0\}$,
(2) the 1-dimensional skeleton of the complex will be in bijection with the set of generators, say $K_1 = \{e_1^1, \ldots, e_d^1\}$, whose boundary will be glued to e^0, and
(3) the 1-dimensional skeleton of the complex will be in bijection with the set of relators, say $K_2 = \{e_1^2, \ldots, e_n^2\}$. The identification morphism is so that ∂e_i^2 is glued to the 1-cell $r_i(\bar{e}^1)$.

REMARK 2.3. Note that the 2-dimensional CW-complex is associated with a presentation, not with the group. However, certain transformations in the presentation are allowed keeping the homotopy type. Such transformations are called *Tietze transformations* of type (I) and (II) (cf. [**27**]):

(I) Adding (deleting) a generator g_i and a relation of the type $g_i = w(g_1, \ldots, g_{i-1}, g_{i+1}, \ldots, g_d)$.
(II) Replacing a relation $r = 1$ by a relation $r = wsw^{-1}$, where w is any word and $s = 1$ is another relation.
(III) Adding (deleting) the relation $1 = 1$.

Tietze transformations of type (III) change the homotopy type of the complex since it means attaching (resp. detaching) a 2-dimensional sphere and this increases (resp. decreases) the Euler characteristic of the complex by one.

THEOREM 2.4 (Libgober [**48**]). *The 2-dimensional complex associated with the Zariski presentation has the homotopy type of* $\mathbb{C}^2 \setminus \mathcal{C}$.

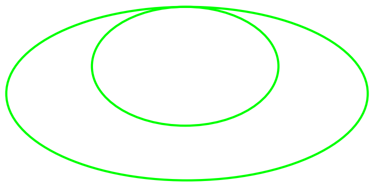
PROOF. The proof of this result is based on two local results.

LEMMA 2.5. *The 2-dimensional complex associated with the Wirtinger presentation of a link* $K \subset \mathbb{S}^3$ *has the homotopy type of* $\mathbb{S}^3 \setminus K$.

LEMMA 2.6. *The 2-dimensional complex associated with the Artin presentation of a link* $K \subset \mathbb{S}^3$ *has the homotopy type of* $\mathbb{S}^3 \setminus K$.

□

EXAMPLE 2.7. Let us consider the affine version of Example 4.1.



The following is a Zariski presentation (see (10)):

$$\pi_1(\mathbb{C}^2 \setminus \mathcal{C}) = \left\langle g_1, g_2, g_3, g_4 : \ [(g_2 g_1)^4, g_1] = 1, \ \begin{array}{l} g_2 = g_3, \quad g_4 = g_2 g_1 g_2^{-1}, \\ g_2 = g_3, \quad g_4 = g_2 g_1 g_2^{-1} \end{array} \right\rangle \equiv$$

$$\equiv \langle g_1, g_2 : [(g_2 g_1)^4, g_1] = 1, 1 = 1, 1 = 1 \rangle$$

Hence, according to Theorem 2.4, $\mathbb{C}^2 \setminus \mathcal{C}$ has the same homotopy type than $(\mathbb{S}^3 \setminus K_{2,8}) \vee \mathbb{S}^2 \vee \mathbb{S}^2$.

Note that both spaces obviously have the same fundamental group and the same Euler characteristic. It is easy to check that $\chi(\mathcal{C}_0 \cup \mathcal{C}) = 1$, and hence $\chi(\mathbb{C}^2 \setminus (\mathcal{C}_0 \cup \mathcal{C})) = 2$. Note that $\chi(\mathbb{S}^3 \setminus K_{2,8}) = 0$.

The following is an open problem.

QUESTION 2.8. *Does the fundamental group and the Euler characteristic determine the homotopy structure of complements to affine curves? That is, given two affine curves $\mathcal{C}_1$ and $\mathcal{C}_2$ with isomorphic fundamental groups and same Euler characteristic, are necessarily their homotopy types the same?*

Note that Question 2.8 has a negative answer in the general case of 2-dimensional complexes [**27**].

QUESTION 2.9. *Does Theorem 2.4 also hold for projective curves for some preferred presentation?*

3. Line Arrangements

Very interesting examples of curves are line arrangements. For line arrangements there are algorithms to construct complexes that share the same homotopy type as the line complement, without resorting to the Zariski-Van Kampen method. Basically the idea is that singularities of line arrangements are all of type $y^k = x^k$ and can be all found as solutions of linear equations.

Very extensive literature has been written on this topic (see [**62, 63, 8, 9, 67, 66, 31, 22, 21, 40, 30, 20**] among others).

Note that, even though some of the topology of complements to line arrangements depends on the combinatorial information of the arrangement (that is, the way lines intersect each other). Combinatorics are not enough to determine fundamental groups as stated by Rybnikov [**65**] (see also [**4**]).

Our focus of attention in this survey will be to describe a simple method to obtain a braid monodromy representation of complexified real arrangements, which will be later extended to other real curves and general line arrangements.

3.1. Wiring Diagrams.

DEFINITION 3.2. A line arrangement $\mathcal{L} = \mathcal{L}_1 \cup \cdots \cup \mathcal{L}_d$ is called a *real line arrangement* if there is a projective system of coordinates such that $\mathcal{L}$ admits a real equation. If in addition, $\mathcal{L}_1, \ldots, \mathcal{L}_d$ admit real equations, then $\mathcal{L}$ is called a *strongly real line arrangement*.

REMARK 3.3. Definitions of *real* and *strongly real* are not consistent throughout the literature, so doublecheck the definitions before reading a result on *real* arrangements.

Note that both concepts are not equivalent, since MacLane arrangement (see [**51**]) is real, but not strongly real.

The following result is immediate, but it is worth mentioning for clarity.

LEMMA 3.4. *Consider the affine situation of a strongly real line arrangement and the vertical projection (onto the first coordinate) as in §31. The following properties hold:*

(1) *The set of singularities of the arrangement have real coordinates,*
(2) *the plane $\mathbb{R}^2 \subset \mathbb{C}^2$ is such that $\mathbb{R}^2 \cap \mathcal{L}$ is isomorphic to a graph Γ with the following structure (see Figure 3).*

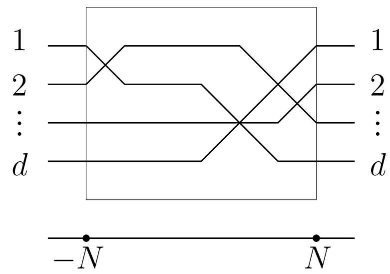


FIGURE 3. Wiring Diagram

(a) *There exists a certain* $N > 0$ *such that* $\Gamma \cap (-\infty, -N] \times \mathbb{R}$ *and* $\Gamma \cap [N, \infty) \times \mathbb{R}$ *are given by* d *parallel rays and* $\Gamma \cap [-N, N]$ *is a union of segments and stars given as a union of segments (the intersection point is called a* singularity *of* Γ*),*

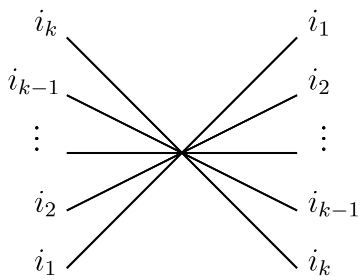


FIGURE 4.

(b) *there are* d *broken lines of segments (as many as affine lines), and*
(c) *each pair of broken lines intersect exactly once.*

Associated with any wiring diagram Γ one can construct a finite list of braids in $\mathbb{B}_d$ as follows.

Let $(x_1, y_1), \ldots, (x_n, y_n)$ denote the singular points of Γ ordered such that $x_1 > \cdots > x_n$. Denote by $\bar{\delta}^i := \delta_1^i, \ldots, \delta_{k_i}^i$ the segments intersecting at (x_i, y_i). One defines $\beta_i \in \mathbb{B}_d$ as

$$\beta_i := \left(\prod_{j=1}^{i-1} \Delta_{\bar{\delta}^j} \right) * \Delta^2_{\bar{\delta}^i}$$

where locally $\Delta_{(1,\ldots,k)} = (\sigma_1 \ldots \sigma_{k-1})(\sigma_1 \cdots \sigma_{k-2})(\sigma_1\sigma_2)\sigma_1$ is a halftwist of the strings $(1, \ldots, k)$. Globally, one needs to keep track of the position of the segments $\delta_1, \ldots, \delta_k$.

EXAMPLE 3.5. The braid monodromy of the wiring diagram of Figure 3 is given as follows:

$$\begin{array}{c} \beta_1 = \sigma_2^2 \\ \beta_2 = (\sigma_2) * \sigma_1^2 \\ \beta_3 = ((\sigma_2)(\sigma_1)) * (\sigma_2\sigma_3)^3 \\ \beta_4 = ((\sigma_2)(\sigma_1)(\sigma_3\sigma_2\sigma_3)) * \sigma_1^2 \end{array} \tag{11}$$

One can prove the following result.

THEOREM 3.6. *The braid monodromy representation associated to the wiring diagram of a line arrangement coincides with the braid monodromy representation of the line arrangement.*

This gives one a simple algorithmic method to compute braid monodromy representations of complexified strongly real line arrangements.

Finally, note that, despite the simplicity of the method, braid monodromy representations of strongly real arrangements are not determined by the combinatorics. In [**3**] the authors present two strongly real arrangements of lines with the same combinatorics but whose braid monodromies are not Hurwitz equivalent (see § 3.4).

This diagram has two possible generalizations: braided wiring diagrams for complex line arrangements and decorated wiring diagrams for complexified strongly real curves, which will be treated separately.

3.7. Braided Wiring Diagrams. Consider a line arrangement $\mathcal{L}$ in the affine situation as in §31. We recall that the projection $\pi|_{\mathcal{L}}$ has a finite set of critical values denoted by $Z_n \subset \mathbb{D}_x$.

Choose a piecewise linear path starting at a base point on $\partial\mathbb{D}_y$ with no self-intersections and joining all the points in Z_n such that the segment is not broken at the points in Z_n. For instance, one can follow the lexicographic order in the complex numbers as in Figure 5:

$$z = x + y\sqrt{-1} > z' = x' + y'\sqrt{-1} \Leftrightarrow \begin{cases} x > x', & \text{or} \\ x = x', y > y'. \end{cases}$$

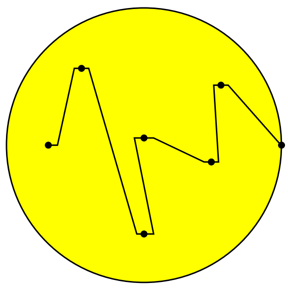


FIGURE 5.

The preimage of each segment will be an *open braid* and not a planar graph as for wiring diagrams. The rest is analogous to the wiring case: when crossing a point in Z_n, where the lines $\bar{\delta} = (\delta_1, \ldots, \delta_k)$ converge, a braid of local type $\Delta_{\bar{\delta}}$ will be generated and when ending at a point in Z_n a braid of local type $\Delta^2_{\bar{\delta}}$ will appear.

EXAMPLE 3.8. Consider the Hesse arrangement:

$$\mathcal{H} := \{\ell_1 \ell_2 \ell_3 \cdots \ell_{12} = 0\},$$

where

$$\begin{array}{ccc} \ell_1 = \{y = 0\}, & \ell_2 = \{(x + \omega^2 y + \omega^2 z) = 0\}, & \ell_3 = \{(x + \omega y + \omega z) = 0\}, \\ \ell_4 = \{(x + \omega^2 y + \omega z) = 0\}, & \ell_5 = \{(x + \omega y + \omega^2 z) = 0\}, & \ell_6 = \{(x + \omega^2 y + z) = 0\}, \\ \ell_7 = \{(x + \omega y + z) = 0\}, & \ell_8 = \{x = 0\}, & \ell_9 = \{(x + y + z) = 0\}, \\ \ell_{10} = \{z = 0\}, & \ell_{11} = \{(x + y + \omega z) = 0\}, & \ell_{12} = \{(x + y + \omega^2 z) = 0\}, \end{array}$$

where ω is a root of $z^2 + z + 1 = 0$. Assume ℓ_{10} is the line at infinity and project from the quadruple point $P := [1 : -1 : 0]$. The lines ℓ_9, ℓ_{11}, and ℓ_{12} become vertical. Figure 6 represents the non-generic braided wiring diagram for the Hesse arrangement and this particular projection:

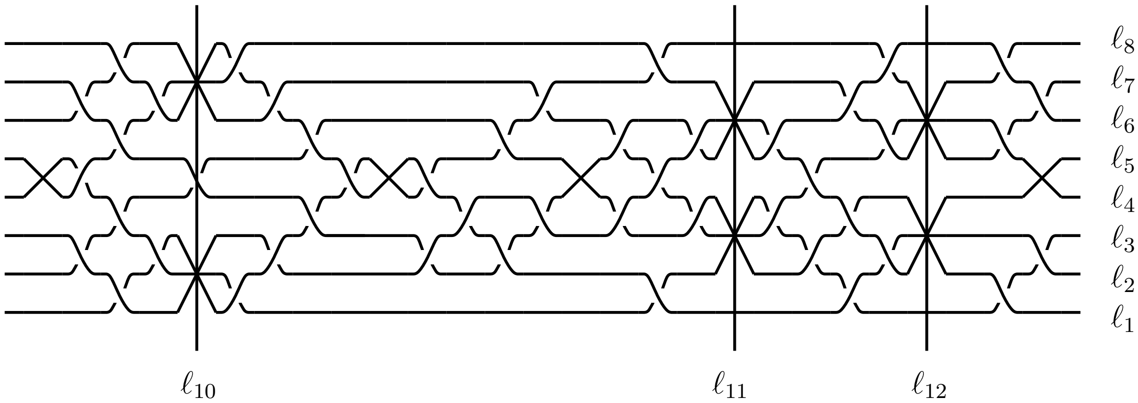


FIGURE 6. Hesse Wiring Diagram

Recall that $f_1 := \ell_4\ell_5\ell_9$ and $f_2 := \ell_1\ell_8\ell_{10}$ generate a pencil of cubics containing $f_3 := \ell_2\ell_7\ell_{11}$ and $f_4 := \ell_3\ell_6\ell_{12}$ as members of the pencil. Note that each reducible cubic $f_1, \ldots, f_4$ consists of three lines in general position (three lines joining all the inflexion points of a smooth cubic, each one containing three of them). Using the results in [**45**] on braid factorizations, one can prove that the Hesse arrangement cannot degenerate onto a pseudoholomorphic Hesse arrangement, where the cubics f_i become three concurrent lines.

3.9. Decorated Wiring Diagrams.

DEFINITION 3.10. A plane curve $\mathcal{C} = \mathcal{C}_1 \cdots \mathcal{C}_r$ is called *real* if there is a projective system of coordinates such that $\mathcal{C}$ admits real equations.

If, in addition, $\mathcal{C}_1, \ldots, \mathcal{C}_r$ admit real equations, the singular points and vertical tangencies have real coordinates, and the tangent cone at each singularity is a strongly real line arrangement, we call $\mathcal{C}$ a *strongly real curve*.

Consider the affine situation of a strongly real curve of degree d and the vertical projection (onto the first coordinate) as in §31. One can write a diagram with solid lines and dashed lines, where:

(1) There exists a certain $N > 0$ such that $\Gamma \cap (-\infty, -N] \times \mathbb{R}$ and $\Gamma \cap [N, \infty) \times \mathbb{R}$ are given by $d - 2k$ parallel rays and $\Gamma \cap [-N, N]$ is a union of solid and dashed paths,
(2) the solid paths are isomorphic to $\mathcal{C} \cap \mathbb{R}^2$,
(3) the dashed paths represent the real parts of non-real (conjugated) branches,
(4) every time two dashed paths intersect, one of them overcrosses if their imaginary parts are larger (dashed paths always overcross solid paths).

EXAMPLE 3.11. The following is the wiring of the quartic from Example 4.1

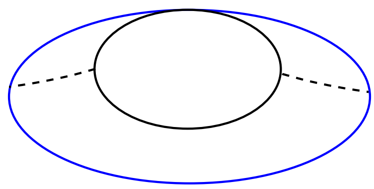


FIGURE 7.

Note, for example, notice the braids generated at the local picture:

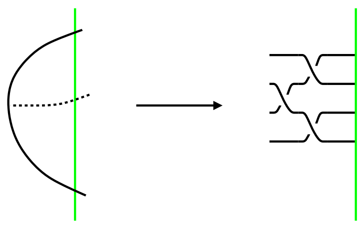


FIGURE 8.

and compare with the monodromy obtained in Example 4.1.

4. Conjugated Curves

One last application of braid monodromies is the study of the different Galois embeddings of a curve given by equations in a number field in the spirit of [**69, 1**]. More precisely, let $\mathcal{C}$ be a plane curve whose equation can be defined on the ring of polynomials with coefficients on a number field $K \supset \mathbb{Q}$. Any Galois transformation σ of the number field will produce another curve $\mathcal{C}^\sigma$, whose equation is again defined on $K[X, Y, Z]$, with the same number and degrees of irreducible components, same type of singularities,... that is same *combinatorial type*.

However, since σ cannot necessarily be extended to a homeomorphism of the total space (think of the automorphism $\sqrt{2} \mapsto -\sqrt{2}$) the question arises whether or not $(\mathbb{P}^2, \mathcal{C})$ and $(\mathbb{P}^2, \mathcal{C}^\sigma)$ are topologically equivalent.

Also note that any such example cannot be detected by means of *algebraic invariants* such as the *algebraic fundamental group* (that is, the profinite completion of the fundamental group), Alexander polynomials, or any kind of invariant related with finite coverings.

Consider the arrangements $\mathscr{C}^+$ and $\mathscr{C}^-$ given by the following equations (see Figure 9).

$$M_1^\pm : z = 0, \qquad M_2^\pm : x = 0, \qquad M_3^\pm : x = z,$$

$$M_4^\pm : x = -(\gamma + 1)z, \qquad M_5^\pm : x = (\gamma + 2)z,$$

$$L_1^\pm : y = x, \qquad L_2^\pm : y = \gamma(x - z), \qquad L_3^\pm : y = \gamma x + z,$$

$$L_4^\pm : y = z, \qquad L_5^\pm : y = 0, \qquad N^\pm : \gamma^\pm x + (\gamma^\pm + 1)y + z = 0,$$

where $\gamma^\pm$ are the roots of $X^2 + X - 1 = 0$.

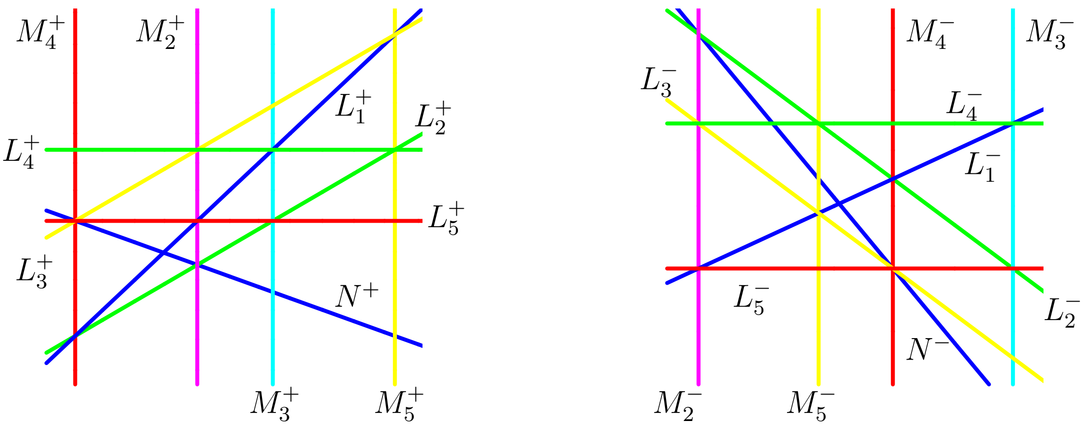


FIGURE 9.

First, one can compute the braid monodromy of the *horizontal* (= non-vertical) lines $\bigcup L_i^\pm$. In order to show that they are not Hurwitz equivalent, representations of the braid group onto finite groups can be used. Once such a representation is fixed, the Hurwitz action only produces a finite number of elements, and hence the problem becomes effectively solvable.

In our particular example, one can use the Burau representation of $\mathbb{B}_5$ into $\mathrm{GL}(5; \mathbb{Z}[t^{\pm 1}])$. Replacing t by $2 \mod 5$ one obtains a representation $\beta : \mathbb{B}_5 \to \mathrm{GL}(5; \mathbb{Z}/5\mathbb{Z})$ such that the braid monodromy representations produce different orbits after the Hurwitz action.

Therefore, using Theorem 1.11 one obtains the following.

THEOREM 4.1 ([**3**]). *There is no homeomorphism between* $(\mathbb{P}^2, \mathscr{C}^+)$ *and* $(\mathbb{P}^2, \mathscr{C}^-)$.

REMARK 4.2. Still, the question whether or not the fundamental groups $G^+ := \pi_1(\mathbb{P}^2 \setminus \mathscr{C}^+)$ and $G^- := \pi_1(\mathbb{P}^2 \setminus \mathscr{C}^-)$ are isomorphic remains open. As mentioned above, the reader should notice that

$$\pi_1^{\mathrm{alg}}(\mathbb{P}^2 \setminus \mathscr{C}^+) \cong \pi_1^{\mathrm{alg}}(\mathbb{P}^2 \setminus \mathscr{C}^-).$$

In other words, G^+ and G^- have the same profinite completion (that is, the same *structure* of finite index subgroups).

This is a paradigmatic example in the sense that it shows the power of the braid monodromy representation of a curve in and of itself and not as a mere instrument to obtain fundamental groups.

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