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A survey on Zariski pairs

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Dedicated to Professor Kenji Ueno on his sixtieth birthday.

§ Introduction

In 1929, O. Zariski published a paper entitled *"On the Problem of Existence of Algebraic Functions of Two Variables Possessing a Given Branch Curve"* [130] where the following question was considered:

Does an algebraic function z of x and y exist, possessing a preassigned curve f as branch curve?

As Zariski pointed out in the *Introduction* of [130], this question was first considered by Enriques and the problem is reduced to finding the fundamental group of the complement of the given curve (the word *complement* is understood and often omitted for short). Zariski considered some explicit cases and *proved* important results. Here we detail some of the most relevant:

(Z1) If two curves lie in a connected family of equisingular curves, then they have isomorphic fundamental groups.

(Z2) If a continuous family $\{\mathcal{C}_t\}_{t\in[0,1]}$ is equisingular for $t \in (0,1]$ and $\mathcal{C}_0$ is reduced, then there is a natural epimorphism $\pi_1(\mathbb{P}^2 \setminus \mathcal{C}_0, p_0) \twoheadrightarrow \pi_1(\mathbb{P}^2 \setminus \mathcal{C}_t, p_t)$, where the base point p_t ($t \in [0,1]$) depends on t continuously.

(Z3) The fundamental group of an irreducible curve of order n, possessing ordinary double points only, is cyclic of order n ([130, Theorem 7]), see Remark 1.

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(Z4) Consider the projection from the general cubic surface in $\mathbb{P}^3$ onto $\mathbb{P}^2$, centered at a general point outside the surface. Its branch locus is a sextic $\mathcal{C}_6$ with six cusps whose fundamental group is isomorphic to $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$.

(Z5) He noted that the six cusps of any sextic described in (Z4) satisfy the extra condition of lying on a conic – without decreasing the dimension of their family. Moreover, if $\mathcal{C}_6$ is a sextic with six cusps and its fundamental group has a representation onto the symmetric group of three letters, then $\mathcal{C}_6$ is the branch curve of a cubic surface and its six cusps lie on a conic. In particular if a sextic $\mathcal{C}'_6$ with six cusps not on a conic exists, then $\pi_1(\mathbb{P}^2 \setminus \mathcal{C}_6, p_o) \not\cong \pi_1(\mathbb{P}^2 \setminus \mathcal{C}'_6, p'_o)$.

(Z6) If Q is a three-cuspidal quartic, then $\pi_1(\mathbb{P}^2 \setminus Q, p_o)$ is isomorphic to the binary 3-dihedral group, which is a non-Abelian finite group of order 12 presented by

$$\langle a, b \mid aba = bab, \quad a^2b^2 = 1\rangle. \tag{1}$$

Remark 1. Zariski's proof of (Z3) depended on the following claim of Severi: *The family of irreducible curves of degree n possessing a given number of ordinary double points is irreducible* [104]. Severi's proof was not correct, and the first rigorous proof of his claim was given in 1986 by J. Harris [54]. In the meantime (Z3) was known as the *Zariski conjecture* until the 70's, when it was proved by Deligne and Fulton in [40] and [48].

Zariski proved the commutativity of the fundamental group for certain smooth curves and then he used (Z1) for general smooth curves. He was also able to prove the commutativity for nodal arrangements of lines. He found degenerations of nodal curves into nodal arrangements of lines, thus proving the commutativity of the fundamental group for certain nodal curves. A combination of (Z1) with Severi's claim allowed him to complete a proof of (Z3).

In [131], Zariski proved another result regarding (Z5): in modern language, the Alexander polynomial of $\mathcal{C}_6$ equals $t^2 - t + 1$ and the Alexander polynomial of $\mathcal{C}'_6$ is 1 (provided $\mathcal{C}'_6$ exists); the key point for both claims is the position of the cusps. The story (almost) ends in [132] where Zariski shows the existence of curves $\mathcal{C}'_6$ using deformation arguments that allow him to prove that their fundamental group is Abelian; explicit examples were found much later [2, 90]. He also claims that there are only two irreducible families of sextics with six cusps. It is not hard to prove that the family of sextics with six cusps on a conic is irreducible, and an analogue for the other family is announced by Degtyarëv in [39, Theorem 5.3.2].

Another important result of [130] is (Z6); for a long time the three-cuspidal quartic (the only quartic with a non-Abelian fundamental group) was the only example of a curve whose complement has a non-Abelian finite fundamental group. In the early nineties, several such examples have been found by A. Degtyarëv [38], M. Oka [91] and I. Shimada [106].

Fundamental groups of curves up to degree five are well known (see [36, 38]), but for now little is known about their structure in the general case. In this sense, questions like the one raised by Zariski ([134, Chapter VIII, §1]) on the residual finiteness of such groups are still open.

These results and open questions motivated many mathematicians to study the topology of the complements of plane curves. One of the most surprising phenomena in this field was the one found by Zariski and stated in (Z5), where two irreducible curves with the same singularities have non-isomorphic fundamental groups. This leads us to the definition of *Zariski pairs* which are, roughly speaking, two curves that have the same local topology but do not have the same embedded topology. Let us give a more precise definition of a Zariski pair.

Definition 2 ([2]). A pair $(\mathcal{C}_1, \mathcal{C}_2)$ of reduced plane curves in $\mathbb{P}^2$ is called a *Zariski pair* if it satisfies the following conditions:

(1) There exist tubular neighborhoods $T(\mathcal{C}_i)$ $(i = 1, 2)$ and a homeomorphism $h : T(\mathcal{C}_1) \to T(\mathcal{C}_2)$ such that $h(\mathcal{C}_1) = \mathcal{C}_2$.

(2) There exists no homeomorphism $f : \mathbb{P}^2 \to \mathbb{P}^2$ with $f(\mathcal{C}_1) = \mathcal{C}_2$.

Analogously $(\mathcal{C}_1, \ldots, \mathcal{C}_k)$ is a *Zariski k-plet* if $(\mathcal{C}_i, \mathcal{C}_j)$ is a Zariski pair for any $i \neq j$.

Remark 3. The first condition in Definition 2 is replaced by the one about the combinatorial data on $\mathcal{C}_i$ $(i = 1, 2)$. More precisely, the *combinatorial type* of a curve $\mathcal{C}$ is given by a 7-tuple

$$(\mathrm{Irr}(\mathcal{C}), \deg, \mathrm{Sing}(\mathcal{C}), \Sigma_{\mathrm{top}}(\mathcal{C}), \sigma_{\mathrm{top}}, \{\mathcal{C}(P)\}_{P \in \mathrm{Sing}(\mathcal{C})}, \{\beta_P\}_{P \in \mathrm{Sing}(\mathcal{C})}),$$

where:

- $\mathrm{Irr}(\mathcal{C})$ is the set of irreducible components of $\mathcal{C}$ and $\deg : \mathrm{Irr}(\mathcal{C}) \to \mathbb{Z}$ assigns to each irreducible component its degree.
- $\mathrm{Sing}(\mathcal{C})$ is the set of singular points of $\mathcal{C}$, $\Sigma_{\mathrm{top}}(\mathcal{C})$ is the set of topological types of $\mathrm{Sing}(\mathcal{C})$, and $\sigma_{\mathrm{top}} : \mathrm{Sing}(\mathcal{C}) \to \Sigma_{\mathrm{top}}(\mathcal{C})$ assigns to each singular point its topological type.
- $\mathcal{C}(P)$ is the set of local branches of $\mathcal{C}$ at $P \in \mathrm{Sing}(\mathcal{C})$, (a local branch can be seen as an arrow in the dual graph of the minimal resolution of $\mathcal{C}$ at P, see [42, Chapter II.8] for details) and $\beta_P : \mathcal{C}(P) \to \mathrm{Irr}(\mathcal{C})$ assigns to each local branch the global irreducible component containing it.

We say that two curves $\mathcal{C}_1$ and $\mathcal{C}_2$ have the *same combinatorial data* (or simply the *same combinatorics*) if their combinatorial data are equivalent, that is, if $\Sigma_{\text{top}}(\mathcal{C}_1) = \Sigma_{\text{top}}(\mathcal{C}_2)$, and there exist bijections $\varphi_{\text{Sing}} : \text{Sing}(\mathcal{C}_1) \to \text{Sing}(\mathcal{C}_2)$, $\varphi_P : \mathcal{C}_1(P) \to \mathcal{C}_2(\varphi_{\text{Sing}}(P))$ (restriction of a bijection of dual graphs) for each $P \in \text{Sing}(\mathcal{C}_1)$, and $\varphi_{\text{Irr}} : \text{Irr}(\mathcal{C}_1) \to \text{Irr}(\mathcal{C}_2)$ such that $\deg_2 \circ \varphi_{\text{Irr}} = \deg_1$, $\sigma_{\text{top}_2} \circ \varphi_{\text{Sing}} = \sigma_{\text{top}_1}$, and $\beta_{2,\varphi_{\text{Sing}}(P)} \circ \varphi_P = \varphi_{\text{Irr}} \circ \beta_{1,P}$.

In the irreducible case, two curves have the same local topology if they have the same degree and the same topological types for local singularities. On the other extreme, for line arrangements, combinatorial type is just the set of incidence relations.

The fact that two curves have the same combinatorial data if and only if they satisfy Definition 2(1) is a consequence of Waldhausen graph manifold theory [125, 126]. The dual graph of the minimal resolution of the singularities of $\mathcal{C}$ is a plumbing graph. Waldhausen theory was developed in terms of plumbing graphs by Neumann [89]. His main result states that minimal normalized graphs are determined by the manifold. It is not hard to see that the topological type of $(\mathbb{P}^2, \mathcal{C})$ determines the combinatorial data. Since the graph coming from the minimal resolution may be not minimal, it is possible to find curves $\mathcal{C}_1, \mathcal{C}_2$ whose complements $\mathbb{P}^2 \setminus \mathcal{C}_i$, $i = 1, 2$ are homeomorphic but such that they do not have the same combinatorics using, for example, Cremona transformations. Jiang and Yau [60] proved that the homeomorphism type of the complement of a line arrangement determines its combinatorics. The connection between homeomorphism type of complements to curves and combinatorics was studied by Di Pasquale in [100], but not much is known about it.

Also, curves with the same combinatorics form a quasi-projective variety in a certain projective space $\mathbb{P}_d$ of dimension $\frac{d(d+3)}{2}$, where d is the total degree of the curves. We will refer to such a variety as the *combinatorial stratum of curves*. A *rigid isotopy* between curves is a smooth path in a combinatorial stratum.

A connected family of equisingular curves is contained in the connected component of the combinatorial stratum of curves determined by any curve of that family. Therefore, the topology of the pair $(\mathbb{P}^2, \mathcal{C})$ is an invariant of such a component. In particular $\pi_1(\mathbb{P}^2 \setminus \mathcal{C}, p_o)$ is also an invariant and hence (Z5) provides the first example of a Zariski pair. In the early nineties, some new examples were found ([2, 37]). Since then, the variety of such examples has been very broad and subtle.

In what follows we will give an insight on the different nature of each of these phenomena, the techniques used, and some open questions on the general study. The study of Zariski pairs may consist of two parts:

(I) To locate curves $\mathcal{C}_1$, $\mathcal{C}_2$ in different irreducible components of a combinatorial stratum, that is, two non rigidly-isotopic curves.

(II) To find an effective invariant $\mathcal{P}$ of the topology of the embedding, so that, if $\mathcal{P}(\mathcal{C}_1) \neq \mathcal{P}(\mathcal{C}_2)$ then $(\mathbb{P}^2, \mathcal{C}_1) \not\approx (\mathbb{P}^2, \mathcal{C}_2)$, that is, there is no homeomorphism $f : \mathbb{P}^2 \to \mathbb{P}^2$ with $f(\mathcal{C}_1) = \mathcal{C}_2$.

As for (I), different strategies have been used, for instance:

(I1) Certain geometrical properties such as position of singularities (e.g. sextics whose six cusps belong to a conic), or in more generality the existence of non-zero global sections for an ideal sheaf $\mathcal{I}$ on $\mathbb{P}^2$ twisted by a certain $\mathcal{O}(d)$. The expected dimension of such vector space of global sections is 0 (like the six cusps on a sextic, which in principle are not supposed to belong to a conic). This method is often used in combination with birational transformations in order to lower the degrees of the curves. In that case, the geometrical properties have to be *translated* into new properties on the transformation ([12, 91]). Also geometrical properties of flex points play an important role in finding irreducible components of the combinatorial strata ([95, 96]).

(I2) Arithmetical properties of the components. Such is the typical case with line arrangements, when built up by pasting together *smaller* arrangements whose combinatorial strata are disconnected ([102, 11, 10]). Also, in the special case of sextics, many of these arithmetical properties come from the existence of a double covering ramified along the curve. Classification of $K3$-surfaces with a given Picard group and some computer work finish the task in the case of simple singularities ([127]). Birational transformations of the covering and appropriate blowdowns can be used to produce equations ([6]).

(I3) Direct computation of strata. This can be used either in the negative sense (proving the irreducibility of a stratum) which tells us where there are no Zariski pairs ([61, 109]), or actually finding equations of the strata. This method often requires big Milnor numbers, so that the dimension of the strata is small and the problem becomes computationally feasible ([9, 98]).

As for (II), several methods have been developed:

(II1) Zariski-van Kampen Theorem. This is the classical method to find the fundamental group of the complement to a given curve from its braid monodromy ([130, 59, 28]). This technique will be treated in detail in §1. Though a very rich invariant, the fundamental group of a curve contains the topological information of the embedding in an intricate manner. The undecidability of the isomorphism problem in group theory justifies the need to construct new invariants that are effectively computable, simple to compare, and fine enough to keep the essential data.

(II2) Alexander invariants such as the Alexander module, Alexander polynomials, and characteristic varieties. This type of invariant shows the connection between algebraic curves and knot theory, since many of these invariants have been adopted from knot theory. Conversely, much of the original interest and techniques of knot theory had the study of algebraic curves at their root. This technique will be developed in §2.

(II3) Braid monodromy equivalence, also referred to as braid monodromy factorization. It is a much stronger invariant of the topology of the embedding than the monodromy group. Braid monodromy factorization was only recently proved to be an invariant of the (not-necessarily-rigid) isotopy class ([70] for cuspidal curves and [25] for any plane curve). This technique has been proved to be specially useful to study conjugated curves [9, 10], branch curves of surfaces of general type and the Chisini conjecture ([85, 70, 69]), as well as symplectic isotopies of curves and the realization problem ([99, 62, 64]). Definition and more details will be given in §1.

(II4) Branched Galois coverings. Based on geometric versions of the inverse Galois problem for certain elementary non-Abelian groups. This technique will be treated in detail in §3.

(II5) Nikulin theory of integral lattices. Recently developed by Degtyarëv [39] and Shimada [107, 108]. Let $\mathcal{C}$ be a sextic curve with at worst simple singularities (see [20] for simple singularities). Let X be the double cover of $\mathbb{P}^2$ ramified along $\mathcal{C}$, and $\bar{X}$ its minimal resolution. Degtyarëv obtains a quadruple $Q := (L, h, \sigma, \omega)$ where $L = H_2(\bar{X})$, which is isomorphic to the integral lattice of the singularities of $\mathcal{C}$, $h \subset L$ is the pull-back of the hyperplane section class $[\mathbb{P}^1] \in H_2(\mathbb{P}^2)$, $\sigma \subset L$ is the set of classes of exceptional divisors appearing in the resolution $\bar{X} \to X$, and $\omega \subset L \otimes \mathbb{R}$ denotes the oriented 2-subspace spanned by the real and imaginary parts of the class of a holomorphic 2-form on $\bar{X}$. He proves that $Q_{\mathcal{C}_1} \cong Q_{\mathcal{C}_2}$ if and only if $\mathcal{C}_1$ and $\mathcal{C}_2$ are rigidly isotopy and the pairs $(\mathbb{P}^2, \mathcal{C}_1)$, $(\mathbb{P}^2, \mathcal{C}_2)$ are regularly diffeomorphic, that is, there is a diffeomorphism between $(\mathbb{P}^2, \mathcal{C}_1)$ and $(\mathbb{P}^2, \mathcal{C}_2)$ that can be extended to a homeomorphism between the $K3$-surfaces $\bar{X}_1$ and $\bar{X}_2$. Shimada proves that the isomorphism class of L is an invariant of the Γ-equivalence of the pair $(\mathbb{P}^2, \mathcal{C})$. This implies that L is an invariant of the homeomorphism class of $(\mathbb{P}^2, \mathcal{C})$ (for a discussion on Γ-equivalence see below).

In light of the previous list of strategies, one can also describe more precisely different examples of Zariski pairs according to which invariants are equal and which are different for each curve.

- **Alexander polynomial**. It is associated with a group and a homomorphism onto $\mathbb{Z}$ and with cyclic coverings ramified along each component with the same ramification index. A Zariski pair that can be distinguished using this invariant is a *classical Zariski pair*, otherwise it will be called an *Alexander-equivalent Zariski pair*. Examples of classical Zariski pairs are abundant in the literature ([131, 37, 2, 95] among many others). Alexander-equivalent Zariski pairs can be found in ([5, 92, 93]).
- **Characteristic varieties and Oka polynomials**. The first ones (introduced by A. Libgober in [75] for curves) are associated with a group and its abelianization morphism whereas the second ones (introduced by Oka in [94]) are associated with a group and any homomorphism onto $\mathbb{Z}$. They are both associated with Alexander modules and Abelian coverings ramified (or not) along each component with any ramification index and basically provide the same information. The existence of certain irreducible components of characteristic varieties can be described in terms of algebraic conditions of the singular locus. Analogously we have *Libgober-Oka-equivalent Zariski pairs*.
- **Non-Abelian coverings and twisted Alexander polynomials**. The first one is given by the existence or not of certain non-Abelian coverings ramified along components. Algebraic conditions can be given for the existence of such coverings. Twisted Alexander polynomials are associated with a group and a representation. Zariski pairs whose algebraic fundamental groups are isomorphic are called *algebraically-equivalent Zariski pairs*. The main source of examples of algebraically-equivalent Zariski pairs is found among conjugated curves ([9, 10, 107, 108]) in a number field; we will call them *arithmetic Zariski pairs*. There are still open questions whether or not some pairs of conjugate, non rigidly-isotopic curves are Zariski pairs ([44]). Also, a famous example of a Zariski pair of line arrangements was produced by G. Rybnikov [102]. A proof was published in [11] using arguments of homological rigidity

(see Definition 2.30), and a final argument of Alexander modules. Such Zariski pair is distinguished by the fundamental group of the complement, and it is an Alexander-equivalent, but it is not known whether it is a Libgober-Oka-equivalent or algebraically-equivalent Zariski pair.

- **Fundamental group of the complement** $\pi_1(\mathbb{P}^2 \setminus \mathcal{C}, p_o)$. A presentation of it can be obtained via the Zariski-Van Kampen Theorem from the action of generic (and sometimes even non-generic) braid monodromy groups of the curve $\mathcal{C}$. Sometimes groups can be compared directly ([45, 92, 93]), but oftentimes this is too hard of a task. Zariski pairs whose fundamental groups are isomorphic will be called *π_1-equivalent Zariski pairs*. Note that fundamental groups of algebraic-equivalent Zariski pairs have the same profinite completion. Finitely presented groups of infinite order with the same profinite completion are hard to distinguish. In fact, it is not known whether or not any of the algebraic-equivalent Zariski pairs is also a π_1-equivalent Zariski pair. Examples of π_1-equivalent Zariski pairs can be obtained from the list of arithmetic Zariski pairs given by Shimada [108]. For example, sextics with singularities $\mathbb{A}_{18} + \mathbb{A}_1$, $\mathbb{A}_{16} + \mathbb{A}_3$, and $\mathbb{A}_{16} + \mathbb{A}_2 + \mathbb{A}_1$ have Abelian fundamental groups isomorphic to $\mathbb{Z}/6\mathbb{Z}$ (see [6, Remark 5.9]).
- **The homotopy type of the complement** $\mathbb{P}^2 \setminus \mathcal{C}$. It can be described as the homotopy type of the CW-complex associated with a presentation of $\pi_1(\mathbb{P}^2 \setminus \mathcal{C}, p_o)$ obtained from a very particular braid monodromy of $\mathcal{C}$ that will be referred to as the *Puiseux-braid monodromy of* $\mathcal{C}$ (see [73]). Analogously we have *homotopy type-equivalent Zariski pairs*. Known examples of *homotopy type-equivalent Zariski pairs* are related to Cremona transformations and conjugated curves. In fact, it would be very interesting to see whether or not any of the non-regular-diffeomorphic examples mentioned above are homotopy type-equivalent.
- **The topology of the complement** $\mathbb{P}^2 \setminus \mathcal{C}$. Any example of this sort will be called *complement-equivalent Zariski pairs*. The main information lost between the embedding and the complement is the *peripheral information*, that is the information on the location of meridians of the irreducible components. A complement-equivalent Zariski pair can be obtained from the problem proposed by Eyral-Oka in [44]. Also, in Example 1.39 we show in detail a complement-equivalent Zariski pair.

- **Γ-equivalence**. This equivalence relation, introduced by Shimada [107], has to do with the *peripheral information* mentioned above. Two curves $\mathcal{C}_1$ and $\mathcal{C}_2$ with the same combinatorics and homeomorphic complements are called Γ-equivalent, if the homeomorphism induces an isomorphism of fundamental groups preserving meridians. Examples of this sort will be called Γ-*equivalent Zariski pairs*.
- **The topology of** $(\mathbb{P}^2, \mathcal{C})$. Since it is determined by any generic braid monodromy factorization of $\mathcal{C}$ (as mentioned above), the ultimate tool to check for a Zariski pair is its generic braid monodromy. These techniques have been used in ([9, 10]).

The main purpose of this article is to review these different methods and to explain how they are used in the study of Zariski pairs.

Contents

§1. Fundamental group and braid monodromy

As indicated in the Introduction, the main goal of Zariski's foundational paper [130] is to study the fundamental group of the complement of a projective plane curve. A method for its computation is outlined in [130]. In [59], E.R. van Kampen gave a more rigorous presentation of this method which is now known as the *Zariski-van Kampen method*. Roughly speaking, Zariski showed that such a group is generated by meridians of a generic line and then described some relations by *moving* this generic line around non-generic lines in a pencil; van Kampen stated and proved his well-known theorem on fundamental groups (now known as the Seifert-van Kampen Theorem) and used it to prove that Zariski's relations define a system of relations for the fundamental group. D. Chéniot gave a modern approach to this method in [28]. In [29], O. Chisini realized that this method contains a finer invariant of the curve if one interprets the *motions* of the generic line in terms of a representation of a free group in a braid group. Much later, B. Moishezon [84] called this invariant *braid monodromy* and used it to study projective complex surfaces via ramification curves of projections.

1.1. Preliminaries

Before explaining the Zariski-van Kampen method and the braid monodromy invariant, let us introduce some settings and notations.

Let G be a group. Let $a, b \in G$. For simplicity, we introduce the following notations:

$$\begin{aligned} [a,b] &= a^{-1}b^{-1}ab \\ a^b &= b^{-1}ab \\ b * a &= bab^{-1}. \end{aligned}$$

We denote the free group with n generators $x_1, \ldots, x_n$ by $\mathbb{F}_n$ and the braid group on n strings by $\mathbb{B}_n$ given by the following presentation:

$$\mathbb{B}_n := \left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \underset{1<i+1<j<n}{[\sigma_i, \sigma_j] = 1}, \; \underset{1 \leq i \leq n-2}{\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}} \right\rangle \tag{2}$$

$\mathbb{B}_n$ naturally acts on $\mathbb{F}_n$ on the right as follows:

$$\Phi(x_i, \sigma_j) = x_i^{\sigma_j} := \begin{cases} x_{i+1} & \text{if } i = j \\ x_i * x_{i-1} & \text{if } i = j+1 \\ x_i & \text{if } i \neq j+1. \end{cases} \tag{3}$$

for $i = 1, \ldots, n-1$, and $j = 1, \ldots, n$.

We call the above right action of $\mathbb{B}_n$ on $\mathbb{F}_n$ the *Hurwitz action.* In particular, the actions of the generators σ_i $(i = 1, \ldots, n-1)$ on $x_1, \ldots, x_n$ are called *Hurwitz moves.* Clearly Φ induces an antihomomorphism $\mathbb{B}_n \to \operatorname{Aut}(\mathbb{F}_n)$, $\sigma \in \mathbb{B}_n \mapsto \Phi(\bullet, \sigma)$. This homomorphism is injective and its image, identified with $\mathbb{B}_n$, is characterized by the following result:

Proposition 1.1 ([22, Theorem 1.9.])**.** *Let* $\tau \in \operatorname{Aut}(\mathbb{F}_n)$. *Then* $\tau \in \mathbb{B}_n$ *if and only if*

$$\tau(x_n \cdot \ldots \cdot x_1) = x_n \cdot \ldots \cdot x_1$$

and there exists a permutation σ *of* n*-letters such that*

$$\tau(x_i) = y_i x_{\sigma(i)} y_i^{-1}, \quad y_i \in \mathbb{F}_n, \quad i = 1, \ldots, n.$$

We will now present a geometric interpretation of $\mathbb{B}_n$ and its action $\mathbb{F}_n$. For details in the following, we refer to [7] and [22].

Let us fix a subset, $\mathbf{Y} = \{t_1, \ldots, t_n\}$, of $\mathbb{C}$ consisting of n distinct elements. Let $\Delta \subset \mathbb{C}$ be a sufficiently big closed disk, *i.e.*, $\{z \in \mathbb{C} \mid |z| \leq R\}$ $R \gg 0$, such that $\mathbf{Y}$ is contained in its interior. Choose a point $\star$ on $\partial\Delta = \{z \in \Delta \mid |z| = R\}$.

Definition 1.2. We define some special elements of the fundamental group $\pi_1(\mathbb{C} \setminus \mathbf{Y}, \star)$, called *meridians.* Meridians are obtained as follows:

- Take a small disk S centered at $t \in \mathbf{Y}$ containing no other elements of $\mathbf{Y}$ and choose a point $\hat{\star} \in \partial S$.
- Consider a path α in $\mathbb{C} \setminus \mathbf{Y}$ joining $\hat{\star}$ and $\star$, and denote by $\eta_{\hat{\star},S}$ the closed path based at $\hat{\star}$ that runs counterclockwise along ∂S.
- The homotopy class of the loop $\alpha^{-1} \cdot \eta_{\hat{\star},S} \cdot \alpha$ is called a meridian of t in $\pi_1(\mathbb{C} \setminus \mathbf{Y}, \star)$. If the base point is understood, then we will simply speak of a meridian of t in $\mathbb{C} \setminus \mathbf{Y}$.
- It is easily checked that the set of meridians of $t \in \mathbf{Y}$ coincides with a conjugacy class in $\pi_1(\mathbb{C} \setminus \mathbf{Y}, \star)$ completely determined by t.
- It is also well known that suitable collections of n meridians in $\mathbb{C} \setminus \mathbf{Y}$ (one for each point of $\mathbf{Y}$) define bases of $\pi_1(\mathbb{C} \setminus \mathbf{Y}, \star)$.
- This construction of meridians also applies to the fundamental group of the complement of a divisor in a surface, see Figure 1.

Definition 1.3. Let Δ and $\star$ be as above. A *geometric basis* of $\pi_1(\mathbb{C} \setminus \mathbf{Y}, \star)$ is an ordered basis $(\gamma_1, \ldots, \gamma_n)$ of $\pi_1(\mathbb{C} \setminus \mathbf{Y}, \star)$ consisting of meridians such that $\gamma_n \cdot \ldots \cdot \gamma_1$ is homotopic to the loop $\gamma_\star$, the closed path based at $\hat{\star}$ that runs counterclockwise along $\partial\Delta$.

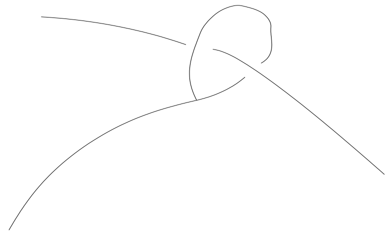

Fig. 1. A meridian

Definition 1.4. A *pseudo-geometric basis* of $\pi_1(\mathbb{C}\setminus\mathbf{Y},\star)$ is an ordered basis $(\gamma_1,\dots,\gamma_n)$ of meridians such that $\gamma_n\cdot\ldots\cdot\gamma_1$ is homotopic to the inverse of a meridian of $\{\infty\}$ in $\mathbb{C}\setminus\Delta$. The product $\gamma_n\cdot\ldots\cdot\gamma_1$ is called the *pseudo-Coxeter* element of the basis.

Note that a geometric basis $\Gamma:=\{\gamma_1,\dots,\gamma_n\}$ is a free basis of $\pi_1(\mathbb{C}\setminus\mathbf{Y},\star)\cong\mathbb{F}_n$. Given any element $\sigma\in\mathbb{B}_n$, the Hurwitz action of σ on Γ produces another geometric basis. By Proposition 1.1, one has the following:

Proposition 1.5 (Artin)**.** *The Hurwitz action of the group $\mathbb{B}_n$ on the set of all geometric bases of $\pi_1(\mathbb{C}\setminus\mathbf{Y},\star)$ is free and transitive.*

Definition 1.6. A $\mathbf{Y}$*-special homeomorphism* is an orientation-preserving homeomorphism $f:\mathbb{C}\to\mathbb{C}$ such that

(i) $\mathbf{Y}$ is fixed as a set, not necessarily pointwise, and
(ii) f is the identity on $\overline{\mathbb{C}\setminus\Delta}$.

A $\mathbf{Y}$*-special isotopy* is an isotopy $H:\mathbb{C}\times[0,1]\to\mathbb{C}$ such that $H(\bullet,t)$ is a $\mathbf{Y}$-special homeomorphism for all $t\in[0,1]$. We denote the set of classes of $\mathbf{Y}$-special homeomorphisms up to $\mathbf{Y}$-special isotopy by $\mathbf{B}_{\mathbf{Y}}$.

Let f_1 and f_2 be $\mathbf{Y}$-special homeomorphisms. We denote their classes in $\mathbf{B}_{\mathbf{Y}}$ by $[f_i]$ $(i=1,2)$, respectively. The product $[f_1][f_2]:=[f_1\circ f_2]$ makes $\mathbf{B}_{\mathbf{Y}}$ a group which acts on $\pi_1(\mathbb{C}\setminus\mathbf{Y},\star)$ on the left.

Let $M_n:=\mathbb{C}^n\setminus\Sigma=$ the set of ordered n-distinct points. The symmetric group of n letters acts on M_n freely via permutation of coordinates. Let $S_n(M)$ be the quotient space of M_n with respect to this action. One can regard $S_n(M)$ as the set of unordered n-distinct points; and $\mathbf{Y}\in S_n(M)$. Consider $B_{\mathbf{Y}}:=\pi_1(S_n(M),\mathbf{Y})$, and note that any $\gamma\in B_{\mathbf{Y}}$ is the homotopy class, relative to $\{0,1\}$, of a set of n-paths $\{\gamma_1(t),\dots,\gamma_n(t)\}$, $\gamma_i:[0,1]\to\mathbb{C}$ such that $\mathbf{Y}=\{\gamma_1(0),\dots,\gamma_n(0)\}=\{\gamma_1(1),\dots,\gamma_n(1)\}$, $\gamma_1(t),\dots,\gamma_n(t)$ are all distinct for $t\in[0,1]$. An element of $B_{\mathbf{Y}}$ is called a *braid based at* $\mathbf{Y}$. It is known that $B_{\mathbf{Y}}\cong\mathbb{B}_n$

(see [22] for details). For technical purposes, it is oftentimes convenient to consider *open braids starting at* $\mathbf{Y}_1$ *and ending at* $\mathbf{Y}_2$, where $\mathbf{Y}_1, \mathbf{Y}_2 \in S_n(M)$ as the homotopy class, relative to $\{0,1\}$, as before where $\mathbf{Y}_1 = \{\gamma_1(0), \ldots, \gamma_n(0)\}$, and $\mathbf{Y}_2 = \{\gamma_1(1), \ldots, \gamma_n(1)\}$. The obvious composition operations endoweds the class of open braids with a groupoid structure (see [7] for details).

Our purpose next is to describe natural identifications between $\mathbf{B_Y}$ and $B_{\mathbf{Y}}$.

Lemma 1.7. $\mathbf{B_Y}$ *and* $B_{\mathbf{Y}}$ *are naturally anti-isomorphic.*

Proof. Note that any orientation preserving homeomorphism of $\mathbb{C}$ is isotopic to the identity (for a proof, see [53, Lemma 5.6.]). Also, let $[h] \in \mathbf{B_Y}$ be an arbitrary element of $\mathbf{B_Y}$ and let $H : \mathbb{C} \times [0,1] \to \mathbb{C}$ be an isotopy such that $H(\bullet, 0) = h$ and $H(\bullet, 1) = id_{\mathbb{C}}$. Then, $H(t_1, t), \ldots, H(t_n, t)$, $t \in [0,1]$, are paths such that

(*i*) $\{H(t_1, 0), \ldots, H(t_n, 0)\} = \{H(t_1, 1), \ldots, H(t_n, 1)\} = \mathbf{Y}$, and

(*ii*) $\{H(t_1, t), \ldots, H(t_n, t)\}$ ($t \in [0,1]$) is a set of n distinct points for all $t \in [0,1]$, i.e., $\gamma_h := \{H(t_1, t), \ldots, H(t_n, t)\}$ gives an element of $B_{\mathbf{Y}}$.

This correspondence $[h] \mapsto \gamma_h$ gives the announced anti-isomorphism (for more details, see [53, Theorem 5.4.]). Q.E.D.

Lemma 1.7 assures the existence of a right action of $\mathbf{B_Y}$ on $\pi_1(\mathbb{C} \setminus \mathbf{Y}, \star)$, which coincides with the Hurwitz action for a proper choice of the base point and the geometric basis (see for instance [7, Example 1.8]). In particular, any homomorphism from a group G to $\mathbb{B}_n$ induces a right action of G on $\pi_1(\mathbb{C} \setminus \mathbf{Y}, \star)$.

1.2. Zariski-van Kampen method and braid monodromy

Let $\mathcal{C} \subset \mathbb{P}^2$ be a reduced projective plane curve of degree d. Choose a line $L \pitchfork \mathcal{C}$ and a point $P \in L \setminus \mathcal{C}$, and consider the projection centered at P. Let us take homogeneous coordinates $[X : Y : Z]$ in $\mathbb{P}^2$ such that $L : Z = 0$ and $P = [0 : 1 : 0]$. In what follows we will focus on $\mathbb{C}^2 := \mathbb{P}^2 \setminus L$ with affine coordinates $x := \frac{X}{Z}$, $y := \frac{Y}{Z}$, and whose projection may be written as $\Pi : \mathbb{C}^2 \to \mathbb{C}$, $\Pi(x, y) := x$. The affine curve $\mathcal{C}^{\text{aff}} := \mathcal{C} \cap \mathbb{C}^2 = \mathcal{C} \setminus L$ can be defined by a reduced monic polynomial $f(x, y)$ in the variable y such that $\deg_y f = \deg f = d$.

Let $\mathcal{D}_f := \{x \in \mathbb{C} \mid \operatorname{disc}_y f(x) = 0\}$ and let $\mathbf{L}$ be the union of the lines $L_t : x = t$, $t \in \mathcal{D}_f$. The main point of the Zariski-van Kampen method is the following:

Lemma 1.8. *The restriction* $\pi := \Pi_{|\mathbb{C}^2 \setminus (\mathcal{C}^{\text{aff}} \cup \mathbf{L})} : \mathbb{C}^2 \setminus (\mathcal{C}^{\text{aff}} \cup \mathbf{L}) \to \mathbb{C} \setminus \mathcal{D}_f$ *is a locally trivial fibration whose fiber is isomorphic to* $\mathbb{C}$ *with* d *punctures.*

This follows from Ehresmann's Fibration Theorem (see [41, p.15]), since the vertical lines having less than d distinct intersection points with $\mathcal{C}^{\mathrm{aff}}$ have been removed.

Since π is a locally trivial fibration whose fiber is diffeomorphic to $\mathbb{C}$ with d punctures, the polynomial f induces an algebraic mapping $\tilde{f}:\mathbb{C}\setminus\mathcal{D}_f\to S_n(M)$ given by $x_0\mapsto\{y\in\mathbb{C}\mid f(x_0,y)=0\}\in S_n(M)$. In order to define braid monodromy we need to consider $*\in\mathbb{C}\setminus\mathcal{D}_f$ a regular value on the boundary of a disk Δ_f containing $\mathcal{D}_f$ in its interior and denote by $\mathbf{Y}^*$ the set of roots of the polynomial $f(*,y)=0$.

Definition 1.9.

(i) The homomorphism $\nabla_*:\pi_1(\mathbb{C}\setminus\mathcal{D}_f,*)\to B_{\mathbf{Y}}:=\pi_1(S_n(M),\mathbf{Y}^*)$ induced by $\tilde{f}$ is called a *braid monodromy.*

(ii) Fix an isomorphism $\iota_{\mathbf{Y}^*}:B_{\mathbf{Y}^*}\cong\mathbb{B}_d$ (Note that the isomorphism depends on the choice of $\mathbf{Y}^*$, i.e. $*$) and define $\nabla_{*,\mathbf{Y}^*}:=\iota_{\mathbf{Y}^*}\circ\nabla_*$. Given any geometric basis $\gamma_1,\dots,\gamma_r$ of $\pi_1(\mathbb{C}\setminus\mathcal{D}_f,*)$, where $r:=\#(\mathcal{D}_f)$, the r-tuple $(\nabla_{*,\mathbf{Y}^*}(\gamma_1),\dots,\nabla_{*,\mathbf{Y}^*}(\gamma_r))\in(\mathbb{B}_d)^r$ is called a *factorization of the braid monodromy of* $(\mathcal{C},L,P)$ or simply a *braid monodromy factorization of* $(\mathcal{C},L,P)$.

(iii) The image by $\nabla_{*,\mathbf{Y}^*}$ of a pseudo-Coxeter element (Definition 1.4) will be referred to as a *pseudo-Coxeter braid.*

Note that a braid monodromy factorization depends on the choice of the geometric basis of $\pi_1(\mathbb{C}\setminus\mathcal{D}_f,\star)$ and $\iota_{\mathbf{Y}}$. By Proposition 1.5, any change of geometric basis is given by a Hurwitz move. We will expound upon this in section §1.3.

Now we are in the position to state the results obtained by Zariski-van Kampen in [130, 59] with the purpose of describing a presentation of the fundamental group of an affine curve. In order to do so consider Δ_f as above and Δ_y a closed disc such that $\mathcal{C}^{\mathrm{aff}}\cap\Pi^{-1}(\Delta_f)\subset\Delta_f\times\Delta_y$, and $\mathcal{C}^{\mathrm{aff}}\cap(\partial\Delta_f)\times(\partial\Delta_y)=\emptyset$. Let $\Gamma:=\{\gamma_1,\dots,\gamma_r\}$ be a geometric basis of $\pi_1(\mathbb{C}\setminus\mathcal{D}_f,*)$ and choose a base point $(*,\hat{*})$ ($\hat{*}\in\partial\Delta_y$). Since $\pi^{-1}(\Delta_f)\cap(\Delta_f\times\Delta_y)\to\Delta_f$ has a section, there exists a lifting α_t for each γ_t. Note that $\pi_1(\mathbb{C}\setminus\mathcal{D}_f,*)\cong\pi_1(\Delta_f\setminus\mathcal{D}_f,*)$ and α_t is a meridian of L_t. Under these conditions, and using the long exact sequence of homotopy, one has the following:

Proposition 1.10. *Let $(\rho_1,\dots,\rho_r)\in(\mathbb{B}_d)^r$ be a braid monodromy factorization of $(\mathcal{C},L,P)$ and let $\{g_1,\dots,g_d\}$ be a geometric basis of $\pi_1(\mathbb{C}\setminus\mathbf{Y}^*,\hat{*})$. Then $\pi_1(\mathbb{C}^2\setminus(\mathcal{C}^{\mathrm{aff}}\cup\mathbf{L}),(*,\hat{*}))$ has the following presentation:*

$$\left\langle g_1,\dots,g_d,\alpha_1,\dots,\alpha_r \;\middle|\; g_j^{\rho_i}=g_j^{\alpha_i},\quad i=1,\dots,r,\quad j=1,\dots,d\right\rangle. \tag{4}$$

As a corollary to Proposition 1.10, one obtains the celebrated Zariski-van Kampen Theorem:

Corollary 1.11 (**Zariski-van Kampen Theorem**)**.** *Under the previous hypotheses, $\pi_1(\mathbb{C}^2 \setminus \mathcal{C}^{\text{aff}})$ has the following presentation:*

$$\left\langle g_1, \dots, g_d \;\middle|\; g_j^{\rho_i} = g_j, \quad i = 1, \dots, r, \quad j = 1, \dots, d \right\rangle. \tag{5}$$

A presentation of $\pi_1(\mathbb{P}^2 \setminus \mathcal{C})$ is given by

$$\left\langle g_1, \dots, g_d \;\middle|\; g_d \cdot \ldots \cdot g_1 = 1,\ g_j^{\rho_i} = g_j,\ i = 1, \dots, r,\ j = 1, \dots, d \right\rangle. \tag{6}$$

The main tool of the proof of Corollary 1.11 is in [47, Lemma 4.18].

Lemma 1.12 ([47])**.** *Let $A \subset X$ be a divisor in a smooth quasi-projective variety X, and let $B \subset X$ be an irreducible divisor not contained in A. Then, the inclusion induces an epimorphism $\pi_1(X \setminus (A \cup B)) \twoheadrightarrow \pi_1(X \setminus A)$ whose kernel is generated by the meridians of B.*

Proof of Corollary 1.11. It is an iterated application of Lemma 1.12. For (5), we use that the loops $\alpha_1, \dots, \alpha_r$ in (4) are meridians of the lines L_t, $t \in \Delta$. For (6), note that $(g_d \cdot \ldots \cdot g_1)^{-1}$ is a meridian of the projection point P in $\overline{L}_\star = L_\star \cup \{P\} \subset \mathbb{P}^2$, i.e. a meridian of the line at infinity L in $\mathbb{P}^2$. Q.E.D.

Since we are mostly interested in the isomorphism class of the fundamental group, and since the spaces studied are connected, unless necessary for technical reasons, the base point of the fundamental group of a curve complement will be omitted.

Remark 1.13. We can decrease the number of relations in (4), (5) and (6).

(a) Since the Hurwitz action fixes the product $g_d \cdot \ldots \cdot g_1$, it is enough to consider $j = 1, \dots, d-1$.

(b) Since L is transversal to $\mathcal{C}$ it is well known that $\rho_r \cdot \ldots \cdot \rho_1 = \Delta_d^2$, that is the full twist, where $\Delta_d^2 = (\sigma_{d-1} \cdot \ldots \cdot \sigma_1)^d$, which is a generator of the center of $\mathbb{B}_d$. Thus its Hurwitz action coincides with the conjugation by $g_d \cdot \ldots \cdot g_1$. Therefore, in (5) it is enough to consider $i = 1, \dots, r-1$, $j = 1, \dots, d-1$, and add relations to make $g_d \cdot \ldots \cdot g_1$ into a central element. Analogously, in (6) it is enough to consider $i = 1, \dots, r-1$, $j = 1, \dots, d-1$, and $g_d \cdot \ldots \cdot g_1 = 1$.

(c) There is another possible reduction in the number of relations which was already indicated in [130]. Let us explain it in modern terms.

One can identify $\pi_1(S_d(M), \mathbf{d})$ (where $\mathbf{d} := \{-1, \ldots, -d\}$) and $\mathbb{B}_d$ as follows: each generator σ_i ($i = 1, \ldots, d{-}1$) represents a positive half-twist interchanging $-i$ and $-(i+1)$. Note that any open braid τ starting at $\mathbf{Y}^*$ and ending at $\mathbf{d}$ defines an isomorphism from $\pi_1(S_d(M), \mathbf{Y}^*)$ onto $\mathbb{B}_d$ by conjugation.

Any meridian γ of $t \in \Delta$ can be decomposed as $\alpha^{-1} \cdot \eta_{\hat{\star},S} \cdot \alpha$ as in Definition 1.2. On the other hand, any open braid τ' starting at the set of roots of $f(\hat{\star}, y)$ and ending at $\mathbf{d}$, can be decomposed as $\nabla(\gamma) = \rho^{-1} \cdot \beta \cdot \rho$, where $\rho, \beta \in \mathbb{B}_d$ satisfying:

- ρ is obtained by the juxtaposition of τ'^{-1}, $\tilde{f}_*(\alpha)$ and τ;
- β is obtained by the juxtaposition of τ'^{-1}, $\tilde{f}_*(\eta_{\hat{\star},S})$ and τ'.

The braid β reflects the local structure of the singularities of the projection Π with respect to $\mathcal{C}$ at t. Let us use the following notation:

- $L_t \cap \mathcal{C} := \{p_1, \ldots, p_s\}$,
- $(L_t \cdot \mathcal{C})_{p_i} = \ell_i + 1$.

Note that $s + \sum_{i=1}^{s} \ell_i = d$. Moreover, for generic projections one can assume that $\ell_i > 0 \Leftrightarrow i = 1$, also if p_1 is smooth then $\ell_1 = 1$, and finally if p_1 is singular then $\ell_1 + 1$ is the multiplicity of $\mathcal{C}$ at p_1.

The braid β is obtained as the unlinked union of s braids $\beta_i \in \mathbb{B}_{\ell_i+1}$, $i = 1, \ldots, s$. Note that each β_i is the local algebraic braid obtained via the Puiseux expansion of the branches of $\mathcal{C}$ at p_i with respect to the variable x. Also note that each β_i is a positive braid (i.e., represented as a word in positive powers of σ_i's). A braid β obtained as an unlinked union of local algebraic braids is called a *Puiseux braid*. In this scenario, the following relations suffice:

$$g_{n_i+k}^{\beta\cdot\rho} = g_{n_i+k}^{\rho}, \quad k = 1, \ldots, \ell_i, \quad i = 1, \ldots, s, \tag{7}$$

where $n_i := \sum_{j=1}^{i-1} (\ell_j + 1)$. Each point p_i produces ℓ_i relations which are *transported* via ρ. Regular points for the projection give no relations.

Example 1.14. Examples of braids β for several arrangements of $\{p_1, \ldots, p_s\}$ are presented below:

(a) $(\mathcal{C}, p_1)$ has a local equation $y^2 - x = 0$ and $\mathcal{C} \pitchfork_{p_i} L_t$, $i = 2, \ldots, d-1$. This means that L_t is a simple tangent to $\mathcal{C}$. Thus $\beta = \sigma_1$.

(b) $(\mathcal{C}, p_i)$ has local equations $y^2 - x = 0$, $i = 1, 2$, and $\mathcal{C} \pitchfork_{p_i} L_t$, $i = 3, \ldots, d-2$. This means that L_t is a simple bitangent to $\mathcal{C}$. Thus $\beta = \sigma_1\sigma_3$.

(c) $(\mathcal{C}, p_1)$ has a local equation $y^3 - x = 0$ and $\mathcal{C} \pitchfork_{p_i} L_t$, $i = 2, \ldots, d-2$. This means that L_t is a tangent at an ordinary inflection point of $\mathcal{C}$. Thus $\beta = \sigma_2\sigma_1$.

(d) $(\mathcal{C}, p_1)$ has local a equation $y^k - x = 0$ and $\mathcal{C} \pitchfork_{p_i} L_t$, $i = 2, \ldots, d-k+1$. This means that L_t is a tangent at a higher order flex of $\mathcal{C}$. Thus $\beta = \sigma_{k-1} \cdot \ldots \cdot \sigma_1$.

(e) $(\mathcal{C}, p_1)$ has a local equation $y^2 - x^{k+1} = 0$ and $\mathcal{C} \pitchfork_{p_i} L_t$, $i = 2, \ldots, d-1$. This means that L_t intersects $\mathcal{C}$ transversally at an $\mathbb{A}_k$-point. Thus $\beta = \sigma_1^{k+1}$.

(f) $(\mathcal{C}, p_1)$ has a local equation $y^3 - x^2 = 0$ and $\mathcal{C} \pitchfork_{p_i} L_t$, $i = 2, \ldots, d-2$. This means that L_t is tangent to $\mathcal{C}$ at an ordinary cusp. Thus $\beta = (\sigma_2\sigma_1)^2$.

(g) $(\mathcal{C}, p_1)$ has a local equation $y^3 - x^4 = 0$ and $\mathcal{C} \pitchfork_{p_i} L_t$, $i = 2, \ldots, d-2$. This means that L_t is transversal to $\mathcal{C}$ at an $\mathbb{E}_6$-point. Thus $\beta = (\sigma_2\sigma_1)^4$. One can check that local equation $y^3 + x^4 = 0$ provides a different braid, namely, $\beta = (\sigma_1\sigma_2)^4$.

(h) $(\mathcal{C}, p_1)$ has a local equation $y^k - x^k = 0$ and $\mathcal{C} \pitchfork_{p_i} L_t$, $i = 2, \ldots, d-k+1$. This means that L_t intersects $\mathcal{C}$ transversally at a k-fold ordinary point. Thus $\beta = (\sigma_{k-1} \cdot \ldots \cdot \sigma_1)^k = \Delta_k^2$.

A general algorithm to obtain a positive braid from Puiseux factorizations has been developed by S. Martínez Juste.

In general, the computation of a braid monodromy factorization is a hard numerical task. Computer-based algorithms have been produced by D. Bessis and J. Michel [21], and J. Carmona [25]. There are some particular cases where a more or less direct computation is possible:

(C1) Arrangements of lines by Arvola [18] via wiring diagrams.

(C2) Strongly real curves [3, 4, 5, 123, 12, 9, 88], i.e., curves with real equation such that the real picture and the topological type of the singular points contain all the topological information of the embedding of the curve.

(C3) A combination of (C1) and (C2) was considered by M. Salvetti in [103]. Explicit constructions can be found in [10].

The computation of the fundamental group allows us to compute the first homology group.

Proposition 1.15. *Let $d_1, \ldots, d_r$ be the degrees of the irreducible components of $\mathcal{C}$, $d := \deg \mathcal{C} = \sum_{i=1}^r d_i$, and $d_0 := \gcd(d_1, \ldots, d_r)$. Then*

$$H_1(\mathbb{P}^2 \setminus \mathcal{C}, \mathbb{Z}) = \langle x_1, \ldots, x_r \mid d_1x_1 + \cdots + d_rx_r = 0\rangle \cong \mathbb{Z}^{r-1} \times \mathbb{Z}/d_0\mathbb{Z}. \tag{8}$$

There is a natural mapping $\mathrm{Mer} : \pi_1(\mathbb{P}^2 \setminus \mathcal{C}) \to \mathbb{Z}/d\mathbb{Z}$ *sending any meridian to* $1 \bmod d$.

Proof. It is enough to abelianize (6) and recall that two meridians are in the same conjugacy class if and only if they are meridians of the same irreducible component (Definition 1.2). Q.E.D.

In some sense $\pi_1(\mathbb{P}^2 \setminus \mathcal{C})$ and $\pi_1(\mathbb{C}^2 \setminus \mathcal{C}^{\text{aff}})$ determine each other (it is important to recall that we assume in this subsection that $L \pitchfork \mathcal{C}$).

Proposition 1.16. *The fundamental group* $\pi_1(\mathbb{C}^2 \setminus \mathcal{C}^{\text{aff}})$ *is the pull-back of*

$$\begin{array}{ccc} & & \mathbb{Z} \\ & & \downarrow \\ \pi_1(\mathbb{P}^2 \setminus \mathcal{C}) & \to & \mathbb{Z}/d\mathbb{Z} \end{array} \tag{9}$$

where the horizontal arrow is the homomorphism Mer *in Proposition* 1.15 *and the vertical one is the standard projection. In particular, if a presentation of* $\pi_1(\mathbb{P}^2 \setminus \mathcal{C})$ *is given whose generators* $g_1, \ldots, g_\delta$ *are meridians of* $\mathcal{C}$, *whose relations are of type* $g_{\ell_i}^{w_i(\bar{g})} = g_{m_i}$, *and* $\prod_{j=1}^d g_{n_j}^{z_j(\bar{g})} = 1$, *where* $\ell_i, m_i, n_j \in \{1, \ldots, \delta\}$, *and* $w_i(\bar{g}), z_j(\bar{g})$ *are words in* $g_1, \ldots, g_\delta$, *then*

$$\pi_1(\mathbb{C}^2 \setminus \mathcal{C}^{\text{aff}}) = \left\langle h_1, \ldots, h_\delta \;\middle|\; h_{\ell_i}^{w_i(\bar{h})} = h_{m_i}, \left[\prod_{j=1}^{d} h_{n_j}^{z_j(\bar{h})}, h_k\right] = 1, 1 \le k \le \delta \right\rangle, \tag{10}$$

where $w_i(\bar{h})$ *and* $z_j(\bar{h})$ *are words in* $h_1, \ldots, h_\delta$ *obtained from* $w_i(\bar{g}), z_j(\bar{g})$ *replacing* g_i*'s by* h_i*'s, respectively.*

Proof. The pull-back of (9) is given by

$$G := \{(g, n) \in \pi_1(\mathbb{P}^2 \setminus \mathcal{C}) \times \mathbb{Z} \mid \ \mathrm{Mer}(g) \equiv n \bmod d\}.$$

Note that $\pi_1(\mathbb{C}^2 \setminus \mathcal{C}^{\text{aff}})$ induces a commutative diagram in (9) using as a vertical arrow the mapping coming from inclusion and as a horizontal arrow the mapping that sends any meridian to $1 \in \mathbb{Z}$. Hence there is a natural mapping $\pi_1(\mathbb{C}^2 \setminus \mathcal{C}^{\text{aff}}) \xrightarrow{\psi} G$. Using Reidemeister-Schreier, it is easily seen that (10) is a presentation of G.

For the first part, one may assume that the presentations of $\pi_1(\mathbb{P}^2 \setminus \mathcal{C})$ and $\pi_1(\mathbb{C}^2 \setminus \mathcal{C}^{\text{aff}})$ are those of (6) and (5), respectively. Since L is transversal to $\mathcal{C}$, Remark 1.13(b) implies that presentation (5) can also be obtained from (6). For convenience, the generators of $\pi_1(\mathbb{P}^2 \setminus \mathcal{C})$ will be denoted by $\tilde{g}_i$. Note that $\psi(g_i) = (\tilde{g}_i, 1)$ and $\psi(g_d \cdot \ldots \cdot g_1) = (1, d)$. Since G and $\pi_1(\mathbb{C}^2 \setminus \mathcal{C}^{\text{aff}})$ have the same presentations, ψ is an isomorphism. Q.E.D.

Presentations of fundamental groups of affine plane curves $\pi_1(\mathbb{C}^2 \setminus \mathcal{C}^{\text{aff}})$ have special properties. For instance, $\pi_1(\mathbb{C}^2 \setminus \mathcal{C}^{\text{aff}})$ can be generated by meridians and any two meridians of the same irreducible component are in the same conjugacy class (Definition 1.2). A useful type of presentations of $\pi_1(\mathbb{C}^2 \setminus \mathcal{C}^{\text{aff}})$ is the following:

Let $\mathcal{C}_1, \ldots, \mathcal{C}_r$ be the irreducible components of $\mathcal{C}^{\text{aff}}$ and let us fix a meridian x_i of $\mathcal{C}_i$. Note that, as mentioned above, any other meridian $\tilde{x}_i$ of $\mathcal{C}_i$ is in the conjugacy class of x_i, in particular, it can be written as $x_i \cdot y$, $y \in \pi_1(\mathbb{C}^2 \setminus \mathcal{C}^{\text{aff}})'$, $\pi_1(\mathbb{C}^2 \setminus \mathcal{C}^{\text{aff}})'$ being the commutator subgroup of $\pi_1(\mathbb{C}^2 \setminus \mathcal{C}^{\text{aff}})$. We call a presentation of a group satisfying these properties a Zariski presentation, whose precise definition is as follows:

Definition 1.17. Let G be group. We denote its commutator group by G'. A presentation $\langle x_1, \ldots, x_r, y_1, \ldots, y_u \mid w_i(\bar{x}, \bar{y}) = 1 \rangle$, $\bar{x} := (x_1, \ldots, x_r)$, $\bar{y} := (y_1, \ldots, y_u)$, of a group G is called a *Zariski presentation* if

- (ZP$_1$) The Abelian group $G/G' \cong \mathbb{Z}^q \oplus \mathbb{Z}/p_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p_s\mathbb{Z}$ $(q+s=r)$ is generated by the classes of $x_1, \ldots, x_r$, where $x_{q+i}^{p_i} \in G'$ for $i = 1, \ldots, s$.
- (ZP$_2$) The classes of $y_1, \ldots, y_u$ in G/G' vanish.
- (ZP$_3$) The words $w_i(\bar{x}, \bar{1})$ are products of commutators in $\bar{x}$ and $x_{q+i}^{p_i}$.

To be more precise, we have just proved the following.

Proposition 1.18. *Under the conditions above, $\pi_1(\mathbb{C}^2 \setminus \mathcal{C}^{\text{aff}})$ admits a Zariski presentation, where $s \leq d - r$, and $d := \deg \mathcal{C}^{\text{aff}}$.*

1.3. Braid monodromy and topology

In this section we want to examine in more detail the type of objects presented in Definition 1.9 in the previous subsection. They have been used to obtain presentations of fundamental groups, but they actually contain much more information.

Consider $(\tau_1, \ldots, \tau_r) \in (\mathbb{B}_d)^r$ a braid monodromy factorization of the triple $(\mathcal{C}, L, P)$ obtained from the geometric basis $\gamma_1, \ldots, \gamma_r$. As mentioned after Definition 1.9, any change of geometric basis produces a new braid monodromy factorization, which is given by a Hurwitz move. Since the family of possible geometric bases is parametrized by $\mathbb{B}_r$ (Proposition 1.5), the braid group induces a *Hurwitz action* on $(\tau_1, \ldots, \tau_r)$. Also, a change of the base point $\iota_{\mathbf{Y}^*}$ produces a new braid monodromy factorization, which is given by conjugation $(\tau_1^\beta, \ldots, \tau_r^\beta)$, where β is a certain braid in $\mathbb{B}_d$.

The different factorizations derived from all the different choices can be described as natural $\mathbb{B}_r$ and $\mathbb{B}_d$ actions on $(\mathbb{B}_r)^r$ on the right as follows:

$\mathbb{B}_r$: Let $(\tau_1,\ldots,\tau_r)\in(\mathbb{B}_d)^r$ and let σ_i $(i=1,\ldots,r-1)$ be canonical generators. Then the action is given by

$$\tau_i^{\sigma_j} := \begin{cases} \tau_{i+1} & \text{if } i=j \\ \tau_i * \tau_{i-1} & \text{if } i=j+1 \\ \tau_i & \text{if } i\neq j+1. \end{cases} \tag{11}$$

for $i=1,\ldots,n-1, j=1,\ldots,n$.

$\mathbb{B}_d$: Let $(\tau_1,\ldots,\tau_r)\in(\mathbb{B}_d)^r$ and let $\beta\in\mathbb{B}_d$. Then the action is given by

$$(\tau_1,\ldots,,\tau_r)^\beta := (\tau_1^\beta,\ldots,\tau_r^\beta)$$

These actions of $\mathbb{B}_r$ and $\mathbb{B}_d$ commute; and hence they define a $\mathbb{B}_r\times\mathbb{B}_d$ action on $(\mathbb{B}_d)^r$.

For more details, see [7]. Summing up, we have the following:

Proposition 1.19. *Let $(\tau_1,\ldots,\tau_r)$ be a braid monodromy factorization of $(\mathcal{C},L,P)$. Then $(\tilde\tau_1,\ldots,\tilde\tau_r)\in(\mathbb{B}_d)^r$ is another braid monodromy factorization of $(\mathcal{C},L,P)$ if and only if $(\tilde\tau_1,\ldots,\tilde\tau_r)$ is in the orbit of $(\tau_1,\ldots,\tau_r)$ with respect to the $\mathbb{B}_r\times\mathbb{B}_d$ action as above.*

Remarks 1.20.

(1) The definitions and results in § 3.1 and § 1.2 where *geometric bases* are required can be substituted by *pseudo-geometric bases*. In particular, the concept of braid monodromy factorization can also be obtained from a pseudo-geometric basis and Proposition 1.10, and Corollary 1.11 remain true.

(2) Note that a continuous change of the generic line L and the projection P produces a situation similar to the change of the base point in the following sense. A continuous change of the generic line L and fixing P (that is, in the pencil of lines through P) only produces a new pseudo-geometric basis, whose associated factorization is the same as the original one. On the other hand, a continuous change of P on L also defines a new pseudo-geometric basis, whose associated factorization is conjugated of the original one by the braid defined by the motion of P on the generic fiber. Combinations of these two motions allow one to move from $(\mathcal{C},L,P)$ to any other triple $(\mathcal{C},L',P')$.

This motivates the following:

Definition 1.21. Two triplets $(\mathcal{C}, L, P)$ and $(\mathcal{C}, L', P')$ as above are said to have *equivalent braid monodromies* if their braid monodromy factorizations are in the same orbit. We will refer to the class of equivalent braid monodromies as the *braid monodromy of the curve* $\mathcal{C}$.

A special –and very useful– type of braid monodromy factorization can be obtained as follows. Let us fix a pseudo-geometric basis $(\gamma_1, \ldots, \gamma_r)$ of $\pi_1(\mathbb{C} \setminus \Delta; \star)$ and decompose each γ_i in the form of a conjugation as in Definition 1.2. Following the ideas in Remark 1.13(c), one obtains a braid monodromy factorization of the form $(\rho_1 \cdot \beta_1 \cdot \rho_1^{-1}, \ldots, \rho_r \cdot \beta_r \cdot \rho_r^{-1})$, where $\beta_1, \ldots, \beta_r$ are Puiseux braids. These are not difficult to obtain from a Puiseux expansion of each local singularity. The difficult numerical part comes from the computation of the braids ρ_i.

Definition 1.22. We say that $((\rho_1, \beta_1), \ldots, (\rho_r, \beta_r))$ is a *Puiseux-braid monodromy factorization* of $\mathcal{C}$ if $(\rho_1^{-1} \cdot \beta_1 \cdot \rho_1, \ldots, \rho_r^{-1} \cdot \beta_r \cdot \rho_r)$ is a braid monodromy factorization of $\mathcal{C}$ and the braids ρ_i, β_i are obtained from the decomposition of a meridian (in a pseudo-geometric basis) as in Remark 1.13(c).

Remark 1.23. Note that if $((\rho_1, \beta_1), \ldots, (\rho_r, \beta_r))$ is a Puiseux-braid monodromy factorization of $\mathcal{C}$, then in particular $\beta_1, \ldots, \beta_r$ are Puiseux braids. Also, if $(\mathcal{C}_1, \mathcal{C}_2)$ is a Zariski pair one can obtain Puiseux-braid monodromy factorizations $((\rho_i^1, \beta_i))_{i=1}^r$ and $((\rho_i^2, \beta_i))_{i=1}^r$ such that the Puiseux braids coincide.

Remark 1.24. Note that Definition 1.22 is very restrictive. Let $((\rho_1, \beta_1), \ldots, (\rho_r, \beta_r))$ be a r-tuple of pairs of braids such that $(\rho_1^{-1} \cdot \beta_1 \cdot \rho_1, \ldots, \rho_r^{-1} \cdot \beta_r \cdot \rho_r)$ is a braid monodromy factorization of $\mathcal{C}$ and $\beta_1, \ldots, \beta_r$ are Puiseux braids. This is a necessary condition for $((\rho_1, \beta_1), \ldots, (\rho_r, \beta_r))$ to be a Puiseux-braid monodromy. Since Definition 1.22 imposes that the factorization $\rho_i^{-1} \cdot \beta_r \cdot \rho_i$ must come from a very particular geometrical decomposition of a meridian, one cannot ensure that $((\rho_1, \beta_1), \ldots, (\rho_r, \beta_r))$ is a Puiseux-braid monodromy factorization of $\mathcal{C}$.

The construction leading to the Zariski-van Kampen method is closer to Puiseux-braid than to general braid monodromy, but it is not easy to define good equivalence relations between Puiseux-braid monodromy factorizations. From Corollary 1.11 one deduces that braid monodromy factorizations determine $\pi_1(\mathbb{P}^2 \setminus \mathcal{C})$ and $\pi_1(\mathbb{C}^2 \setminus \mathcal{C}^{\text{aff}})$.

In [73], Libgober showed the relationship between Puiseux-braid monodromy factorizations and homotopy type as follows.

Theorem 1.25 (Libgober [73])**.** *The CW-complex associated with the finite presentation of* $\pi_1(\mathbb{C}^2 \setminus \mathcal{C}^{\text{aff}})$ *obtained from* (5)*, with the reduction in the number of relations described in Remark* 1.13(c)*, has the* *homotopy type of* $\mathbb{C}^2 \setminus \mathcal{C}^{\text{aff}}$. *In particular, a Puiseux-braid monodromy factorization of* $\mathcal{C}$ *determines the homotopy type of* $\mathbb{C}^2 \setminus \mathcal{C}^{\text{aff}}$.

The following results show the strength of Theorem 1.25 and are evidence of the importance of braid monodromy.

Theorem 1.26 (Kulikov-Teicher [70])**.** *If* $\mathcal{C}$ *has only ordinary nodes and cusps, then a braid monodromy factorization of* $\mathcal{C}$ *determines the diffeomorphism type of* $(\mathbb{P}^2, \mathcal{C})$.

Theorem 1.27 (Carmona [25])**.** *A braid monodromy factorization of* $\mathcal{C}$ *determines the oriented homeomorphism types of the pairs* $(\mathbb{C}^2, \mathcal{C}^{\text{aff}})$ *and* $(\mathbb{P}^2, \mathcal{C})$.

Carmona uses the local results of M. Namba and M. Takai [87] to prove that one can produce a topological model of the pairs $(\mathbb{C}^2, \mathcal{C}^{\text{aff}})$ and $(\mathbb{P}^2, \mathcal{C})$ from any decomposition as in Remark 1.24. With this model one can apply Libgober's techniques in [73] to prove that $\mathbb{C}^2 \setminus \mathcal{C}^{\text{aff}}$ has the homotopy type of the CW-complex associated with a presentation obtained from a Puiseux-braid factorization, after the reduction explained in Remark 1.13(c).

1.4. Generic and non-generic braid monodromies

Up to now, all the statements in §1 assume a generic projection, i.e., $P \notin \mathcal{C}$ and $L \pitchfork \mathcal{C}$. We also assume that the lines L_t, $t \in \Delta$, satisfy the following:

- Either L_t passes through a singular point of $\mathcal{C}$ and L_t does not belong to its tangent cone, or
- L_t is an ordinary tangent to a smooth point of $\mathcal{C}$ (i.e., not a flex).

Also, all other intersections of L_t are transversal, i.e., in the notation of Remark 1.13(c), one has $\ell_i = 0$ if $i > 1$. The braid monodromy obtained under these hypotheses does not depend on the particular choice of L and P, $P \in L$ (Remark 1.20(2)). Moreover, it is an invariant of the connected components of the combinatorial strata of curves.

Oftentimes, non-generic braid monodromies arise in a natural way, and in general they are very useful. Let us explain different situations where a braid monodromy is non-generic.

(NG1) We can choose a non-generic line L and P generic in L; in particular, $P \notin \mathcal{C}$ and $L \not\pitchfork \mathcal{C}$. In this case Proposition 1.10 and Corollary 1.11 are still true, because Libgober's proof of

Theorem 1.25 also applies here and so does Carmona's Theorem 1.27. One can eliminate relations for the fundamental group using Remark 1.13(c), but not (b).

(NG2) The line L is generic but P is non-generic and $P \notin \mathcal{C}$, i.e., for some $t \in \Delta$ either we have several non-transversal intersections or L_t is tangent to $\mathcal{C}$ at a flex. All the results of §1.2 and §1.3 are true word-for-word. In general, this braid monodromy is not an invariant of connected components of combinatorial strata of curves.

(NG3) P is a smooth (non-flex) point of $\mathcal{C}$ and L is the tangent line. In this case the affine parts of §1.2 and §1.3 remain true. This is due to the fact that $\mathcal{C}^{\text{aff}}$ admits an equation $f(x, y) = 0$, where f is monic in y. In general we cannot eliminate the relations as in Remark 1.13(b). This braid monodromy defines an invariant of connected components of combinatorial strata of curves.

(NG4) The point $P \in \mathcal{C}$ is either a flex or a singularity whose tangent cone is irreducible. In either case we choose L to be the tangent line. The braid monodromy in this case behaves as in (NG3) (because $\mathcal{C}$ admits a monic equation in y) though in general it will not be an invariant of the connected component of combinatorial strata of curves. In this case, as well as in (NG3), one can also compute $\pi_1(\mathbb{P}^2 \setminus \mathcal{C})$, but some additional information about the behavior of the strings of the braid at infinity is needed, see [5, 9, 12] for examples.

(NG5) Choose P, L in order to have vertical asymptotes, i.e., $P \in \mathcal{C}$ and at least one tangent line to $\mathcal{C}$ at P is not L. This case has been deeply studied in [25]. Braid monodromy and some additional data allow one to apply a modified version of the Zariski-van Kampen Theorem and to codify the embedding of $\mathcal{C}$ in $\mathbb{P}^2$.

Why are non-generic braid monodromies interesting? There are at least two reasons. The first one comes from effectiveness of computation. For a curve $\mathcal{C}$, generic braid monodromy factorizations are orbits in $(\mathbb{B}_d)^r$, where r is the sum of the degree of the dual curve and the number of singular points of $\mathcal{C}$. This number could be reduced significantly by considering non-generic projections. Also, under certain circumstances, a generic braid monodromy factorization can be recovered from a non-generic one. The second reason has to do with a partial converse of Theorem 1.27 and will be developed in page 26.

Proposition 1.28. *Let us assume that L_t, $t \in \Delta$, is as in* (NG2). *Let τ be the braid associated with L_t. If L_t, $t \in \Delta$, has h non-transversal intersections, then τ is the product of h pairwise commuting braids.*

Let us assume now that any L_t has only one non-transversal intersection $p_1 \in L_t \cap \mathcal{C}$. Then:

(1) *If L_t is tangent to $\mathcal{C}$ at a smooth point p_1 with $(\mathcal{C} \cdot L_t)_{p_1} = n$, then r decreases by $n-2$. Moreover, τ decomposes into $n-1$ braids conjugated to σ_1 when P is slightly moved in L.*

(2) *If L_t, $t \in \Delta$, passes through $p_1 \in \mathrm{Sing}(\mathcal{C})$ of multiplicity m and $(\mathcal{C} \cdot L_t)_{p_1} = n$, then r decreases by $n-m$ and τ decomposes into $n-m+1$ braids, $n-m$ of them conjugated to σ_1 when P is slightly moved in L.*

Proof. The situation becomes generic by slightly moving the point P in L, which implies a change in the projection direction. In this case, if L_t intersects non-transversally at h points, then after changing the projection direction, L_t splits into h non-transversal lines in a small neighborhood of L_t, which correspond to disjoint and unlinked strings. Thus the first statement follows. For example, in the case of a bitangent (Figure 2) as in Example 1.14(b), $\tau = \rho \cdot (\sigma_3 \cdot \sigma_1) \cdot \rho^{-1}$ decomposes into the commuting braids $\rho \cdot \sigma_3 \cdot \rho^{-1}$ and $\rho \cdot \sigma_1 \cdot \rho^{-1}$.

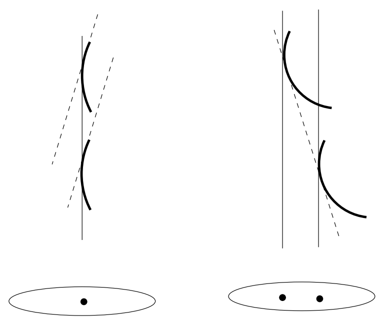

Fig. 2. Bitangent

Let us prove (1). Note that, in this situation $\tau = \rho\cdot(\sigma_{n-1}\cdot\ldots\cdot\sigma_1)\cdot\rho^{-1}$ as in Example 1.14(d), and hence τ decomposes into $n-1$ braids $\rho\cdot\sigma_i\cdot\rho^{-1}$, $i = 1, \ldots, n-1$ –see Figure 3.

If p_1 is a singular point of multiplicity m and $(\mathcal{C} \cdot L_t)_{p_1} = n$, then a perturbation of the projection produces $m-n+1$ non-transversal lines close to L_t: one of them passing through p_1 (not in the tangent cone) and the other ones ordinary tangents. This gives the statement of (2). Let us see what happens if p_1 is a cusp as in Example 1.14(f), where

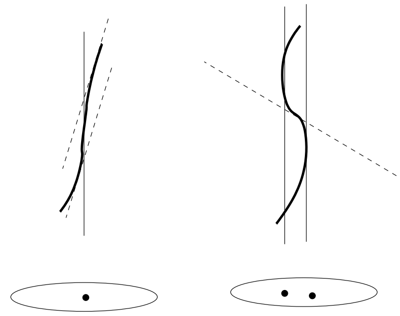

Fig. 3. Flex

$\tau = \rho \cdot (\sigma_2 \cdot \sigma_1)^2 \cdot \rho^{-1}$. The decomposition is given by $\rho \cdot \sigma_2^3 \cdot \rho^{-1}$ and $\rho \cdot \sigma_2^{-1} \cdot \sigma_1 \cdot \sigma_2 \cdot \rho^{-1}$. Q.E.D.

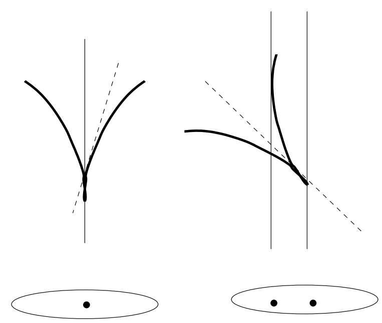

Fig. 4. Cusp

Also, for projections as in (NG1) one can prove the following.

Proposition 1.29. *For each non-transversal point $p \in \mathcal{C} \cap L$, with $\operatorname{mult}_p \mathcal{C} = m$ and $(\mathcal{C} \cdot L)_p = n$, the decreasing of r is given by:*

(1) $n-1$ *if L is tangent to $\mathcal{C}$ at a smooth point p.*

(2) $n-m+1$ *if L passes through $p \in \operatorname{Sing}(\mathcal{C})$.*

In this case the line at infinity can be deformed into a generic line through P. The new braid factorization is of type (NG2) *and has an extra term, which is obtained as the product of Δ_d^2 by the inverse of the product of the original braid factorization.*

Therefore a combination of Propositions 1.28 and 1.29 allows one to obtain a generic braid monodromy factorization.

Remark 1.30. If $P \in \mathcal{C}$ and $\operatorname{mult}_P \mathcal{C} = m$, then not only r decreases but also its braid monodromy takes values in $\mathbb{B}_{d-m}$. As for the fundamental group, one needs additional data in order to recover the generic braid monodromy but the complete description of these additional data and the recovering process has not been developed yet.

The second reason stands on this partial converse of Theorem 1.27. Let us introduce some notation. Let $L \nsubseteq \mathcal{C}$ be a line and $P \in L$. Let us also assume that the tangent cone of $\mathcal{C}$ at P is contained in L (the tangent cone is empty if $P \notin \mathcal{C}$). In other words, keeping the notations of §1.2, we are assuming that f is monic in y. For $t \in \Delta$ let $\bar{L}_t$ be the projective closure of L_t (recall that $P \in \bar{L}_t$).

Definition 1.31. A triple $(\mathcal{C}, L, P)$ as above is called a *horizontal triple*. The braid monodromy obtained choosing P as projection point and L as line at infinity is called the *braid monodromy of the horizontal triple*. The *fibered curve* $\mathcal{C}^{\varphi}$ associated with the horizontal triple is $\mathcal{C} \cup L \cup \bigcup_{t \in \Delta} \bar{L}_t$.

We recall the following results by the first two authors and Carmona.

Theorem 1.32 ([7])**.** *Let $(\mathcal{C}_1, L, P)$ and $(\mathcal{C}_2, L, P)$ be two horizontal triples. Let $F : \mathbb{P}^2 \to \mathbb{P}^2$ be an orientation-preserving homeomorphism such that:*

(i) *$F(P) = P$, i.e, F respects the base point of the fibration.*
(ii) *$F(L) = L$ preserving orientations.*
(iii) *$F(\mathcal{C}_1^{\varphi}) = \mathcal{C}_2^{\varphi}$ preserving orientations.*

Then $(\mathcal{C}_1, L, P)$ and $(\mathcal{C}_2, L, P)$ have the same braid monodromy.

For the special case of line arrangements this result has an *ordered* version.

Definition 1.33. An *ordered arrangement of lines* $\mathcal{L}$ is an ordered list of lines in $\mathbb{P}^2$. A triple $(\mathcal{L}, L, P)$ is called a *horizontal triple arrangement* if it defines a horizontal triple and the lines $\{\bar{L}_t\}_{t \in \Delta}$ are ordered. The *fibered arrangement* $\mathcal{L}^{\varphi}$ associated with the horizontal triple arrangement is the ordered arrangement $\mathcal{L} + (L) + (\bar{L}_t)_{t \in \Delta}$.

Let us consider a braid monodromy factorization of $(\mathcal{L}, L, P)$. Since $\mathcal{L}$ is an arrangement of lines, this representative belongs to $(\mathbb{PB}_d)^r$, where $\mathbb{PB}_d$ is the pure braid group. Moreover, by the choice of an order in $\mathcal{L}$ only conjugations by elements of $\mathbb{PB}_d$ are allowed (otherwise meridians of different components are interchanged). Moreover, since $\{\bar{L}_t\}_{t \in \Delta}$ is also ordered by the incidence relations, only *pure* Hurwitz moves are allowed.

Definition 1.34. A *pure braid monodromy* is an orbit of $(\mathbb{PB}_d)^r$ by the action of $\mathbb{PB}_d \times \mathbb{PB}_r$. A *horizontal triple arrangement* $(\mathcal{L}, L, P)$ has associated with it a pure braid monodromy.

Theorem 1.35 ([10])**.** *Let $(\mathcal{L}_1, L, P)$ and $(\mathcal{L}_2, L, P)$ be two horizontal triple arrangements. Let $F : \mathbb{P}^2 \to \mathbb{P}^2$ be an orientation-preserving homeomorphism such that:*

(i) $F(P) = P$.
(ii) $F(L) = L$ *preserving orientations.*
(iii) $F(\mathcal{L}_1^\varphi) = \mathcal{L}_2^\varphi$ *preserving orientations and orders.*

Then $(\mathcal{L}_1, L, P)$ and $(\mathcal{L}_2, L, P)$ have the same pure braid monodromy.

1.5. Fundamental group, braid monodromy and Zariski pairs

Fundamental groups are a primary tool in the problem of finding Zariski pairs. In the example mentioned in the Introduction (Z5), Zariski proved that the combinatorially-equivalent curves had different fundamental groups ($\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$). In fact, before completing all the computations, Zariski had found other weaker invariants that also served to distinguish the members of the pair. The first one is the study of branched Galois coverings ramified along the curve: in only one case do D_6-coverings exist – a proof in modern terms can be found in §3.3. In §3, we present how the algebraic study of Galois coverings is a powerful tool to study Zariski pairs, obtaining information about fundamental groups without having to compute them. The second invariant is the Alexander polynomial which, along with some generalizations, is studied in §2.

These two kinds of invariants are useful for several reasons. As we have seen in this section, computation of fundamental groups can be a very tricky task. On one hand, some algebraic properties of the curves can give rise to invariants of the fundamental group, and thus a difference in such invariants means a difference in fundamental groups. On the other hand, even if the fundamental group is computed, what one obtains is a finite presentation of it. The undecidability of the isomorphism problem makes this task feasible only in the simplest examples.

As explained in the Introduction note that, even though $\pi_1(\mathbb{P}^2 \setminus \mathcal{C})$ is an invariant of the embedded topology of a curve $\mathcal{C}$, any homeomorphism of pairs $(\mathbb{P}^2, \mathcal{C})$ should send meridians to meridians. Therefore, $\pi_1(\mathbb{P}^2 \setminus \mathcal{C})$ with a peripheral structure given by the conjugacy class of meridians of the irreducible components of $\mathcal{C}$ is sometimes a more useful invariant. For example, if we are counting the number of irreducible representations of $\pi_1(\mathbb{P}^2 \setminus \mathcal{C})$ onto a given finite group, the peripheral structure introduces some restrictions on the images of the meridians. Analogously, if we are considering Betti numbers of Abelian coverings, the peripheral structure allows us to describe such coverings canonically and thus point out possible differences. In general they will not be invariants of the sole fundamental group, but they will be useful for detecting Zariski pairs. These techniques fail when we look for *arithmetic Zariski pairs*, since in this case most invariants of finite coverings are of an arithmetic nature.

We end up this section with referring examples for various kinds of Zariski pairs. For details, see the cited references

Example 1.36. Once one has an example of a Zariski pair which is distinguished by the fundamental group, it is possible to give infinite families of Zariski pairs using Cremona transformations and covering maps. These techniques have been used by Oka [91], Shimada [105, 106] and A.M. Uludağ [123].

Example 1.37 ([62])**.** In their paper Kharlamov-Kulikov use braid monodromy factorizations to find a special kind of Zariski pair, called *oriented Zariski pair*. An oriented Zariski pair is characterized by the non-existence of orientation-preserving homeomorphisms. Note that complex conjugation preserves the orientation of $\mathbb{P}^2$, but reverses the orientations of the curves. In [63], they also find examples of complex-conjugated surfaces such that the complex conjugation does not preserve canonical divisors, thus they do not admit orientation-preserving homeomorphisms. If we apply Chisini's conjecture, the branch curves of generic coverings give oriented Zariski pairs. For each m the produced Zariski pair involves curves of degree $333m^2$.

Example 1.38 ([9])**.** Let us consider the combinatorial stratum of curves with the following combinatorics: sextics with two irreducible components of degrees 5 and 1. The quintic curve has three singular points of types $\mathbb{E}_6$, $\mathbb{A}_3$ and $\mathbb{A}_2$ and the line intersects the quintic at two smooth points with intersection multiplicities 4 and 1.

It is not hard to prove that this space has two connected components (each one is isomorphic to $\mathrm{PGL}(3;\mathbb{C})$). For one component there is a representative $\mathcal{C}_+$ with equation in $\mathbb{Q}(\sqrt{2})[X,Y,Z]$. Its conjugate $\mathcal{C}_-$ belongs to the other connected component. Let us denote by $(\mathcal{C}_\pm, L_\pm, P_\pm)$ the horizontal triples, where $P_\pm$ is the $\mathbb{E}_6$ point of $\mathcal{C}_\pm$ and $L_\pm$ is the tangent line.

Using techniques for strongly real curves one can compute a non-generic braid monodromy factorization of type (NG4) of $(\mathcal{C}_+, L_+, P_+)$:

$$(\sigma_2^8, \sigma_2^4\sigma_1^2\sigma_2^{-4}, \sigma_2^3\sigma_1^3\sigma_2^{-3}, \sigma_2\sigma_1^4\sigma_2^{-1}, \sigma_1^{-3}\sigma_2\sigma_1^3),$$

and one of $(\mathcal{C}_-, L_-, P_-)$:

$$(\sigma_2^3, (\sigma_2\sigma_1^{-1}\sigma_2)\sigma_1(\sigma_2\sigma_1^{-1}\sigma_2)^{-1}, \sigma_2\sigma_1^8\sigma_2^{-1}, \sigma_1^{-2}\sigma_2^4\sigma_1^2, \sigma_1^{-3}\sigma_2^2\sigma_1^3).$$

In fact, these are Puiseux-braid monodromy factorizations. Since the additional data at infinity are easy to obtain, one can compute $\pi_1(\mathbb{P}^2 \setminus \mathcal{C}_\pm)$. It turns out that both groups are isomorphic to $\mathbb{Z} \times \mathrm{GL}(2; \mathbb{F}_7)$.

The main point is that these two braid monodromy factorizations are non-equivalent. Taking representations of the braid group onto $\mathrm{GL}(2, \mathbb{Z}/32\mathbb{Z})$, the image of the braid monodromies becomes a finite set. One can simply check using GAP [49] that both images are disjoint. By Theorem 1.32, $(\mathcal{C}_+^\varphi, \mathcal{C}_-^\varphi)$ is an arithmetic Zariski pair. Similar examples in [9] also provide examples of oriented Zariski pairs.

Example 1.39. Let us consider the combinatorial stratum of sextic curves with 4 singular points of types $\mathbb{E}_7$, $\mathbb{E}_6$, $\mathbb{A}_4$, and $\mathbb{A}_2$. As in the previous example, this space consists of two irreducible components, each one isomorphic to $\mathrm{PGL}(3; \mathbb{C})$. Representatives can be taken in each component with equations in $\mathbb{Q}(\sqrt{5})[x, y, z]$ as follows:

$$\begin{aligned} f_s(x,y,z) := &-(200+90\,s)\,x^6 - (1575+705\,s)\,x^5y - (552+254\,s)\,zx^5 \\ &-(3963+1779\,s)\,zx^4y - (456+222\,s)\,z^2x^4 \\ &-(63+27\,s)\,zx^3y^2 - (2817+1251\,s)\,z^2x^3y \\ &-(56+21\,s)\,z^3x^3 + (666+324\,s)\,z^2x^2y^2 \\ &+(-45+15\,s)\,z^3x^2y + (48+16\,s)\,z^4x^2 + (1737+783\,s)\,z^3xy^2 \\ &+(384+192\,s)\,z^4xy + 54z^3y^3 + (1008+432\,s)\,z^4y^2, \end{aligned}$$

where $s^2 = 5$. Let us consider the triples associated with the $\mathbb{E}_7$-point $P = [0 : 1 : 0]$ and the tangent line $L = \{z = 0\}$ at P. First we will compute their braid monodromy factorization based on the real picture, since both curves are strongly real (see (C2)).

Figure 5 shows the real picture of $\mathcal{C}_{\sqrt{5}} := \{f_{\sqrt{5}} = 0\}$, the choice of the generic line $L_\star$, and the choice of the generators of the braid group based on $L_\star \setminus \mathcal{C}_{\sqrt{5}}$. We recall that the way the σ_i are chosen in general corresponds to the lexicographic order in $\mathbb{C}$ where $a_1 + b_1\sqrt{-1} < a_2 + b_2\sqrt{-1}$ if and only if $a_1 < a_2$, or $a_1 = a_2$ and $b_1 < b_2$.

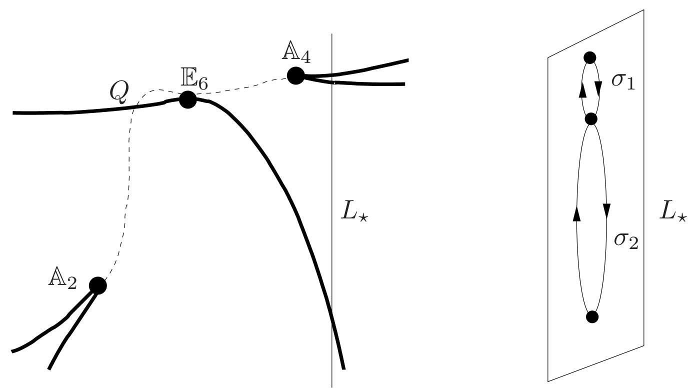

Fig. 5. Real graph of $\mathcal{C}_{\sqrt{5}}$.

The only singular fibers occur at $x = -\frac{32}{5}$ (where the $\mathbb{A}_2$-point $(-\frac{32}{5}, -\frac{1584}{5} - 144\sqrt{5})$ lies), at $x = -1$ (where the $\mathbb{E}_6$-point $(-1, 0)$ lies), and at $x = 0$ (where the $\mathbb{A}_4$-point $(0, 0)$ lies). This can be checked by factorizing the discriminant of $f_{\sqrt{5}}$ with respect to y.

The dotted curve in Figure 5 represents the real parts of the complex conjugated branches. When at most two branches are complex conjugated per fiber of the projection (as is our case) this picture plus the local braids contain all the necessary information to compute the braid monodromy factorization.

In our case, the local braids around the $\mathbb{A}_4$ and the $\mathbb{A}_2$ are obvious because the branches involved are real. Therefore the first factor of the factorization should be σ_1^5 and the last one should be a conjugated of σ_2^3 (Example 1.14(e)). In this case note that half a turn around the $\mathbb{A}_4$ point corresponds to σ_1^2. Therefore the factorization this far looks like $(\sigma_1^5, \sigma_1^2 * \beta, (\sigma_1^2\beta_1\alpha) * \sigma_2^3)$, where β is the local braid around the $\mathbb{E}_6$-point, β_1 is half the braid around the $\mathbb{E}_6$-point, and α is the braid from $\mathbb{E}_6$ to $\mathbb{A}_2$.

To obtain α it is enough to note that a local crossing of type Q as in Figure 5 corresponds to $\sigma_2^{-1}\sigma_1$ (always according to our lexicographic order in $\mathbb{C}$) as shown in Figure 6. Since there are no more crossings between $\mathbb{E}_6$ and $\mathbb{A}_2$ one has that $\alpha = \sigma_2^{-1}\sigma_1$.

For the $\mathbb{E}_6$-point one has to work a little bit more. One first considers a parametrization for the local branches of f_s at this point: something of the form $y = \omega_1 x + \omega_2 \xi^k x^{\frac{4}{3}}$, $k = 0, 1, 2$, where $\xi^2 + \xi + 1 = 1$.

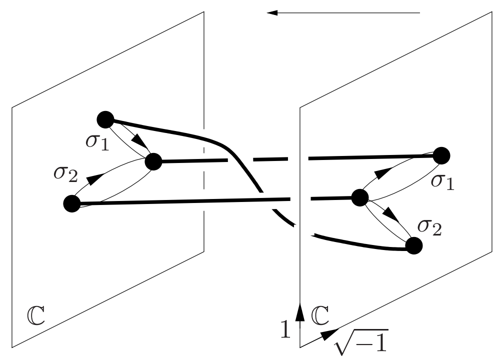

Fig. 6. Description of the crossing at Q.

Basically the sign of the real part of ω_2 determines the local braid as shown in Example 1.14(g). In our case one obtains $\beta = (\sigma_1\sigma_2)^4$ and hence $\beta_1 = (\sigma_1\sigma_2)^2$.

Therefore the braid monodromy factorization in this example is

$$(\sigma_1^5, \sigma_1^2 * (\sigma_1\sigma_2)^4, (\sigma_1^2(\sigma_1\sigma_2)^2(\sigma_2^{-1}\sigma_1)) * \sigma_2^3).$$

Using the relation $\sigma_1 * \sigma_2^k = \sigma_2^{-1} * \sigma_1^k$ and the obvious $\sigma_i^r * \sigma_i^k = \sigma_i^k$, the last term can be reduced to $\sigma_1 * \sigma_2^3$, obtaining Table 1.

σ_1^5	$\sigma_1^2 * (\sigma_1\sigma_2)^4$	$\sigma_1 * \sigma_2^3$

Table 1.

Claim 1.40. *The (non-generic) braid monodromy factorizations of* $(\mathcal{C}_{\sqrt{5}}, L, P)$ *and* $(\mathcal{C}_{-\sqrt{5}}, L, P)$ *coincide.*

Proof. For $\mathcal{C}_{-\sqrt{5}}$ one has an analogous situation as shown in Figure 7, which has the same local and global information of the strongly real picture. Therefore, their braid monodromy factorizations coincide. Q.E.D.

Moreover, note that, even though the combinatorial stratum consists of two irreducible components, the associated affine curves are isomorphic. In particular

$$\tilde{f}_{\sqrt{5}}(\omega(x,y)) = \left(51841 + 23184\sqrt{5}\right) \tilde{f}_{-\sqrt{5}}(\ell(x,y)),$$

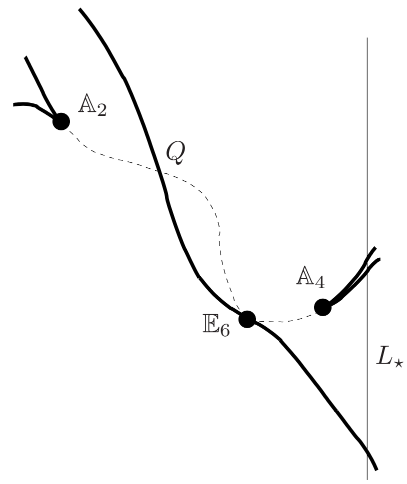

Fig. 7. Real graph of $\mathcal{C}_{-\sqrt{5}}$.

where $\tilde{f}_s$, denote the affine equation of f_s,

$$\omega(x,y) := \left(x, -y + \left(1620 + 648\sqrt{5}\right) x - \left(\frac{25}{2} + \frac{35\sqrt{5}}{6}\right) x^2\right)$$

is a Jung automorphism, and

$$\ell(x,y) := \left(x, \left(\frac{8075}{2} - \frac{10657\sqrt{5}}{6}\right) x + \left(-\frac{47}{2} + \frac{21\sqrt{5}}{2}\right) y\right)$$

is a linear automorphism of $\mathbb{C}^2$. Note that this does not give another proof of Claim 1.40 via Theorem 1.27 since the latter can only be applied in principle to generic projections. But it does tell us that the generic braid monodromies of $\mathcal{C}_{\sqrt{5}} \cup L$ and $\mathcal{C}_{-\sqrt{5}} \cup L$ are equivalent.

A geometrical interpretation of the Jung automorphism can be given as follows. In Figure 8, we show the dual graph of the total transform of L after an embedded resolution of the $\mathbb{E}_7$-point. The successive exceptional divisors are denoted by E_i, $i = 1,2,3$. The branch B_1 denotes the strict transform of the cusp of $\mathbb{E}_7$, the branch B_2 denotes the strict transform of the smooth branch of the $\mathbb{E}_7$ and the branch B_3 denotes the branch at the smooth point of $\mathcal{C}$ on $L \cap \mathcal{C}_s$.

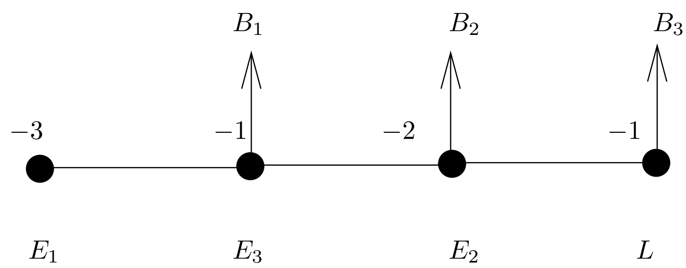

Fig. 8.

Note that, contracting the strict transform of L one achieves a combinatorially-symmetric situation where B_2 and B_3 cannot be distinguished. Note that, by the Jung automorphism, B_2 and B_3 are interchanged.

We can still recuperate valuable information to add to equivalent braid monodromies that can distinguish the different behavior at infinity. The idea is to *color* the different branches at infinity. This idea will be developed in what follows.

Remark 1.41. Let $(\mathcal{C}, L, P)$ be a horizontal triple of degree d and let us choose L_*, a generic member of $\mathcal{H}_P$, as the base line of the pencil. Let us also fix a continuous uniparametric family of lines L_t ($t \in [0,1]$) in the pencil such that $L_0 = L_*$ and $L_1 = L$. The continuity of L_t allows us to associate a branch of $\mathcal{C}$ at L to each point of $\mathcal{C} \cap L_*$. The combinatorics of $\mathcal{C}$ at L defines a partition on the set of such branches and hence induces a partition $\mathscr{P}^*$ on $\mathcal{C} \cap L_*$ (which turns out to be independent of the chosen path L_t). Ordering the points of $\mathcal{C} \cap L_*$ induces a partition $\mathscr{P}$ in $\{-1, \ldots, -d\}$. Let $\Sigma_{\mathscr{P}}$ be the subgroup of Σ_d preserving the partition and let $\mathbb{B}_{\mathscr{P}}$ be the preimage of $\Sigma_{\mathscr{P}}$ in $\mathbb{B}_d$. By restriction to $\mathbb{B}_{\mathscr{P}}$ one can define the $\mathscr{P}$-braid monodromy of $(\mathcal{C}, L, P)$. The same proof of Theorem 1.32 can be applied to this particular scenario to obtain the following.

Theorem 1.42. *The statement of Theorem* 1.32 *also holds if we replace braid monodromy by $\mathscr{P}$-braid monodromy, where $\mathscr{P}$ has the same combinatorial meaning at infinity for both triples.*

Now we can show that $\left(\mathbb{P}^2, \mathcal{C}^{\varphi}_{\sqrt{5}} \cup L\right)$ and $\left(\mathbb{P}^2, \mathcal{C}^{\varphi}_{-\sqrt{5}} \cup L\right)$ form an arithmetic-Zariski pair.

Proposition 1.43. *Let us consider the curves $\mathcal{C}_{\sqrt{5}}$ and $\mathcal{C}_{-\sqrt{5}}$ in Example* 1.39. *There is no homeomorphism between the pairs $(\mathbb{P}^2, \mathcal{C}^{\varphi}_{\sqrt{5}} \cup L)$ and $(\mathbb{P}^2, \mathcal{C}^{\varphi}_{-\sqrt{5}} \cup L)$, but their complements are homeomorphic, that is $\mathbb{P}^2 \setminus \left(\mathcal{C}^{\varphi}_{\sqrt{5}} \cup L\right) \cong \mathbb{P}^2 \setminus \left(\mathcal{C}^{\varphi}_{-\sqrt{5}} \cup L\right)$.*

Proof. Let us prove the last statement first. Note that the homeomorphism between $\mathbb{P}^2 \setminus \left(\mathcal{C}_{\sqrt{5}} \cup L\right)$ and $\mathbb{P}^2 \setminus \left(\mathcal{C}_{-\sqrt{5}} \cup L\right)$ preserves the pencil of lines through P, therefore, it induces a homeomorphism between $\mathbb{P}^2 \setminus \left(\mathcal{C}^{\varphi}_{\sqrt{5}} \cup L\right)$ and $\mathbb{P}^2 \setminus \left(\mathcal{C}^{\varphi}_{-\sqrt{5}} \cup L\right)$.

Let us prove now that the pairs are not homeomorphic. In order to do so we will consider the partitions of Remark 1.41. We order the points of $\mathcal{C} \cap L_*$ as in Figure 6. One has the following situation:

for $\mathcal{C}_{\sqrt{5}}$: $(1,2,3) \leftrightarrow (B_1, B_2, B_3)$.
for $\mathcal{C}_{-\sqrt{5}}$: $(1,2,3) \leftrightarrow (B_1, B_3, B_2)$.

Hence, the group $\mathbb{B}_{\mathscr{P}}$ is simply the pure braid group. Let us denote by $n_{\sqrt{5}}$ the braid monodromy in Table 1. It defines a $\mathscr{P}$-braid monodromy for $\mathcal{C}_{\sqrt{5}}$. In order to have a representative of the $\mathscr{P}$-braid monodromy for $\mathcal{C}_{-\sqrt{5}}$ we have to permute the second and third strings, for instance $n_{-\sqrt{5}} := \sigma_2 * n_{\sqrt{5}}$.

Let H_s, $s^2 = 5$, be the monodromy groups in $\mathbb{B}_3$. If these curves have the same $\mathscr{P}$-braid monodromy, then $H_{\sqrt{5}}$ and $H_{-\sqrt{5}}$ are conjugated by an element in $\mathbb{B}_{\mathscr{P}}$. Using GAP4 [49], it is easily seen that this is the case and that $H_{\sqrt{5}} \neq H_{-\sqrt{5}}$.

Let c_s be the pseudo-Coxeter braid of n_s, $s^2 = 5$, (see Definition 1.9*(iii)*). Note that

$$c := c_{\sqrt{5}} = c_{-\sqrt{5}} = (\sigma_1 * \sigma_2^3)(\sigma_1^2 * (\sigma_1\sigma_2)^4)\sigma_1^5 = \sigma_1\sigma_2^3\sigma_1\Delta_3^2\sigma_1\sigma_2\sigma_1^3 =$$

$$\Delta_3^2\sigma_1\sigma_2^3\sigma_1^2\sigma_2\sigma_1^3 = \Delta_3^2\sigma_1\sigma_2^2\sigma_1^{-1}\Delta_3^2\sigma_1^2 = \Delta_3^4\sigma_1\sigma_2^2\sigma_1,$$

where $\Delta_3^2 = (\sigma_1\sigma_2)^3 = (\sigma_1\sigma_2\sigma_1)^2$ is the generator of the center of $\mathbb{B}_3$ (see Remark 1.13(b)).

If $n_{-\sqrt{5}}$ and $n_{\sqrt{5}}$ are $\mathscr{P}$-equivalent, there exists a pure braid τ such that $\tau * H_{\sqrt{5}} = H_{-\sqrt{5}}$ and $[c, \tau] = 1$. It can be easily computed (for innstance, via the standard representation in the special linear group $\mathrm{SL}(2,\mathbb{Z})$) that the intersection of the pure braid group and the commutator of c is the subgroup generated by c and Δ_3^2. This group is contained in the normalizer of $H_{\sqrt{5}}$ and hence, such a τ cannot exist. Q.E.D.

Example 1.44 ([10])**.** This is the first example of arithmetic Zariski pairs of lines. It consists of two arrangements $\mathscr{M}_{\pm}$ of eleven lines having conjugate equations in with coefficients in $\mathbb{Q}(\sqrt{2})$. In particular, their fundamental groups have isomorphic profinite completions.

The real pictures of $\mathscr{M}_\pm$ are shown in Figure 9 (lines at infinity included). In order to prove that they provide an arithmetic Zariski pair, one can proceed by contradiction as follows.

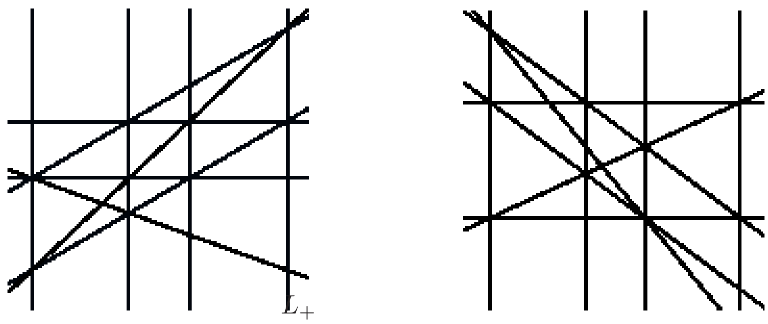

Fig. 9. Zariski pair of real line arrangements

Let us assume that a homeomorphism $\psi : (\mathbb{P}^2, \mathscr{M}_+) \to (\mathbb{P}^2, \mathscr{M}_-)$ exists. This homeomorphism must preserve the orientation of $\mathbb{P}^2$. Using standard intersection theory, one can assume that it either preserves the orientations of all the lines in $\mathscr{M}_\pm$ or it reverses them. Taking complex conjugation into account, one can assume that ψ preserves the orientations of the lines in $\mathscr{M}_\pm$. For combinatorial reasons $\psi(L_+) = L_-$ in Figure 9. Let us consider the arrangements $\mathscr{L}_\pm$ obtained by removing both the vertical lines and $L_\pm$ from $\mathscr{M}_\pm$. Since $\mathscr{M}_\pm$ has a unique point of multiplicity five, it is easy to see that $\psi(\mathscr{L}_+) = \mathscr{L}_-$. Thus, one can order these arrangements in such a way that the i^{th} line of $\mathscr{L}_+$ and $\mathscr{L}_-$ are conjugate in $\mathbb{Q}(\sqrt{2})$. The choice of the lines $L_\pm$ implies that ψ preserves the order. Moreover, the vertical lines can be ordered so as to fulfill the same property.

Let P be the point of intersection of the vertical lines and let L_∞ be the line at infinity. Then $(\mathscr{L}_\pm, L_\infty, P)$ are horizontal triple arrangements such that $\mathscr{L}_\pm^\varphi = \mathscr{M}_\pm \setminus \{L_\pm\}$. By Theorem 1.35, $(\mathscr{L}_\pm, L_\infty, P)$ have the same pure braid monodromy, but this contradicts [10, Theorem 4.19].

Note that no *ordered* homeomorphism exists from $\mathscr{L}_+^\varphi$ to $\mathscr{L}_-^\varphi$, but it is not hard to prove that there exists a projective transformation in $\mathrm{PGL}(3, \mathbb{C})$ sending $\mathscr{L}_+^\varphi$ to $\mathscr{L}_-^\varphi$ (thus not respecting orders).

Whether or not this is an example of a π_1-equivalent Zariski pair (that is, if the groups are actually isomorphic) or a complement-equivalent

Zariski pair (that is, if the complements are homeomorphic) remains an open problem.

Example 1.45. [88] Example 1.44 and Rybnikov's example are particularly interesting cases of Zariski pairs, since they come from line arrangements. What happens with conic arrangements? A nice example of a Zariski pair of conic arrangements has been provided by Namba and Tsuchihashi [88]. An elementary and exhaustive approach to it occupies §4.

§2. Alexander invariants

2.1. Alexander polynomials

Alexander polynomials have been largely used for knots and links in connection with cyclic branched coverings of their complement (see [50] for a survey on the matter). The first application of cyclic coverings to complements of plane curves was already proposed by Zariski (as mentioned in the Introduction), and later formalized by Libgober [71]. Since then the bibliography on the subject has become extensive. In what follows, we will give the basic definitions and present the main results on this invariant.

Consider $X_{\mathcal{C}} := \mathbb{P}^2 \setminus \mathcal{C}$, where $\mathcal{C} = \mathcal{C}_0 \cup \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_r$, $\mathcal{C}_i$ is an irreducible curve of degree d_i with equation $\{C_i = 0\}$, and $d_0 = 1$ (this last condition is purely technical to simplify notation). Note that, under these conditions,

$$H_1(X_{\mathcal{C}}; \mathbb{Z}) = \frac{\bigoplus_{i=0}^{r} \gamma_i \mathbb{Z}}{\langle \gamma_0 + d_1\gamma_1 + \cdots + d_r\gamma_r \rangle} \approx \mathbb{Z}^r,$$

where γ_i is the homology class of a meridian of $\mathcal{C}_i$. Let $\varepsilon : H_1(X_{\mathcal{C}}; \mathbb{Z}) \to \mathbb{Z}$ be an epimorphism. This epimorphism is defined by $(\varepsilon_1, \ldots, \varepsilon_r) \in \mathbb{Z}^r$, where $\varepsilon_i := \varepsilon(\gamma_i)$.

The kernel K_ε of the composition $G := \pi_1(X_{\mathcal{C}}) \xrightarrow{\text{ab}} H_1(X_{\mathcal{C}}; \mathbb{Z}) \xrightarrow{\varepsilon} \mathbb{Z}$ defines a covering of $X_{\mathcal{C}}$, say $\pi_\varepsilon : X_{\mathcal{C},\varepsilon} \to X_{\mathcal{C}}$, whose group of deck transformations is $G/K_\varepsilon = \mathbb{Z}$.

Remark 2.1. Given $n \in \mathbb{N}$ the composition of $\varepsilon \circ \text{ab}$ with the natural quotient $\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ produces an n-fold cyclic finite covering $\pi_{\varepsilon,n} : X^n_{\mathcal{C},\varepsilon} \to X_{\mathcal{C}}$ whose group of deck transformations is $\mathbb{Z}/n\mathbb{Z}$. Note that if n divides ε_i then $\pi_{\varepsilon,n}$ could be extended above $\mathcal{C}_i \setminus \bigcup_{j\neq i} \mathcal{C}_j$ as an unramified covering.

The group $G/K_\varepsilon = \mathbb{Z}$ acts on $K_\varepsilon/K'_\varepsilon = H_1(X_{\mathcal{C},\varepsilon};\mathbb{Z})$ by conjugation as follows

$$\begin{array}{rccc} * : & G/K_\varepsilon \times K_\varepsilon/K'_\varepsilon & \to & K_\varepsilon/K'_\varepsilon \\ & (\varepsilon(g), \bar{k}) & \mapsto & \varepsilon(g) * \bar{k} := \overline{g \cdot k \cdot g^{-1}}. \end{array}$$

Note that if $g' = gh_1$ ($h_1 \in K_\varepsilon$) and $k' = kh_2$ ($h_2 \in K'_\varepsilon$), then

$$\begin{aligned} \left(g' \cdot k' \cdot g'^{-1}\right)\left(g \cdot k^{-1} \cdot g^{-1}\right) &= \left((gh_1)\cdot(kh_2)\cdot(h_1^{-1}g^{-1})\right)\left(g \cdot k^{-1} \cdot g^{-1}\right) \\ &= g \cdot \left(h_1 k h_2 h_1^{-1} k^{-1}\right) \cdot g^{-1} \\ &= g \cdot \left((h_1 k)\cdot h_2 \cdot (h_1 k)^{-1}\ h_1 k h_1^{-1} k^{-1}\right) g^{-1} \\ &\in K'_\varepsilon. \end{aligned}$$

Hence "$*$" does not depend on the choice of g mod K_ε or k mod K'_ε.

This action endows $M^{\mathbb{Z}}_{\mathcal{C},\varepsilon} := H_1(X_{\mathcal{C},\varepsilon};\mathbb{Z})$ with a Λ_ε-module structure, where $\Lambda_\varepsilon := \mathbb{Z}[G/K_\varepsilon] \approx \mathbb{Z}[t^{\pm 1}]$. One can tensor such a module by a field $\mathbb{K} = \mathbb{Q}, \mathbb{C}, \mathbb{F}_p, \ldots$ to obtain a module $M^{\mathbb{K}}_{\mathcal{C},\varepsilon}$ over $\Lambda^{\mathbb{K}}_\varepsilon = \mathbb{K}[t^{\pm 1}]$. Since G is finitely presented, $M^{\mathbb{K}}_{\mathcal{C},\varepsilon}$ is finitely generated as a $\Lambda^{\mathbb{K}}_\varepsilon$-module (by as many 1-cells as generators of G). The rings $\Lambda^{\mathbb{K}}_\varepsilon$ are principal ideal domains and hence one can define $\Delta^{\mathbb{K}}_{\mathcal{C},\varepsilon}(t)$ as the *order* of $M^{\mathbb{K}}_{\mathcal{C},\varepsilon}$. We recall that, if R is a principal ideal domain, the *order* of an R-module M, is defined as

$$\Delta := \begin{cases} 0 & \text{if } M \text{ has a free summand,} \\ 1 & \text{if } M = 0, \\ \prod_{i=1}^m \lambda_i & \text{if } M \approx \frac{R}{(\lambda_1)} \oplus \cdots \oplus \frac{R}{(\lambda_m)}. \end{cases} \tag{12}$$

Such a polynomial can be assumed to be unique by adding the extra condition $\lambda_i(0) = 1$. This is known as the *Alexander polynomial* of $\mathcal{C}$ associated with ε. In general, if $\mathbb{K} = \mathbb{Q}$ or $\mathbb{C}$, then the reference to the field will be omitted.

The *classical Alexander polynomial* (denoted $\Delta_{\mathcal{C}}(t)$) corresponds to the special case when $\mathbb{K} = \mathbb{Q}$, $\mathcal{C}_0$ is transversal to $\mathcal{C}_i$ for any $i = 1, \ldots, r$, and ε is the epimorphism that sends any meridian γ_i around $\mathcal{C}_i$ to 1, except for $i = 0$, where $\varepsilon(\gamma_0) = -d$, where $d := \sum_{i=1}^r d_i$. We will refer to this morphism as the *trivial morphism*. If $\varepsilon(\gamma_i) \neq \pm 1$ for any $i = 0, 1, \ldots, r$ we will call ε a *non-coordinate* epimorphism. The *Oka polynomials* (denoted $\Delta_{\mathcal{C},\varepsilon}(t)$) correspond to $\mathbb{K} = \mathbb{Q}$, and a transversal $\mathcal{C}_0$ ([94]).

Remark 2.2.

(1) Note that $M^{\mathbb{Z}}_{\mathcal{C},\varepsilon}$ is not necessarily a torsion module in general. For example, if $\mathcal{C}$ is a union of $r+1$ lines passing through a common point and ε is the trivial morphism, then $\pi_1(X_{\mathcal{C}}) = \mathbb{Z} * \cdots * \mathbb{Z}$ a free product of rank r, and it is easy to see that $M^{\mathbb{K}}_{\mathcal{C},\varepsilon} = \left(\Lambda^{\mathbb{K}}_{\varepsilon}\right)^{r-1}$.

(2) Also note that $M^{\mathbb{Z}}_{\mathcal{C},\varepsilon}$ depends only on $G = \pi_1(X_{\mathcal{C}})$ and ε. Hence one can associate an Alexander polynomial $\Delta_{G,\varepsilon}(t)$ to any finitely presented group G and epimorphism $\varepsilon : G/G' \to \mathbb{Z}$. In fact, such a polynomial corresponds to the Alexander polynomial of the CW-complex X_G associated with any finite presentation of G, and $\varepsilon : H_1(X_G; \mathbb{Z}) \to \mathbb{Z}$.

(3) Assume that

$$\left(\Lambda^{\mathbb{K}}_{\varepsilon}\right)^m \xrightarrow{A} \left(\Lambda^{\mathbb{K}}_{\varepsilon}\right)^n \to M^{\mathbb{K}}_{G,\varepsilon}$$

is a free resolution of $M^{\mathbb{K}}_{G,\varepsilon}$, where A is an $n \times m$ matrix with coefficients in $\Lambda^{\mathbb{K}}_{\varepsilon}$. Then $\Delta_{G,\varepsilon}(t)$ can also be defined as 0 if $m < n$, or as the greatest common divisor of all the minors of maximal order of A if $n \leq m$. From (2) above, n can be considered as the number of generators in a presentation of G.

A very useful remark on Alexander polynomials is the following:

Lemma 2.3. [71, Proposition 2.1] *Let $G \overset{\psi}{\twoheadrightarrow} H$ be an epimorphism of finitely presented groups and consider $\varepsilon_H : H/H' \twoheadrightarrow \mathbb{Z}$ another epimorphism. Then $\Delta^{\mathbb{K}}_{H,\varepsilon_H}$ divides $\Delta^{\mathbb{K}}_{G,\varepsilon_G}$, where $\varepsilon_G = \varepsilon_H \circ \psi_1$ and $\psi_1 : G/G' \to H/H'$ is induced by ψ.*

Proof. A presentation of H can be given from one of G just by adding a finite number of relations. Therefore from Remark 2.2(3), a presentation matrix for $M^{\mathbb{K}}_{H,\varepsilon_H}$ is the result of adding a finite number of columns to the presentation matrix of $M^{\mathbb{K}}_{G,\varepsilon_G}$. Therefore the ideal generated by the minors of maximal order of $M^{\mathbb{K}}_{G,\varepsilon_G}$ is contained in the one of $M^{\mathbb{K}}_{H,\varepsilon_H}$. Q.E.D.

This situation appears in a natural way when an equisingular family of curves $\{C_t\}_{t\in(0,\delta]}$ degenerates into a reduced curve $\mathcal{C}_0$.

Proposition 2.4. *Under the above conditions there is an epimorphism of fundamental groups*

$$\pi_1(X_{\mathcal{C}_0}) \overset{j}{\twoheadrightarrow} \pi_1(X_{\mathcal{C}_\delta}).$$

Hence $\Delta^{\mathbb{K}}_{\mathcal{C}_\delta,\varepsilon_1}$ divides $\Delta^{\mathbb{K}}_{\mathcal{C}_0,\varepsilon_2}$, where $\varepsilon_2 = \varepsilon_1 \circ j_1$ as in Lemma 2.3.

Proof. A proof of the first part can be found in [41, Corollary §3 (3.2)]. The second part is an immediate consequence of Lemma 2.3. Q.E.D.

Example 2.5.

(1) Consider a family of $r+1$ lines $\mathcal{C}_t := \ell_{t,0} \cup \cdots \cup \ell_{t,r}$, $t \in (0,1]$ in general position degenerating into $r+1$ lines $\mathcal{C}_0 := \ell_0 \cup \cdots \cup \ell_r$ passing through a common point. If ε is the trivial morphism $\varepsilon(\gamma_i) = 1$, $i = 1, \ldots, r$ and $\varepsilon(\gamma_0) = -r$, then one has the following

$$\begin{array}{lll} M^{\mathbb{K}}_{\mathcal{C}_1,\varepsilon} = \left(\frac{\Lambda^{\mathbb{K}}_{\varepsilon}}{(t-1)}\right)^{r-1} & \Rightarrow & \Delta^{\mathbb{K}}_{\mathcal{C}_1,\varepsilon}(t) = (t-1)^{r-1} \\ M^{\mathbb{K}}_{\mathcal{C}_0,\varepsilon} = \left(\Lambda^{\mathbb{K}}_{\varepsilon}\right)^{r-1} & \Rightarrow & \Delta^{\mathbb{K}}_{\mathcal{C}_0,\varepsilon}(t) = 0. \end{array} \tag{13}$$

(2) Consider the three-cuspidal quartic $\mathcal{C}_1$ presented in (Z6) and a generic line $\mathcal{C}_0$. In order to give a presentation for the fundamental group of $\mathcal{C} := \mathcal{C}_0 \cup \mathcal{C}_1$ one can simply apply Proposition 1.16 to the presentation (1) and obtain

$$\langle a, b \mid aba = bab, \quad [a, a^2b^2] = [b, a^2b^2] = 1\rangle. \tag{14}$$

Note that there is basically only one possible morphism ε, the abelianization morphism, which we will omit in the notation. An easy computation produces

$$M^{\mathbb{K}}_{\mathcal{C}} = \frac{\mathbb{K}[t^{\pm 1}]}{(3, t+1)}$$

and hence

$$\Delta^{\mathbb{K}}_{\mathcal{C}}(t) = \begin{cases} t+1 & \text{if } \operatorname{char}(\mathbb{K}) = 3 \\ 1 & \text{otherwise.} \end{cases}$$

Since the three-cuspidal quartic is dual to a nodal cubic, we know it has a bitangent, say ℓ_0. The fundamental group of $\mathcal{C}' := \ell_0 \cup \mathcal{C}_1$ has the following presentation (see [97, Example 4.5(3)])

$$\langle a_1, a_2, a_3, a_4 \mid a_i a_{i+1} a_i = a_{i+1} a_i a_{i+1} (i = 1, .., 3), \quad a_2 a_4 = a_1 a_2\rangle, \tag{15}$$

which produces

$$M^{\mathbb{K}}_{\mathcal{C}'} = \frac{\mathbb{K}[t^{\pm 1}]}{(t^2 - t + 1)} \oplus \frac{\mathbb{K}[t^{\pm 1}]}{(t^2 - t + 1)}$$

and hence

$$\Delta^{\mathbb{K}}_{\mathcal{C}'}(t) = (t^2 - t + 1)^2.$$

Note that if $\operatorname{char}(\mathbb{K}) = 3$, then $(t^2 - t + 1) = (t+1)^2$, and hence $\Delta^{\mathbb{K}}_{\mathcal{C}'}(t) = (t+1)^4$.

All computations in Example 2.5 above have been performed directly from a presentation of the fundamental group. We refer the reader to Section §2.5 for more details on this.

Remark 2.6. Very often the Alexander polynomial is defined as the torsion of $H_1(X_{\mathcal{C},\varepsilon}, \pi_\varepsilon^{-1}(p); \mathbb{Q})$ for some $p \in X_{\mathcal{C}}$. This definition comes handy for computational purposes, since it can be obtained as the determinantal variety of corank 1 of the Fox derivative matrix associated with the group G, as in the case of knots ([35]).

The geometrical interpretation of the classical Alexander polynomial is given as follows (see [101]). The polynomial $C_1 \cdot \ldots \cdot C_r$ defines a non-isolated singularity at the origin of $\mathbb{C}^3$. The monodromy of the Milnor fiber defines an automorphism on the H_1 and the classical Alexander polynomial is the characteristic polynomial of the monodromy of the Milnor fiber.

Theorem 2.7. [94, Theorem 43] *The Alexander polynomial of $\mathcal{C}$ with respect to the epimorphism $\varepsilon : H_1(X_{\mathcal{C}}) \twoheadrightarrow \mathbb{Z}$ ($\varepsilon_i \geq 0$, $i = 1, \ldots, r$) is equal to the characteristic polynomial of the monodromy $h_* : H_1(F) \to H_1(F)$ where F is the Milnor fiber of the polynomial $C_1^{\varepsilon_1} \cdot \ldots \cdot C_r^{\varepsilon_r}$.*

Since the monodromy has a finite order, this implies the following.

Corollary 2.8. *All the zeroes of the Alexander polynomial $\Delta_{\mathcal{C},\varepsilon}(t)$ of a curve $\mathcal{C}$ with respect to an epimorphism ε are roots of unity.*

Alexander polynomials depend on the local type of singularities of $\mathcal{C}$. To describe this dependency we will consider $L_1, \ldots, L_s$ the local links of the singularities of the affine part $\mathcal{C}^{\text{aff}} := \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_r$ and L_∞ the link at infinity, that is, the intersection of $\mathcal{C}^{\text{aff}}$ with the boundary of a tubular neighborhood of the line at infinity $\mathcal{C}_0$. The inclusion $\mathbb{S}^3 \setminus L_k \hookrightarrow X_{\mathcal{C}}$ induces a map $\pi_1(\mathbb{S}^3 \setminus L_k) \to \pi_1(X_{\mathcal{C}})$. Therefore ε also induces epimorphisms $\pi_1(\mathbb{S}^3 \setminus L_k) \twoheadrightarrow \mathbb{Z}$. The Alexander polynomials associated with such maps will be called local Alexander polynomials and denoted by $\Delta_{L_k,\varepsilon}$ for simplicity.

This dependency can be described for classical Alexander polynomials.

Theorem 2.9 ([71]). *The classical Alexander polynomial of $\mathcal{C}$ divides both the product of the local Alexander polynomials $\prod_{k=1}^s \Delta_{L_k}(t)$ and $\Delta_{L_\infty}(t)$.*

This dependency also has an expression for general Alexander polynomials.

Theorem 2.10. *Under the above conditions*

$$\Big(\prod_{i=1}^{r}(1-t^{\varepsilon_i})^{s_i-\chi(\mathcal{C}_i)}\Big)\prod_{k=1,\dots,s,\infty}\Delta_{L_k,\varepsilon}(t)=\Delta^2_{\mathcal{C},\varepsilon}(t)\cdot\det\varphi^t(\mathcal{C}),$$

where $s_i := \#\operatorname{Sing}(\mathcal{C}^{\text{aff}})\cap\mathcal{C}_i$*,* χ *is Euler characteristic and* $\varphi^t(\mathcal{C})$ *is an intersection form on* $H_2(X_{\mathcal{C},\varepsilon},\mathbb{Q}[t^{\pm1}])$ *with twisted coefficients.*

Proof. It is an immediate consequence of [31, Theorem 5.6] and the fact that $\Delta^{\mathbb{Q}}_{\mathcal{C},\varepsilon}(t^{-1})=\Delta^{\mathbb{Q}}_{\mathcal{C},\varepsilon}(t)$ (by Corollary 2.8). Q.E.D.

The fact that Alexander polynomials are not combinatorial invariants was already known (with a different language) by Zariski as mentioned in the Introduction with the first example of a classical Zariski pair.

A topological interpretation of classical Alexander polynomials can be given as follows: an n-th root of unity ($n>1$) is a root of the classical Alexander polynomial of a curve $\mathcal{C}$ if and only if the cyclic covering of the complement $X_{\mathcal{C}}$ ramified along each irreducible component of $\mathcal{C}^{\text{aff}}$ with order n has a bigger first Betti number than $h_1(X_{\mathcal{C}};\mathbb{C})$ ([71, Corollary 3.2]). Moreover the difference between the two Betti numbers is exactly the sum of the multiplicities of such roots. The reason for this is that the Alexander invariant $M_{\mathcal{C}}$ is *semisimple* in this case, that is, it is a direct sum of modules with no proper submodules (also called *simple modules*).

Analogously, for general Alexander polynomials one has the following:

Theorem 2.11. *Let* $\mathcal{C}$ *be a curve and* $\varepsilon: H_1(X_{\mathcal{C}};\mathbb{Z})\to\mathbb{Z}$ *an epimorphism. If an* n*-th primitive root of unity (*$n>1$*) is a root of the Alexander polynomial* $\Delta_{\mathcal{C},\varepsilon}(t)$ *then the covering* $X^n_{\mathcal{C},\varepsilon}$ *has a bigger first Betti number than* $h_1(X_{\mathcal{C}};\mathbb{C})$*.*

Moreover,

$$h_1(X^n_{\mathcal{C},\varepsilon};\mathbb{C})=h_1(X_{\mathcal{C}};\mathbb{C})+\sum_{i=1}^{m}\alpha_i^n, \tag{16}$$

where α_i^n *is the number of common roots between* $\frac{t^n-1}{t-1}$ *and* $\lambda_i(t)$ *from* (12).

Remark 2.12.

(1) A similar formula for homology with coefficients in other fields exists [83, Theorem 4.6]. The field needs to contain all the n-th roots of unity.

(2) Note that in general one cannot just count multiplicities in order to compute the first Betti number of cyclic (or Abelian) coverings since the Alexander invariant need not be semisimple. For instance, consider the Example 2.5(2) of the three cuspidal quartic and the bitangent line $\mathcal{C}'$. According to Matei-Suciu [83] if ε is the trivial morphism and $n = 2$, then the formula (16) is still valid

$$h_1(X^2_{\mathcal{C}',\varepsilon};\mathbb{F}_3) = h_1(X_{\mathcal{C}'};\mathbb{F}_3) + \sum_{i=1}^{2} \alpha_i^2,$$

but in this case $h_1(X^2_{\mathcal{C}',\varepsilon};\mathbb{F}_3) - h_1(X_{\mathcal{C}'};\mathbb{F}_3) = 2$ even though $t = -1$ has multiplicity 4 in $\Delta^{\mathbb{F}_3}_{\mathcal{C}'}(t)$.

2.2. Characteristic varieties

Characteristic varieties were introduced by Hillman [55] for links, then systematically studied by Arapura [1] for Kähler manifolds, and first applied to algebraic curves by Libgober [74]. They can be defined analogously to Alexander polynomials as follows.

Let $\mathcal{C} := \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_r$ similarly as at the beginning of this section, except that we are not asking any component to be a line. Let $\tau := \gcd(d_1, \ldots, d_r)$. Then

$$H_1(X_{\mathcal{C}};\mathbb{Z}) = \frac{\bigoplus_{i=1}^{r} \gamma_i \mathbb{Z}}{\langle d_1\gamma_1 + \cdots + d_r\gamma_r \rangle} \approx \mathbb{Z}^{r-1} \oplus \frac{\mathbb{Z}}{\tau\mathbb{Z}}, \tag{17}$$

where γ_i is the homology class of a meridian of $\mathcal{C}_i$.

We can study the projective plane curve $\mathcal{C}$ as follows; let $\mathrm{ab} : G := \pi_1(X_{\mathcal{C}}) \to H_1(X_{\mathcal{C}};\mathbb{Z})$ be the abelianization epimorphism. The kernel G' of ab defines the universal Abelian covering of $X_{\mathcal{C}}$, say $X_{\mathcal{C},\mathrm{ab}} \to X_{\mathcal{C}}$, whose group of deck transformations is $G/G' = H_1(X_{\mathcal{C}};\mathbb{Z})$. Such a group acts on $G'/G'' = H_1(X_{\mathcal{C},\mathrm{ab}};\mathbb{Z})$ by conjugation as before endowing $M^{\mathbb{Z}}_{\mathcal{C},\mathrm{ab}} := H_1(X_{\mathcal{C},\mathrm{ab}};\mathbb{Z})$ with a $\Lambda_{\mathcal{C}}$-module structure, where $\Lambda_{\mathcal{C}} := \mathbb{Z}[G/G'] \approx \mathbb{Z}[t_1^{\pm 1}, \ldots, t_r^{\pm 1}]/(t_1^{d_1} \cdot \ldots \cdot t_r^{d_r} - 1)$.

One can tensor $M^{\mathbb{Z}}_{\mathcal{C},\mathrm{ab}}$ by a field $\mathbb{K} = \mathbb{Q}, \mathbb{C}, \mathbb{F}_p, \ldots$ to obtain a module $M^{\mathbb{K}}_{\mathcal{C},\mathrm{ab}}$ over $\Lambda^{\mathbb{K}}_{\mathcal{C}} = \mathbb{K}[t_1^{\pm 1}, \ldots, t_r^{\pm 1}]/(t_1^{d_1} \cdot \ldots \cdot t_r^{d_r} - 1)$ (in general we only ask $\Lambda^{\mathbb{K}}_{\mathcal{C}}$ to be integrally closed and Noetherian). Since G is finitely presented,

$M^{\mathbb{K}}_{\mathcal{C},\mathrm{ab}}$ is again a finitely generated $\Lambda^{\mathbb{K}}_{\mathcal{C}}$-module (by as many 1-cells as generators of G). If $r \geq 2$, then $\Lambda^{\mathbb{K}}_{\mathcal{C}}$ is not a Principal Ideal Domain and hence one has to study the module invariants of $M^{\mathbb{K}}_{\mathcal{C},\mathrm{ab}}$, that is, the Fitting ideals of $M^{\mathbb{K}}_{\mathcal{C},\mathrm{ab}}$.

Let us briefly recall the notion of Fitting ideals. Let R be a commutative Noetherian ring with unity and M a finitely generated R-module. One has a finite free presentation for M, say $\phi : R^m \to R^n$, where $M = \operatorname{coker} \phi$. The homomorphism ϕ has an associated $(n \times m)$ matrix A_ϕ with coefficients in R such that $\phi(x) = A_\phi x^t$.

Definition 2.13. The k-th *Fitting ideal* $F_k(M)$ of M is defined as the ideal generated by

$$\begin{cases} 0 & \text{if } k \leq \max\{0, n-m\} \\ 1 & \text{if } k > n \\ \text{minors of } A_\phi \text{ of order } (n-k+1) & \text{otherwise.} \end{cases}$$

It will be denoted F_k if no ambiguity seems likely to arise.

Definition 2.14. [74] Under the above conditions the *k-th characteristic variety* of M can be defined as

$$\operatorname{Char}_k(M) := \operatorname{Supp}_R\left(R/F_k(M)\right) \subset \operatorname{Spec} R.$$

The subindex k is also known as the *depth* of a characteristic variety.

Similarly, the *k-th projective characteristic variety* $\operatorname{Char}^{\mathbb{K}}_{k,\mathbb{P}}(\mathcal{C})$ of a curve $\mathcal{C}$ is the k-th characteristic variety of $M^{\mathbb{K}}_{\mathcal{C},\mathrm{ab}}$ as a $\Lambda^{\mathbb{K}}_{\mathcal{C}}$-module.

If L is a line not contained in $\mathcal{C}$ then $\Lambda^{\mathbb{K}}_{L\cup\mathcal{C}}$ is naturally isomorphic to $\mathbb{K}[t_1^{\pm1}, \ldots, t_r^{\pm1}]$. Moreover, if $L \pitchfork \mathcal{C}$ then the $\Lambda^{\mathbb{K}}_{L\cup\mathcal{C}}$-module $M^{\mathbb{K}}_{L\cup\mathcal{C},\mathrm{ab}}$ does not depend on L by Proposition 1.16 and $\operatorname{Char}^{\mathbb{K}}_{k,\mathbb{P}}(L\cup\mathcal{C})$ is called the *k-th affine characteristic variety* and denoted simply by $\operatorname{Char}^{\mathbb{K}}_k(\mathcal{C})$.

One can also define the k-th characteristic variety $\operatorname{Char}^{\mathbb{K}}_k(G)$ of a finitely presented group G as the k-th characteristic variety of the module $M^{\mathbb{K}}_G$ obtained by considering the CW-complex associated with a given presentation (of course, such invariant is independent of the finite presentation of G).

In the particular case when $\mathbb{K} = \mathbb{C}$ and $M = M^{\mathbb{C}}_{\mathcal{C},\mathrm{ab}}$ one has:

- $\operatorname{Spec} \Lambda_{L\cup\mathcal{C}} = \mathbb{T}^r = (\mathbb{C}^*)^r$, for the affine case, and
- $\operatorname{Spec} \Lambda_{\mathcal{C}} = \mathbb{T}_{\mathcal{C}} = \{\omega^i\}_{i=0}^{\tau-1} \times (\mathbb{C}^*)^{r-1} = V(t_1^{d_1} \cdot \ldots \cdot t_r^{d_r} - 1) \subset \mathbb{T}^r$, where ω is a τ-th primitive root of unity for the projective case.

In the case of a finitely presented group G where $G/G' = \mathbb{Z}^r \oplus \mathbb{Z}/\tau_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/\tau_s\mathbb{Z}$ we obtain

$$\operatorname{Spec} \Lambda_G = \mathbb{T}_G = \{(\omega_1^{i_1}, \ldots, \omega_s^{i_s}) \mid i_k = 0, \ldots, \tau_k - 1,\ k = 1, \ldots, s\} \times (\mathbb{C}^*)^r,$$

where $\Lambda_G = \mathbb{C}[G/G']$ and ω_i is a τ_i-th primitive root of unity.

One might want to keep the non-reduced structure of the Fitting ideal. In that case we define the *projective (resp. affine) k-th Fitting ideal* of the curve $\mathcal{C}$ over the field $\mathbb{K}$ and denote it as $F_{k,\mathbb{P}}^{\mathbb{K}}(\mathcal{C})$ (resp. $F_k^{\mathbb{K}}(\mathcal{C})$).

Remarks 2.15.

(1) Note that any isomorphism between two finitely presented groups G_1 and G_2 produces an automorphism of the ambient torus $\mathbb{T}_{G_1} = \mathbb{T}_{G_2}$ such that $\operatorname{Char}_k(G_1)$ and $\operatorname{Char}_k(G_2)$ are isomorphic.

Note however, that in some particular cases, like fundamental groups of link complements (which will not be considered here) or curve complements, the ambient torus has a natural system of coordinates. For instance in the latter case, a natural system of coordinates for $\mathbb{T}_{\mathcal{C}}$ is given by the preferred basis of $H_1(X_{\mathcal{C}}; \mathbb{Z})$ described in (17). In that respect, the characteristic varieties of $\mathcal{C}$ as subspaces of the torus $\mathbb{T}_{\mathcal{C}}$ are not directly an invariant of the group G, rather they are invariant of the embedding of $\mathcal{C}$ in $\mathbb{P}^2$, which is the group G with some peripheral information about the homology classes of the meridians of the irreducible components of $\mathcal{C}$.

(2) In the particular case of plane algebraic curves, due to the Hodge decomposition of $H_1(X_{\varepsilon,\mathcal{C}}; \mathbb{C})$ for appropriate branched coverings, the ring homomorphism $\Lambda_{\mathcal{C}}^{\mathbb{Z}} \to \Lambda_{\mathcal{C}}^{\mathbb{Z}}$ given by $t_i \to t_i^{-1}$ induces an automorphism of the components of $\operatorname{Char}_k(\mathcal{C})$ containing ε (see [75, Theorem 3.1.c]).

Proposition 2.16. *If $(\mathbb{P}^2, \mathcal{C})$ and $(\mathbb{P}^2, \mathcal{D})$ are homeomorphic, then $\Lambda^{\mathbb{C}} := \Lambda_{\mathcal{C}}^{\mathbb{C}} = \Lambda_{\mathcal{D}}^{\mathbb{C}}$ in a natural way, and $M_{\mathcal{C},\mathrm{ab}}^{\mathbb{C}} \approx M_{\mathcal{D},\mathrm{ab}}^{\mathbb{C}}$ are isomorphic as $\Lambda^{\mathbb{C}}$-modules. In particular $\operatorname{Char}_k^{\mathbb{C}}(\mathcal{C}) = \operatorname{Char}_k^{\mathbb{C}}(\mathcal{D})$.*

Proof. Let us denote by $f : (\mathbb{P}^2, \mathcal{C}) \to (\mathbb{P}^2, \mathcal{D})$ the homeomorphism of pairs. Note that the image by f of any disk transversal to a component $\mathcal{C}_i$ of $\mathcal{C}$ will be sent to a disk transversal to a component, say $\mathcal{D}_i$, of $\mathcal{D}$. Since irreducible components intersect pairwise and always positively, it is possible to prove that f must either respect or reverse orientations on all the irreducible components of the curves. Therefore meridians will be sent to meridians (up to sign) and the induced homomorphism of groups $f_*: H_1(X_{\mathcal{C}};\mathbb{Z}) \to H_1(X_{\mathcal{D}};\mathbb{Z})$ has the expected property $f_*(\gamma_{\mathcal{C}_i}) = \delta\gamma_{\mathcal{D}_i}$, $i = 1, \ldots, r$ ($\delta = \pm 1$). By Remark 2.22 below, one can assume that $\delta = 1$. Finally note that $d_i := \deg \mathcal{C}_i = \deg \mathcal{D}_i$ is preserved, since it is a topological invariant and hence the first part follows and $\Lambda^{\mathbb{C}} = \mathbb{C}[t_1^{\pm 1}, \ldots, t_r^{\pm 1}]/(t_1^{d_1} \cdot \ldots \cdot t_r^{d_r} - 1)$. The fundamental groups of $X_{\mathcal{C}}$ and $X_{\mathcal{D}}$ are isomorphic and the action of each t_i on both $M^{\mathbb{C}}_{\mathcal{C},\mathrm{ab}} = \Lambda^{\mathbb{C}} \otimes_{\Lambda} \pi_1(X_{\mathcal{C}})'/\pi_1(X_{\mathcal{C}})''$ and $M^{\mathbb{C}}_{\mathcal{D},\mathrm{ab}} = \Lambda^{\mathbb{C}} \otimes_{\Lambda} \pi_1(X_{\mathcal{D}})'/\pi_1(X_{\mathcal{D}})''$ is preserved by f. Therefore the second part follows. Q.E.D.

Remark 2.17.

(1) Note that the isomorphism of $\Lambda^{\mathbb{K}}$-modules exhibited in Proposition 2.16 is of a very special kind, since it comes from an isomorphism of fundamental groups inducing the identity on the abelianization. Since the Alexander invariant G/G'' of a group G can be seen as an extension of G/G' by G'/G'', this type of isomorphisms of $\Lambda^{\mathbb{K}}$-modules will be called *extension isomorphisms*.

(2) Closely related to Remark 2.15.(1), if one wants to say something about whether or not the fundamental groups of two curves are isomorphic, verifying that $M^{\mathbb{K}}_{\mathcal{C},\mathrm{ab}} \not\approx M^{\mathbb{K}}_{\mathcal{D},\mathrm{ab}}$ as modules over an abstract Λ or $\mathrm{Char}_k(\mathcal{C}) \neq \mathrm{Char}_k(\mathcal{D})$ is not enough. Instead, invariants of the isomorphism class of $\mathrm{Char}_k(\mathcal{C})$ and $\mathrm{Char}_k(\mathcal{D})$ should be used such as their total number of irreducible components of a certain dimension (see Section §4), or their combinatorial structure (see §2.3).

Example 2.18.

(1) Let $G = \mathbb{Z}^q * \mathbb{Z}/p_1\mathbb{Z} * \cdots * \mathbb{Z}/p_s\mathbb{Z}$. According to Proposition 2.39

$$M^{\mathbb{K}}_G = \frac{\bigoplus_{1 \leq i < j \leq r} \Lambda^{\mathbb{K}}_G x_{i,j}}{\mathcal{T} + \mathcal{J}},$$

where

- $\Lambda^{\mathbb{K}}_G = \mathbb{K}[t_1^{\pm 1}, \ldots, t_{q+s}^{\pm 1}]/(t_{q+1}^{p_1} - 1, \ldots, t_{q+s}^{p_s} - 1)$,
- $\mathcal{T}$ is the submodule generated by $\frac{t_{q+i}^{p_i} - 1}{t_{q+i} - 1}$, $i = 1, \ldots, s$, and
- $\mathcal{J}$ is the Jacobian submodule.

In this situation it is easy to see that

$$F^{\mathbb{K}}_1(G) = \left(\frac{t_{q+1}^{p_1} - 1}{t_{q+1} - 1}, \ldots, \frac{t_{q+s}^{p_s} - 1}{t_{q+s} - 1} \right).$$

And hence,

$$\mathrm{Char}_1(G) = (\mathbb{C}^*)^q \times \{(\omega_1, \dots, \omega_s) \mid \omega_i^{p_i} = 1, \omega_i \neq 1, i = 1, \dots, s\}.$$

(2) Consider the Hopf link of d components. For convenience we denote its components as $L_{i,j(i)}$, $j(i) = 1, \dots, d_i$, $i = 1, \dots, r$, where $\sum_{i=1}^r d_i = d$. A natural presentation of its fundamental group G is given by meridians $\gamma_{i,j(i)}$ for each component and an extra generator σ, and whose relations are $[\sigma, \gamma_{i,j(i)}] = 1$ and

$$\sigma \prod_{i=1}^r \prod_{j(i)=1}^{d_i} \gamma_{i,j(i)} = 1,$$

i.e, G is the direct product of $\mathbb{Z}$ and the free group in $d-1$ generators. Then, $\Lambda_G^{\mathbb{K}}$ is a ring of Laurent polynomials in the variables s and $t_{i,j(i)}$, $j(i) = 1, \dots, d_i$, $i = 1, \dots, r$, where the product of all variables equals 1. Applying Proposition 2.39 one obtains

$$M_G^{\mathbb{K}} = \frac{\bigoplus_{1 \leq i < j < d} \Lambda_G^{\mathbb{K}} x_{i,j}}{\mathcal{W}_d + \mathcal{J}},$$

where $\mathcal{W}_d$ is the submodule generated by $s-1$, and $\mathcal{J}$ is the Jacobian submodule. Hence

$$F_1^{\mathbb{K}}(G) = (s-1) = (\prod_{i=1}^r \prod_{j(i)=1}^{d_i} t_{i,j(i)} - 1).$$

Remark 2.19. Note that if $0 \to M' \to M \to M'' \to 0$ is an exact sequence of R-modules, then $\mathrm{Char}_1^{\mathbb{K}}(M) = \mathrm{Char}_1^{\mathbb{K}}(M') \cup \mathrm{Char}_1^{\mathbb{K}}(M'')$. Let L be a line transversal to $\mathcal{C}$, and consider $T(L)$ a tubular neighborhood of L. In this situation there is a surjection

$$G_\infty := \pi_1(\partial T(L) \setminus \mathcal{C}) \twoheadrightarrow G_L := \pi_1(\mathbb{P}^2 \setminus (L \cup \mathcal{C})).$$

Following the notations of Example 2.18(2), we assume that the image of $\gamma_{i,j(i)}$ by this surjection is a meridian of $\mathcal{C}_i$.

Since $\Lambda_{G_L}^{\mathbb{K}} = \mathbb{K}[t_1^{\pm 1}, \dots, t_r^{\pm 1}]$, the above morphism induces a ring morphism $\Lambda_{G_\infty}^{\mathbb{K}} \twoheadrightarrow \Lambda_{G_L}^{\mathbb{K}}$ given by $t_{i,j(i)} \mapsto t_i$ and $s \mapsto (t_1^{d_1} \cdot \ldots \cdot t_r^{d_r})^{-1}$. Therefore, the surjection $M_{G_\infty, G_L} := M_{G_\infty} \otimes \Lambda_{G_L}^{\mathbb{K}} \to M_{L \cup \mathcal{C}, \mathrm{ab}}$ induces an inclusion

$$\mathrm{Char}_1^{\mathbb{K}}(\mathcal{C}) = \mathrm{Char}_1^{\mathbb{K}}(G_L) \subset \mathrm{Char}_1^{\mathbb{K}}(M_{G_\infty} \otimes \Lambda_{G_L}^{\mathbb{K}}) = V(t_1^{d_1} \cdot \ldots \cdot t_r^{d_r} - 1),$$

where the last equality can be computed from Example 2.18(2) above.

In other words, even though the affine $\mathrm{Char}_1(\mathcal{C})$ seems to sit in the bigger torus $\mathbb{T}^r$ than the projective $\mathrm{Char}_{1,\mathbb{P}}(\mathcal{C})$, the fact is that both are contained in $\mathbb{T}_{\mathcal{C}}$. Moreover, they coincide as subtori of $\mathbb{T}_{\mathcal{C}}$ ([75, Proposition 1.2.3]). In what follows we will use either one indistinctly.

D. Arapura, in [1, Theorem 1.6] gives the following description of the structure of first characteristic varieties for certain Kähler varieties. We adapt the original statement for our particular case of curve complements. In order to do that we need the following concept.

Definition 2.20. A (fixed component free) pencil of curves in $\mathbb{P}^2$ is said to *completely contain a curve* $\mathcal{C}$, if the induced morphism $\hat{f}$: $\widehat{\mathbb{P}}^2 \to \mathbb{P}^1$ (after blowing-up the corresponding base points) satisfies that $\widehat{\mathcal{C}} \subset \hat{f}^{-1}(P)$, where P is a finite subset of $\mathbb{P}^1$, and $\widehat{\mathcal{C}}$ is the strict transform of $\mathcal{C}$ in $\widehat{\mathbb{P}}^2$. Note that, if a pencil completely contains $\mathcal{C}$, then f restricts to a well-defined holomorphic map $f : \mathbb{P}^2 \setminus \mathcal{C} = X_{\mathcal{C}} \to \mathbb{P}^1 \setminus P$. A pencil is said to *contain a curve* $\mathcal{C}$, if it completely contains at least one irreducible component of $\mathcal{C}$. In addition, a pencil is called *primitive* if the fibers of $\hat{f}$ are connected.

Theorem 2.21 ([1])**.** *There exist a finite number of torsion points $\varepsilon_i \in \mathbb{T}_{\mathcal{C}}$, unitary points $\tilde{\varepsilon}_j \in \mathbb{T}_{\mathcal{C}}$, and primitive pencils containing $\mathcal{C}$, $f_i : X_{\mathcal{C}} \to \mathcal{D}_i = \mathbb{P}^1 \setminus P_i$ such that*

$$\mathrm{Char}_{1,\mathbb{P}}(\mathcal{C}) := \bigcup \varepsilon_i f_i^* H^1(\mathcal{D}_i; \mathbb{C}^*) \cup \bigcup \tilde{\varepsilon}_j.$$

An analogous result follows for the affine case $\mathrm{Char}_1(\mathcal{C})$.

Note that any element of $H^1(\mathcal{D}_i; \mathbb{C}^*) = \mathrm{Hom}(H_1(\mathcal{D}_i; \mathbb{C}), \mathbb{C}^*)$, i.e. any character on $H_1(\mathcal{D}_i; \mathbb{C})$ can be seen as a point of $\mathrm{Spec}(\Lambda^{\mathbb{C}}_{\mathcal{D}_i}) = \mathrm{Spec}(\mathbb{C}[H_1(\mathcal{D}_i; \mathbb{C})]) = \mathbb{T}^{h_1(\mathcal{D}_i;\mathbb{C})}$ and vice versa. Therefore $f_i^* H^1(\mathcal{D}_i; \mathbb{C}^*)$ is a subset of $\mathbb{T}_{\mathcal{C}}$. Also note that $\varepsilon_i f_i^* H^1(\mathcal{D}_i; \mathbb{C}^*)$ refers to coordinatewise multiplication in $\mathbb{T}_{\mathcal{C}}$. Finally, ε_i is a torsion point if $\varepsilon_i^n = (1, \ldots, 1) =: \mathbb{1}_r$ for some $n \in \mathbb{Z}$ and $\tilde{\varepsilon}_j$ is unitary if $\tilde{\varepsilon}_j \in (\mathbb{S}^1)^r \subset \mathbb{T}_{\mathcal{C}}$.

Remark 2.22. According to a recent work by Libgober [78] unitary non-torsion isolated points cannot exist in $\mathrm{Char}_{k,\mathbb{P}}(\mathcal{C})$. Therefore, according to Remark 2.15(2), the ring automorphism $\Lambda^{\mathbb{Z}}_{\mathcal{C}} \to \Lambda^{\mathbb{Z}}_{\mathcal{C}}$ given by $t_i \to t_i^{-1}$ induces a skew automorphism of the corresponding modules $M^{\mathbb{C}}_{\mathcal{C},\mathrm{ab}}$.

Certain components of $\mathrm{Char}_{k,\mathbb{P}}(\mathcal{C})$ can be inherited from subarrangements of $\mathcal{C}$. More specifically, let us assume that $V \subset \mathrm{Char}_{k,\mathbb{P}}(\mathcal{C}_{(i)})$

is a non-empty component of the k-th characteristic variety of $\mathcal{C}_{(i)} = \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_{i-1} \cup \mathcal{C}_{i+1} \cup \cdots \cup \mathcal{C}_r$. Then the natural surjection

$$\begin{array}{ccc} \Lambda_{\mathcal{C}} & \twoheadrightarrow & \Lambda_{\mathcal{C}_{(i)}} \\ t_j & \mapsto & \begin{cases} t_j & \text{if } j \neq i \\ 1 & \text{if } i = j \end{cases} \end{array}$$

induces an inclusion

$$\operatorname{Spec}(\Lambda_{\mathcal{C}_{(i)}}) = \mathbb{T}_{\mathcal{C}_{(i)}} \subset \mathbb{T}^i_{\mathcal{C}}$$

where $\mathbb{T}^i_{\mathcal{C}} = \{(\varepsilon_1, \ldots, \varepsilon_r) \in \mathbb{T}_{\mathcal{C}} \mid \varepsilon_i = 1\}$.

This allows us to see both $M_{\mathcal{C},\mathrm{ab}}$ and $M_{\mathcal{C}_{(i)},\mathrm{ab}}$ as $\Lambda_{\mathcal{C}}$-modules and consider the natural surjection $M_{\mathcal{C},\mathrm{ab}} \twoheadrightarrow M_{\mathcal{C}_{(i)},\mathrm{ab}}$, as a surjection of modules. Remark 2.19 and the previous discussion imply that $\operatorname{Char}_{k,\mathbb{P}}(\mathcal{C}_{(i)}) \subset \operatorname{Char}_{k,\mathbb{P}}(\mathcal{C}) \cap \mathbb{T}^i_{\mathcal{C}}$.

Definition 2.23. We call a component V of $\operatorname{Char}_{k,\mathbb{P}}(\mathcal{C})$ *essential* if it is not contained in any $\operatorname{Char}_{k,\mathbb{P}}(\mathcal{C}_{(i)})$. Otherwise we say V is *non-essential*. We call V a *coordinate component* if it is contained in a coordinate torus $\mathbb{T}^i_{\mathcal{C}}$. Otherwise we say V is *non-coordinate*.

Note that non-coordinate components are necessarily essential by the discussion above.

Libgober proved in [75] that any *positive dimensional* coordinate component is necessarily non-essential. In [75] he also introduced ideal sheaves called *ideals of quasi-adjunction* denoted by $\mathcal{A}^{\bar{X}}_{\mathcal{C}}$, where $\bar{X} \in (0,1)^r$ and $\bar{X}$ is determined by the configuration of singularities of $\mathcal{C}$, and showed that points in $\operatorname{Char}^{\mathrm{nc}}_{1,\mathbb{P}}(\mathcal{C})$ (the non-coordinate components of $\operatorname{Char}_{1,\mathbb{P}}(\mathcal{C})$) can be detected by studying the superabundance (see below for its definition) of a finite family of ideal sheaves $\mathcal{A}^{\bar{X}}_{\mathcal{C}}(d-3-\ell_{\bar{X}})$, where $\ell_{\bar{X}} := \sum_{i=1}^r d_i X_i \in \mathbb{N}$ (see [75] for details).

It is known that, for a given $\mathcal{C}$, $\mathcal{A}^{\bar{X}}_{\mathcal{C}}$ satisfies the following properties.

- The number of points $\bar{X} \in (0,1)^r$ determined by the configuration of singularities of $\mathcal{C}$ is finite.
- $\mathcal{O}_{\mathbb{P}^2}/\mathcal{A}^{\bar{X}}_{\mathcal{C}}$ is supported on the singularities of $\mathcal{C}$.
- Let $\bar{\xi} := (\xi_1, \ldots, \xi_r) \in \mathbb{T}_{\mathcal{C}}$ be a torsion point such that $\xi_1^{d_1} \cdot \ldots \cdot \xi_r^{d_r} = 1$, and $\xi_i \neq 1$. We define $X_i := \frac{\log \xi_i}{2\pi\sqrt{-1}} \in (0,1)$. Under these notations, $\bar{\xi} \in \operatorname{Char}_{1,\mathbb{P}}(\mathcal{C})$ if the homomorphism $\sigma_{\bar{X}}$ is *not* surjective.

$$\begin{gathered} 0 \to H^0(\mathbb{P}^2, \mathcal{A}^{\bar{X}}_{\mathcal{C}}(d-3-\ell_{\bar{X}})) \to \\ \to H^0(\mathbb{P}^2, \mathcal{O}(d-3-\ell_{\bar{X}})) \xrightarrow{\sigma_{\bar{X}}} \bigoplus_{P \in \operatorname{Sing} \mathcal{C}} \mathcal{O}_{\mathbb{P}^2,P}/(\mathcal{A}^{\bar{X}}_{\mathcal{C}})_P, \end{gathered} \tag{18}$$

We say that there is a *superabundance* of dimension superabundance of dimension dim coker $\sigma_{\bar{X}}$. In Example 2.24 below, we describe $(\mathcal{A}_{\mathcal{C}}^{\bar{X}})_P$ in the case when $P \in \operatorname{Sing}(\mathcal{C})$ is a double point.

Moreover, in that case the exponential of the irreducible system of equations given by the local and global conditions provides an irreducible component of $\operatorname{Char}_{1,\mathbb{P}}(\mathcal{C})$ (by an *irreducible system of equations* with integer coefficients we mean an equivalent system where the integer coefficients of the variables are relatively prime).

Example 2.24.

(1) One of the simplest examples of positive dimensional non-coordinate characteristic varieties is the case of two conics $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$ intersecting at two tacnodes. It is not hard to see that the pencil generated by C_1 and C_2 induces an epimorphism $G := \pi_1(X_{\mathcal{C}}) \twoheadrightarrow \mathbb{Z} * \mathbb{Z}/2\mathbb{Z} =: G_2$, where a meridian of $\mathcal{C}_1$ is sent to $(1,0)$ and a meridian of $\mathcal{C}_2$ is sent to $(-1,1)$ (see [14, §4] for more details on this). By Example 2.18(1) $\operatorname{Char}_1^{\mathbb{K}}(\mathcal{C}) \subset \operatorname{Char}_1^{\mathbb{K}}(G_2) = V(\bar{t}_2+1)$, where the embedding comes from the following morphism of rings:

$$\begin{array}{ccc}
\Lambda_{\mathcal{C}}^{\mathbb{K}} := \mathbb{K}[t_1^{\pm 1}, t_2^{\pm 1}]/(t_1^2 t_2^2 - 1) & \to & \Lambda_{G_2}^{\mathbb{K}} := \mathbb{K}[\bar{t}_1^{\pm 1}, \bar{t}_2^{\pm 1}]/(\bar{t}_2^2 - 1) \\
t_1 & \mapsto & \bar{t}_1 \\
t_2 & \mapsto & \bar{t}_1^{-1}\bar{t}_2
\end{array}$$

Since the classical Alexander polynomial of $\mathcal{C}$ is trivial, $\operatorname{Char}_1(\mathcal{C}) \neq \mathbb{T}_{\mathcal{C}}$, and hence $\{(t, -t^{-1}) \mid t \in \mathbb{C}^*\} \subset \mathbb{T}_{\mathcal{C}}$ is an irreducible component of $\operatorname{Char}_1(\mathcal{C})$.

(2) Let us describe the local quasiadjunction ideals of the $\mathbb{A}_k$-singularities (locally described as $y^2 - x^{k+1}$).

(a) If $k = 2s$, then there is only one local branch. In this case, $r = 1$. Given $x_1 \in (0,1)$ we associate

$$(\mathcal{A}_{\mathcal{C}}^{x_1})_P := \begin{cases} (y, x^\delta) & \begin{array}{c}\text{if there exists } \delta \in \mathbb{N} \text{ such that} \\ 2s - 2\delta - 1 < 2(2s+1)x_1 \leq 2s - 2\delta + 1\end{array} \\ \mathcal{O} & \text{if } 2s - 1 < 2(2s+1)x_1, \end{cases}$$

(b) If $k = 2s-1$, then there are two local branches. In this case, $r = 2$. Given $(x_1, x_2) \in (0,1)^2$ we associate

$$(\mathcal{A}_{\mathcal{C}}^{(x_1,x_2)})_P := \begin{cases} (y, x^\delta) & \begin{array}{c}\text{if there exists } \delta \in \mathbb{N} \text{ such that} \\ s - \delta - 1 < s(x_1 + x_2) \leq s - \delta\end{array} \\ \mathcal{O} & \text{if } s - 1 < s(x_1 + x_2), \end{cases}$$

Finally note that the description of ideals given here is local, analytical and not global, algebraic. In other words, if $\mathcal{C}$ has an $\mathbb{A}_k$ singularity at P, then the equation D of a curve $\mathcal{D}$ belongs to the local ideal $\mathfrak{m}_\delta := (y, x^\delta)$ at P if the multiplicity of intersection of $\mathcal{D}$ with each branch of $\mathcal{C}$ at the singular point P is at least δ.

The role of essential coordinate components is oftentimes important. For instance, in [8] an example of an Alexander equivalent Zariski pair was exhibited by computing the characteristic varieties. It turns out that they only differ in essential coordinate components. The problem with such components is that up to now, no algebro-geometrical condition has been found for their existence, so one needs to compute them via a presentation of the fundamental group.

As we mentioned above, non-coordinate components can be detected by considering the singularities of $\mathcal{C}$ both locally and globally. To be more specific, all singularities seem to play a role, except for nodes, as the following result claims.

Proposition 2.25. [31, Proposition 6.1] *Let $\mathcal{C}_\lambda$, $\lambda \in (0,1]$ be an equisingular continuous family of curves degenerating into a curve $\mathcal{C}_0$ with the same non-nodal singularities as $\mathcal{C}_\lambda$. Consider also a continuous family of non-coordinate epimorphisms $\varepsilon_\lambda : H_1(X_{\mathcal{C}_\lambda}; \mathbb{Z}) \to \mathbb{Z}$ degenerating into $\varepsilon_0 : H_1(X_{\mathcal{C}_0}; \mathbb{Z}) \to \mathbb{Z}$, then*

$$\Delta_{\mathcal{C}_\lambda, \varepsilon_\lambda}(t) = \Delta_{\mathcal{C}_0, \varepsilon_0}(t).$$

Moreover, if $\mathcal{C}_0$ has the same number of irreducible components as $\mathcal{C}_\lambda$, then one also has

$$\operatorname{Char}_1^{\mathrm{nc}}(\mathcal{C}_\lambda) = \operatorname{Char}_1^{\mathrm{nc}}(\mathcal{C}_0) \subset \mathbb{T}_{\mathcal{C}_0},$$

where $\operatorname{Char}_1^{\mathrm{nc}}(\mathcal{C})$ denotes the union of the non-coordinate components of $\operatorname{Char}_1(\mathcal{C})$.

Note that the definition of non-coordinate epimorphisms is given in the paragraph preceding Remark 2.2.

Finally we want to compare Fitting ideals, characteristic varieties, and Oka-polynomials.

Theorem 2.26. *Let $\mathcal{C} = \mathcal{C}_0 \cup \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_r$ be a curve where $\mathcal{C}_0$ is a transversal line, denote by G its fundamental group, consider $\varepsilon = (\varepsilon_1, ..., \varepsilon_r) \in \operatorname{Hom}(G, \mathbb{Z})$ an epimorphism, and the evaluation morphism*

$$\begin{array}{rrcc} \varphi_\varepsilon : & \Lambda_{\mathcal{C}}^{\mathbb{K}} = \mathbb{K}[t_1^{\pm 1}, \ldots, t_r^{\pm 1}] & \to & \Lambda_\varepsilon^{\mathbb{K}} = \mathbb{K}[t^{\pm 1}] \\ & t_i & \mapsto & t^{\varepsilon_i}. \end{array}$$

Then $(t-1)^q\varphi_\varepsilon(F_1^{\mathbb{K}}(\mathcal{C}))$ *is a principal ideal generated by the Oka-polynomial* $\Delta^{\mathbb{K}}_{\mathcal{C},\varepsilon}(t)$.

In particular, $\operatorname{Char}_1^{\mathbb{K}} \cap \{(t^{\varepsilon_1},\ldots,t^{\varepsilon_r}) \mid t\in\mathbb{K}^*\} = \operatorname{Supp}\left(\Lambda^{\mathbb{K}}_\varepsilon/\Delta^{\mathbb{K}}_{\mathcal{C},\varepsilon}(t)\right)$.

Proof. Let $M^{\mathbb{K}}_{\mathcal{C},\mathrm{ab}}$ be the Alexander invariant. Let

$$\tilde{\Lambda}^{\mathbb{K}} = \frac{\Lambda^{\mathbb{K}}_{\mathcal{C}}[t^{\pm 1}]}{(t_1-\varphi_\varepsilon(t_1),\ldots,t_r-\varphi_\varepsilon(t_r))}.$$

Using for example the identity

$$M^{\mathbb{K}}_{\mathcal{C},\mathrm{ab}} = \frac{\ker(\partial_1 : C_1(X_{\mathcal{C},\mathrm{ab}})\to C_0(X_{\mathcal{C},\mathrm{ab}}))}{\operatorname{Im}(\partial_2 : C_2(X_{\mathcal{C},\mathrm{ab}})\to C_1(X_{\mathcal{C},\mathrm{ab}}))},$$

and the fact that $\gcd(\varepsilon_1,\ldots,\varepsilon_r)=1$ it is easy to check that

$$\begin{aligned}&\ker(\partial_1 : C_1(X_{\mathcal{C},\mathrm{ab}})\to C_0(X_{\mathcal{C},\mathrm{ab}}))\otimes\tilde{\Lambda}^{\mathbb{K}}\\ &= \ker(\partial_1 : C_1(X_{\mathcal{C},\varepsilon})\to C_0(X_{\mathcal{C},\varepsilon}))\otimes(t-1)\Lambda^{\mathbb{K}}_\varepsilon\end{aligned}$$

and $\operatorname{Im}(\partial_2 : C_2(X_{\mathcal{C},\mathrm{ab}}) \to C_1(X_{\mathcal{C},\mathrm{ab}}))\otimes\tilde{\Lambda}^{\mathbb{K}} = \operatorname{Im}(\partial_2 : C_2(X_{\mathcal{C},\varepsilon}) \to C_1(X_{\mathcal{C},\varepsilon}))$. Hence

$$M^{\mathbb{K}}_{\mathcal{C},\mathrm{ab}}\otimes\tilde{\Lambda}^{\mathbb{K}} = M^{\mathbb{K}}_{\mathcal{C},\varepsilon}\otimes(t-1)\Lambda^{\mathbb{K}}_\varepsilon$$

as $\Lambda^{\mathbb{K}}_\varepsilon$-modules. The results follows from the exact sequence

$$0\to M^{\mathbb{K}}_{\mathcal{C},\varepsilon}\otimes(t-1)\Lambda^{\mathbb{K}}_\varepsilon \to M^{\mathbb{K}}_{\mathcal{C},\varepsilon} \to \frac{M^{\mathbb{K}}_{\mathcal{C},\varepsilon}}{M^{\mathbb{K}}_{\mathcal{C},\varepsilon}\otimes(t-1)\Lambda^{\mathbb{K}}_\varepsilon} = \left(\frac{\Lambda^{\mathbb{K}}_\varepsilon}{(t-1)}\right)^q \to 0$$

and from [55, Lemma III.6]. Q.E.D.

In other words, varying the epimorphisms $\varepsilon\in\operatorname{Hom}(G,\mathbb{Z}^r)$ and computing their corresponding Oka-polynomials, one is able to recuperate $\operatorname{Char}_1(\mathcal{C})$.

2.3. The special case of line arrangements

Characteristic varieties and Alexander polynomials of line arrangements have been largely studied in the recent years by Cohen-Orlik [32], Cohen-Suciu [33, 34], M. Falk [46], E. Hironaka [56, 57], Libgober [75], Libgober-Yuzvinsky [76, 77], M. Marco [81], and S. Yuzvinsky [129] among others. It turns out that the set of positive dimensional components passing through the origin $\mathbb{1}$ of the characteristic variety of a line arrangement is combinatorially determined (this is also the case for rational arrangements [30]). Components not passing through the origin sometimes exist [110, Example 10.6], but it is not known whether or not they are combinatorially determined.

The example we want to describe in more detail was proposed by G.Rybnikov in the mid 90's ([102]). He presented a Zariski pair of line arrangements (in particular, both arrangements had the same combinatorics). His final purpose was to prove that both arrangements had non-isomorphic fundamental groups. An alternative proof was proposed in [11] using the Alexander invariants plus an extra property that made the Alexander invariant an invariant of the fundamental group (this is usually not the case as mentioned in Remark 2.17(2)).

We will briefly recall the concept of combinatorial type or (abstract) line combinatorics:

Definition 2.27. A *combinatorial type* (or simply a *(line) combinatorics*) is a couple $\mathscr{C} := (\mathcal{L}, \mathcal{P})$, where $\mathcal{L}$ is a finite set and $\mathcal{P}$ is a family of subsets of $\mathcal{L}$, satisfying that:

(1) For all $P \in \mathcal{P}$, $\#P \geq 2$;
(2) For any $\ell_1, \ell_2 \in \mathcal{L}$, $\ell_1 \neq \ell_2$, there exists a unique $P \in \mathcal{P}$ such that $\ell_1, \ell_2 \in P$.

An *ordered combinatorial type* $\mathscr{C}^{\text{ord}}$ is a combinatorial type where $\mathcal{L}$ is an ordered set.

In what follows we will present a typical technique that allows one to find Zariski pairs of line arrangements. It is directly related to the Alexander invariant and extension isomorphisms (Remark 2.17(1)). Other possible techniques related to Massey products have also been explored [82]. First we will describe some combinatorial types.

Example 2.28 (Rybnikov's combinatorics)**.** For details on this example see [11]. Let us consider $V := \mathbb{F}_3^2 \setminus \{(0,0)\}$, where $\mathbb{F}_3^2$ is the 2-dimensional affine space on the field $\mathbb{F}_3$ of three elements. We define $\mathscr{C}_{\text{ML}} := (\mathcal{L}_{\mathscr{C}_{\text{ML}}}, \mathcal{P}_{\mathscr{C}_{\text{ML}}})$, where $\mathcal{L}_{\mathscr{C}_{\text{ML}}}$ is the set of points in V and $\mathcal{P}_{\mathscr{C}_{\text{ML}}}$ is the set of affine lines in V. Note that any affine line in V contains either two or three points of V (which implies property (1) in Definition 2.27). Also note that any two points in V define exactly one line in V (which implies property (2) in Definition 2.27). Thus $\mathscr{C}_{\text{ML}}$ is a combinatorial type that will be referred to as *MacLane's combinatorial type*.

Recall that a combinatorics is called *real* if it admits a realization in $\mathbb{CP}^2$ whose global equation has real coefficients, whereas it is called *strongly real* if each line admits a real equation. Note that strongly real combinatorics admit strongly real curves as equations in the sense of (C2).

It is well known that the MacLane combinatorics is real but not strongly real and that its combinatorial stratum is connected, however its ordered combinatorial stratum has two connected components whose representatives have eight complex conjugated lines. Moreover, five of them are real and the remaining three have coefficients in $\mathbb{Q}[\omega]$, where $\omega := \exp\left(\frac{2\pi\sqrt{-1}}{3}\right)$.

We will refer to such ordered realizations as

$$L_\omega := \{\ell_0, \ell_1, \ell_2, \ell_3, \ell_4, \ell_5, \ell_6, \ell_7\} \text{ and } L_{\bar\omega} := \{\ell_0, \ell_1, \ell_2, \ell_3, \ell_4, \bar\ell_5, \bar\ell_6, \bar\ell_7\}.$$

Let us decompose $L_\omega = L_0 \cup L_+$ and $L_{\bar\omega} = L_0 \cup L_-$, where $L_0 := \{\ell_0, \ell_1, \ell_2\}$ and consider a projective transformation ρ fixing the initial ordered set L_0 (that is, $\rho(\ell_i) = \ell_i$ $i = 0, 1, 2$) and such that ρL_+ and ρL_- intersect both L_+ and L_- only in double points. Note that ρ can be chosen with real coefficients. Let us consider the following ordered arrangements of thirteen lines: $R_{\alpha,\beta} = L_0 \cup L_\alpha \cup \rho L_\beta$, where $\alpha, \beta \in \{+, -\}$. They produce a combinatorics $\mathscr{C}_{\text{RYB}}$ with 13 lines, 33 double points, and 15 triple points.

Using complex conjugation one can see that $(\mathbb{P}^2, R_{+,+}) \approx (\mathbb{P}^2, R_{-,-})$ and that $(\mathbb{P}^2, R_{+,-}) \approx (\mathbb{P}^2, R_{-,+})$. Hence, we will only deal with the Alexander invariants M_+ (resp. M_-) of $R_{+,+}$ (resp. $R_{+,-}$).

One can prove ([11, Theorem 3.8]) that there is no extension isomorphism (see Remark 2.17(1)) from M_+ to M_- as Λ-modules, where $\Lambda = \mathbb{Z}[t_1, \ldots, t_{12}]$ and each t_i represents a meridian around each affine line of $R_{\pm,\pm}$ in $\mathbb{P}^2 \setminus \ell_0$.

Let $\mathcal{C} = \mathcal{C}_0 \cup \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_r$ be a line arrangement. We will denote by $\mathcal{I}^{\mathbb{Z}}$ the submodule of $\Lambda^{\mathbb{Z}}_{\mathcal{C},\text{ab}}$ generated by $(t_1 - 1, \ldots, t_r - 1)$ and it will be referred to as the *augmentation* ideal.

Proposition 2.29. *The truncation $M^{\mathbb{Z}}_{\mathcal{C},\text{ab}} \otimes \Lambda^{\mathbb{Z}}_{\mathcal{C}}/\mathcal{I}^{\mathbb{Z}}$ of the Alexander invariant of $\mathcal{C}$ is completely determined by the combinatorics of $\mathcal{C}$.*

Proof. A Zariski presentation of $G := \pi_1(X_{\mathcal{C}})$ can be given as in Definition 1.17, where the set of relations of the presentation are combinatorial up to conjugation. For instance, at each ordinary multiple point P of multiplicity k one obtains the following relations $[x_{i_j}^{\alpha_j}, X] = 1$ where $X := \prod_{j=1}^k x_{i_j}^{\alpha_j}$, $a^b = b^{-1} \cdot a \cdot b$, x_i is a meridian of the line ℓ_i, and $i_1, \ldots, i_k$ are subindices of the k lines intersecting at P. The result is an immediate consequence of [11, Proposition 2.15] which assures that the class of $[x_{i_j}^{\alpha_j}, X]$ in $M^{\mathbb{Z}}_{\mathcal{C},\text{ab}} \otimes \Lambda^{\mathbb{Z}}_{\mathcal{C}}/\mathcal{I}^{\mathbb{Z}}$ only depends on the Abelian class of each $x_{i_j}^{\alpha_j}$, that is x_{i_j}, and hence it is combinatorial. Q.E.D.

Let $\mathcal{L}$ denote the combinatorial type of a line arrangement $\mathcal{C}$. Since $H_1(X_{\mathcal{C}})$ only depends on $\mathcal{L}$, it will be denoted it by $H_{\mathcal{L}}$. Consider $\operatorname{Aut}(H_{\mathcal{L}})$ the set of automorphisms of $H_{\mathcal{L}}$. Any $h \in \operatorname{Aut}(H_{\mathcal{L}})$ induces a transformation of $M^{\mathbb{Z}}_{\mathcal{C},\mathrm{ab}} \otimes \Lambda^{\mathbb{Z}}_{\mathcal{C}}/\mathcal{I}^{\mathbb{Z}}$ as follows $h([x_i,x_j]) = [h(x_i),h(x_j)]$ (since the elements $[x_i,x_j]$ generate G' this defines a transformation, but not necessarily an automorphism). The set of those elements in $\operatorname{Aut}(H_{\mathcal{L}})$ that induce automorphisms of $M^{\mathbb{Z}}_{\mathcal{C},\mathrm{ab}} \otimes \Lambda^{\mathbb{Z}}_{\mathcal{C}}/\mathcal{I}^{\mathbb{Z}}$ (as a group) will be denoted by $\operatorname{Aut}^1(H_{\mathcal{L}})$. Let $\Gamma(\mathcal{L}) \subset \operatorname{Aut}(H_{\mathcal{L}})$ denote the set of automorphisms that preserve the combinatorics. Note that if $\varphi \in \Gamma(\mathcal{L})$, then $\pm\varphi \in \operatorname{Aut}^1(H_{\mathcal{L}})$. Thus $\{\pm 1\} \times \Gamma(\mathcal{L}) \subset \operatorname{Aut}^1(H_{\mathcal{L}})$.

Definition 2.30. A line combinatorics $\mathcal{L}$ is called *homologically rigid* if $\operatorname{Aut}^1(H_{\mathcal{L}}) = \{\pm 1\} \times \Gamma(\mathcal{L})$.

Proposition 2.31. *If $\mathcal{C}$ is a line arrangement whose combinatorics, say $\mathcal{L}$, is homologically rigid, then the Alexander invariant $M^{\mathbb{Z}}_{\mathcal{C},\mathrm{ab}}$ as a $\Lambda^{\mathbb{Z}}_{\mathcal{C}}$-module is an invariant of the fundamental group $\pi_1(X_{\mathcal{C}})$ (up to* extension isomorphisms*).*

Example 2.32. Rybnikov's combinatorics $\mathscr{C}_{\mathrm{RYB}}$ is homologically rigid ([11, Proposition 4.22]). We have mentioned above that no extension isomorphism exists from M_+ to M_-. Therefore one concludes that $\pi_1(X_{R_{+,+}}) \not\approx \pi_1(X_{R_{+,-}})$.

2.4. Twisted Alexander polynomials

Twisted Alexander polynomials have been developed and extensively studied in the mid 90's for knots. In many instances where Abelian invariants were not able to identify a certain property of knots, non-Abelian invariants such as these, were able to do it −see [67, 79, 124]. Later P. Kirk and C. Livingston [65, 66] were able to give partial answers to questions of mutation and concordance for general CW-complexes. Our purpose here is to define and briefly describe twisted Alexander polynomials for curves and some of their recent applications following [31].

Let us consider the general setting of §2.1, that is, a curve $\mathcal{C}$, its complement $X_{\mathcal{C}}$, the epimorphism $\varepsilon : G := \pi_1(X_{\mathcal{C}}) \to \mathbb{Z}$, $K_\varepsilon := \ker \varepsilon$, and the infinite cyclic covering $X_{\mathcal{C},\varepsilon}$. In addition let us consider a $\mathbb{K}$-vector space V of finite dimension and an (anti)representation

$$\rho : G \longrightarrow \mathrm{GL}(V).$$

Note that V inherits a right $\mathbb{K}[G]$-module structure denoted by V_ρ. Let $X_{\mathcal{C},\mathrm{ab}} \to X_{\mathcal{C}}$ denote the universal Abelian covering of $X_{\mathcal{C}}$. Analogously as mentioned in Remark 2.1 and the subsequent discussion, the cellular chain complex $C_*(X_{\mathcal{C},\mathrm{ab}};\mathbb{K})$ also becomes a finitely generated (left) $\mathbb{K}[G]$-module generated by the lifts of the cells of $X_{\mathcal{C}}$. Hence, one defines

$$C_*^{\varepsilon,\rho}(X_{\mathcal{C}};\mathbb{K}[t^{\pm 1}]) := V_\rho \otimes_{\mathbb{K}[K_\varepsilon]} C_*(X_{\mathcal{C},\mathrm{ab}})$$

as a $\mathbb{K}[t^{\pm 1}]$-module, where $\mathbb{K}[t^{\pm 1}]$ is a trivial $\mathbb{K}[G]$-module, as follows:

$$t^n \cdot (v \otimes c) = v\gamma^{-n} \otimes \gamma^n c$$

where $\gamma \in G$ verifies $\varepsilon(\gamma) = t$.

Definition 2.33. The homology of $(X_{\mathcal{C}}, \varepsilon, \rho)$ is defined as the $\mathbb{K}[t^{\pm 1}]$-module

$$H_*^{\varepsilon,\rho}(X_{\mathcal{C}};\mathbb{K}[t^{\pm 1}]) = H_*(C^{\varepsilon,\rho}(X_{\mathcal{C}},\mathbb{K}[t^{\pm 1}])).$$

Definition 2.34. The k-th *twisted Alexander polynomial* $\Delta^k_{\mathcal{C},\varepsilon,\rho}(t)$ of $(X_{\mathcal{C}}, \varepsilon, \rho)$ is the order of $H_k^{\varepsilon,\rho}(X_{\mathcal{C}};\mathbb{K}[t^{\pm 1}])$. For short, we denote by $\Delta_{\mathcal{C},\varepsilon,\rho}(t) = \frac{\Delta^1_{\mathcal{C},\varepsilon,\rho}(t)}{\Delta^0_{\mathcal{C},\varepsilon,\rho}(t)}$ the element of $\mathbb{K}(t)$.

See (12) for a definition of order of a module over a principal ideal domain.

Remark 2.35.

(1) Note that even if $\varepsilon : G \to \mathbb{Z}/m\mathbb{Z}$ was an epimorphism onto the finite cyclic group $\mathbb{Z}/m\mathbb{Z}$, all the definitions can be modified accordingly to suit this case.

(2) Note that $\Delta_{\mathcal{C},\varepsilon,\rho}(t)$ does not have to be a polynomial. For example, we can consider the projective three-cuspidal quartic $\mathcal{Q}$, whose fundamental group G is shown in (1). Since G is finite we can consider the regular representation $\rho : G \to \mathrm{GL}(12,\mathbb{K})$ and the trivial morphism $\varepsilon : G \to \mathbb{Z}/4\mathbb{Z}$. In this situation

$$\Delta_{\mathcal{Q},\varepsilon,\rho}(t) = \begin{cases} \frac{1}{t-1} & \text{if } \operatorname{char}\mathbb{K} = 3 \\ \frac{1}{t^2-1} & \text{otherwise.} \end{cases}$$

Under certain very general conditions, however, $\Delta_{\mathcal{C},\varepsilon,\rho}(t)$ is a polynomial [31, Proposition 5.4].

(3) An alternative definition of $\Delta_{\mathcal{C},\varepsilon,\rho}(t)$ can be given by means of Fox calculus – see [124].

Twisted Alexander polynomials can also be seen ([120, 121]) as the Reidemeister torsion of the complex of vector spaces obtained by tensoring the usual CW-complex C_* describing the homotopy type of $X_{\mathcal{C}}$

by V_ρ and by $\mathbb{K}(t)$. Since the Reidemeister torsion behaves well with respect to surgery of complexes, a division formula for twisted Alexander polynomials that generalizes Theorem 2.10 can be obtained. In order to do so, one needs the following construction:

Suppose that we are given a curve $\mathcal{C}$, an epimorphism $\varepsilon : H_1(X_{\mathcal{C}}) \to \mathbb{Z}$, and a representation $\rho : \pi_1(X_{\mathcal{C}}) \to \mathrm{GL}(V)$. Let $\mathbb{S}^3_1, \ldots, \mathbb{S}^3_s$ be sufficiently small 3-spheres around the singular points $\{P_1, ..., P_s\}$ of $\mathcal{C}$. Denote by $L_k = \mathcal{C} \cap \mathbb{S}^3_k$ the link of the singularity at P_k. Also choose a base point $Q_k \in \mathbb{S}^3_k \setminus L_k$ and denote by $\pi^k = \pi_1(\mathbb{S}^3_k \setminus L_k; Q_k)$ the local fundamental groups at P_k. The inclusion maps $i_k : \pi^k \to \pi_1(X_{\mathcal{C}})$ and (ε, ρ) induce morphisms

$$\varepsilon_k : \pi^k \longrightarrow \mathbb{Z}$$

$$\text{and } \rho_k : \pi^k \longrightarrow \mathrm{GL}(V),$$

for any $k = 1, ..., s$. Analogously, one can consider $\mathbb{S}_\infty$ a sufficiently large 3-sphere, $L_\infty = \mathcal{C} \cap \mathbb{S}_\infty$ the link at infinity and define accordingly π^∞, ε_∞, and ρ_∞.

Theorem 2.36. [31, Theorem 5.6] *Let $\mathcal{C}$ be a curve, ε an epimorphism, and ρ a unitary representation. Suppose also that the induced triples $(X_{\mathcal{C}} \cap \mathbb{S}^3_k, \varepsilon_k, \rho_k)$, $k = 1, ..., s, \infty$ are acyclic. Then*

$$\begin{aligned}&\Big(\prod_{\ell=1}^{r} \det(\mathrm{Id} - \rho(\nu_\ell) t^{q_\ell})^{s_\ell - \chi(\mathcal{C}_\ell)}\Big) \cdot \prod_{k=1,\ldots,s,\infty} \Delta_{L_k,\rho_k} \\ &= \Delta_{\mathcal{C},\varepsilon,\rho} \cdot \overline{\Delta}_{\mathcal{C},\varepsilon,\rho} \cdot \det \varphi^{\varepsilon,\rho}(\mathcal{C}),\end{aligned}$$

where ν_ℓ is the homology class of a meridian of the irreducible component $\mathcal{C}_\ell$, $s_\ell = \#\,\mathrm{Sing}(\mathcal{C}) \cap \mathcal{C}_\ell$, and $\varphi^{\varepsilon,\rho}(\mathcal{C})$ is an intersection form on $H_2^{\varepsilon,\rho}(X_{\mathcal{C}}, \mathbb{Q}[t^{\pm 1}])$ with twisted coefficients.

Remark 2.37. The condition of acyclicity is purely technical and can be expressed as follows. A triple (X, ε, ρ) is *acyclic* if the chain complex $C_*^{\varepsilon,\rho}(X; \mathbb{K}(t))$ is acyclic over $\mathbb{K}(t)$.

In the irreducible case, something is known about the roots of the twisted Alexander polynomial of unitary representations.

Theorem 2.38. [78, Theorem 5.3.] *Let $\mathcal{C}$ be an irreducible curve and L is a line at infinity. Let ρ be a unitary representation of the fundamental group and let $\mathbb{K}$ be the extension of $\mathbb{Q}$ generated by the eigenvalues of $\rho(\gamma)$ where γ is a meridian of $\mathcal{C}$. Then the roots of $\Delta_\rho(C)$ belong to a cyclotomic extension of $\mathbb{K}$.*

Note that in this case the morphism ε is uniquely determined up to orientation.

Finally, let us point out that twisted Alexander polynomials are sensitive to nodal degenerations, that is, Proposition 2.25 is no longer true for twisted Alexander polynomials as the following example illustrates.

We say a plane projective curve $\mathcal{D}$ of degree, say d, is a type-I curve if $\mathcal{D}$ is irreducible and has an ordinary $(d-2)$-ple point at some point, say P. Consider $\mathcal{D}$ a type-I curve and let L_1 and L_2 be lines through P such that either L_i is tangent to a smooth point $P_i \in \mathcal{D}$ or L_i passes through a double point $P_i \neq P$ of type $\mathbb{A}_{2r}$. Let us denote $\mathcal{C} = L_1 + L_2 + \mathcal{D}$. Assume that $\mathcal{D}$ has only nodes as singular points apart from P. We recall the following properties of such curves:

- There exist nodal degenerations $\mathcal{D}_\lambda \to \mathcal{D}_0$ of non-rational type-I curves $\mathcal{D}_\lambda$ $(\lambda > 0)$ into a rational type-I curve $\mathcal{D}_0$ ([13, Corollary 3]).
- Let $\mathcal{D}_\lambda \to \mathcal{D}_0$ be a nodal degeneration as above. This produces a degeneration $\mathcal{C}_\lambda \to \mathcal{C}_0$, where $\mathcal{C}_\lambda = L_1 + L_2 + \mathcal{D}_\lambda$, $\lambda \geq 0$. If G_λ denotes $\pi_1(X_{\mathcal{D}_\lambda})$, then G_λ, $\lambda > 0$ is Abelian, whereas G_0 is not ([13, Proposition 6.1]). Moreover, a presentation of G_0 can be given as follows:

$$G_0 = \langle\ \ell, x_1, x_2 \mid [x_1, x_2] = 1, \ell^{-1}x_1\ell = x_2, \ell^{-1}x_2\ell = x_1\ \rangle,$$

  where ℓ is a meridian around a line and x_1, x_2 are meridians around $\mathcal{D}_0$.

Consider $\mathcal{C}_\lambda \to \mathcal{C}_0$ a degeneration as above. Let us denote by ν_1, ν_2 and ν_ℓ the homology classes of the corresponding generators of G_0. Since x_1 and x_2 are meridians of the same irreducible component, one has that $\nu = \nu_1 = \nu_2$. Our purpose is to find a suitable representation that produces a sensitive twisted Alexander polynomial. Let us consider ε the usual morphism $\varepsilon(\nu_\ell) = \varepsilon(\nu) = 1$, and the rank 2 representation

$$\rho(\ell) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \rho(x_1) = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}, \quad \rho(x_2) = \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}.$$

Using ε and ρ one obtains

$$\Delta_{\mathcal{C}_0,\varepsilon,\rho}(t) = (t^2 - 1). \tag{19}$$

Note that $\rho(G_0) \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$.

Finally note that, by Proposition 2.25, the classical Alexander polynomial $\Delta_{\mathcal{C}_\lambda}(t)$ and the torsion non-coordinate characteristic varieties are invariant for $\lambda \in [0,1]$. Since G_λ is Abelian, this implies that $\Delta_{\mathcal{C}_1,\varepsilon} = \Delta_{\mathcal{C}_0,\varepsilon} = (t-1)$ and $\mathrm{Char}_1^*(\mathcal{C}_1) = \mathrm{Char}_1^*(\mathcal{C}_0) = \emptyset$.

On the other hand, formula (19) shows that $\mathcal{C}_0$ has a non-trivial twisted Alexander polynomial, whereas any twisted Alexander polynomial of $\mathcal{C}_\lambda$, $\lambda \in (0,1]$ is trivial since G_λ is Abelian.

2.5. Computational methods

We have used the following result which is a straightforward generalization of [11, Proposition 2.8].

Proposition 2.39. *Let $\langle \bar{x}, \bar{y}; \bar{W} \rangle$ be a presentation of G such that*

(1) *$\bar{x}$ be the free group generated the words $x_1, ..., x_r$ whose Abelian classes generate $G/G' = \mathbb{Z}^q \oplus \mathbb{Z}/p_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p_s\mathbb{Z}$, $q+s=r$,*
(2) *$\bar{y} = \langle y_1, \ldots, y_u \rangle \subset G'$,*
(3) *$\forall w(\bar{x}, \bar{y}) \in \bar{W}$, one has that $w(\bar{x}, 1)$ is a product of commutators in $\bar{x}$ and $x_{q+i}^{p_i}$.*

Then the module $M_G^{\mathbb{Z}}$ admits a presentation $\tilde{\Gamma}/(\mathcal{T} + \mathcal{J} + \mathcal{W})$, where

$$\tilde{\Gamma} := \left(\bigoplus_{1 \leq i < j \leq r} x_{ij} \Lambda^{\mathbb{Z}} \right) \oplus \left(\bigoplus_{1 \leq k \leq u} y_k \Lambda^{\mathbb{Z}} \right),$$

(1) *$\Lambda^{\mathbb{Z}} = \mathbb{Z}[t_1^{\pm 1}, \ldots, t_r^{\pm 1}]/(t_{q+1}^{p_1} - 1, \ldots, t_r^{p_s} - 1)$,*
(2) *$\mathcal{T}$ is the submodule of $\tilde{\Gamma}$ generated by the torsion relations $\dfrac{t_{i+q}^{p_i} - 1}{t_{i+q} - 1}$, $i = 1, \ldots, s$,*
(3) *$\mathcal{J}$ is the Jacobian submodule of $\tilde{\Gamma}$ generated by the relations*

$$J(i,j,k) := (t_i - 1)x_{jk} - (t_j - 1)x_{ik} + (t_k - 1)x_{ij}, \quad 1 \leq i < j < k \leq r, \text{ and}$$

(4) *$\mathcal{W}$ is the submodule of $\tilde{\Gamma}$ generated by subset of $\tilde{\Gamma}$ obtained by rewriting the relations $\bar{W}$ in terms of $\tilde{\Gamma}$.*

Proof. The same proof used in [11, Proposition 2.8] can be applied using the Reidemeister-Schreier method to obtain a presentation of G'/G'' (which is not finitely presented in general) and then apply the module structure to give the finite presentation as a module. Q.E.D.

By Proposition 1.18, the fundamental group of the complement of any plane curve admits a presentation as in Proposition 2.39. In order to obtain the submodule $\mathcal{W}$ the following properties are very useful.

Proposition 2.40. *The following equalities hold in $M_G^{\mathbb{Z}}$:*

(1) $[x, x] = 0$,
(2) $[x, y] = -[y, x]$,
(3) $[x^{-1}, y] = -t_x^{-1}[x, y]$,

(4) $[x,p] = (t_x - 1)p \quad \forall p \in G'$,
(5) $[xy,z] = [x,z] + t_x[y,z]$,
(6) $[x * y, z] = [y,z] + (t_z - 1)[y,x]$, *where* $x * y = xyx^{-1}$,
(7) $[x_1 \cdot \ldots \cdot x_n, y_1 \cdot \ldots \cdot y_m] = \sum_{i=1}^{n} \sum_{j=1}^{m} T_{ij}[x_i, y_j]$, *where* $T_{ij} = \prod_{k=1}^{i-1} t_{x_k} \cdot \prod_{\ell=1}^{j-1} t_{y_\ell}$.
(8) Jacobi relations:

$$J(x,y,z) := (t_x - 1)[y,z] + (t_y - 1)[z,x] + (t_z - 1)[x,y] = 0,$$

where t_x *represents the image of* $x \in G$ *in* $\Lambda_G^{\mathbb{Z}}$.

Proof. Property (1) is obvious (it is even true in G). For (2) note that $xyx^{-1}y^{-1} = \left(yxy^{-1}x^{-1}\right)^{-1}$. Property (3) follows from $x^{-1}yxy^{-1} = x^{-1}\left(xyx^{-1}y^{-1}\right)^{-1}x$. To prove (4) note that $xpx^{-1}\,p^{-1} = t_x p - p$. Property (5) follows from $[xy,z] = x\left(yzy^{-1}z^{-1}\right)x^{-1}\left(xzx^{-1}z^{-1}\right)$. For (6),

$$\begin{aligned}[x*y,z] &= [[x,y]y,z] \overset{(5)}{=} [[x,y],z] + [y,z] \overset{(4)}{=} [y,z] - (t_z - 1)[x,y] \\ &\overset{(2)}{=} [y,z] + (t_z - 1)[y,x].\end{aligned}$$

Property (7) follows by induction and using property (2). Finally, for the Jacobi relations, note that on the one hand, by (5)

$$[xy,z] = [x,z] + t_x[y,z]. \tag{20}$$

On the other hand, $[xy,z] = [(x*y)x,z]$, then by properties (5) and (6), one has

$$[xy,z] = [(x*y)x,z] \overset{(5)}{=} [(x*y),z] + t_y[x,z] = [y,z] - (t_z - 1)[x,y] + t_y[x,z]. \tag{21}$$

The difference between (20) and (21) equals zero and the result follows. Q.E.D.

§3. Non-Abelian Branched coverings and Zariski pairs

In §2 we have mainly dealt with invariants associated with different sorts of Abelian coverings. In this section we will give an approach to invariants related to non-Abelian coverings. A more group-theoretical approach is given by the *Hall invariants* studied by Matei-Suciu [83] in relation with the Alexander invariant. The Hall invariant $\delta_\Gamma(G)$ of a group G associated with a finite group Γ is defined as the number of epimorphisms from G to Γ up to automorphisms of Γ. Matei-Suciu [83] prove that in the case of metabelian groups $\Gamma = (\mathbb{Z}/q\mathbb{Z})^s \rtimes \mathbb{Z}/p\mathbb{Z}$, where p and q are distinct primes and s is the order of $q \bmod p$ in $(\mathbb{Z}/p\mathbb{Z})^\times$, the Hall invariant $\delta_\Gamma(G)$ can be computed in terms of the characteristic varieties $\operatorname{Char}^{\mathbb{F}_{q^s}}(G)$.

Our approach here is more algebraic, in the sense that we ask ourselves whether or not there are any algebraic conditions on the singular points of a curve $\mathcal{C}$ that can characterize the existence of certain metabelian coverings (in this case dihedral coverings). *A posteriori* torsion points in $\operatorname{Char}^{\mathbb{F}_{q^s}}(\mathcal{C})$ have an algebraic interpretation in terms of position of singularities.

3.1. Preliminaries

Let X and Y be normal varieties. We call X a *(branched) covering of* Y if there exists a finite surjective morphism $\pi : X \to Y$. When needed, the covering morphism will be specified as *a covering* $\pi : X \to Y$.

Let $\pi : X \to Y$ be a covering. The corresponding rational function fields will be denoted by $\mathbb{C}(X)$ and $\mathbb{C}(Y)$, respectively. Note that $\mathbb{C}(X)$ is an algebraic extension of $\mathbb{C}(Y)$ and $\deg \pi = [\mathbb{C}(X) : \mathbb{C}(Y)]$ (see e.g. [86, p. 46, Proposition 3.17]).

Definition 3.1. Let X, Y and $\pi : X \to Y$ be as above.

(i) We call X a *Galois covering* of Y if the field extension is Galois.

(ii) Let G be a finite group. We call X a *G-covering* if X is a Galois covering of Y with $\operatorname{Gal}(\mathbb{C}(X)/\mathbb{C}(Y)) \cong G$.

We say that $x \in X$ is a *ramification point* of π if $\pi^*\mathfrak{m}_{Y,f(x)}\mathcal{O}_{X,x} \neq \mathfrak{m}_{X,x}$, where $\mathfrak{m}_{X,x}$ and $\mathfrak{m}_{Y,f(x)}$ are the maximal ideals of $\mathcal{O}_{X,x}$ and $\mathcal{O}_{Y,f(x)}$, respectively. Geometrically, this means that π is not a local isomorphism around x. The set of all ramification points will be denoted by R_π. Its image $\pi(R_\pi)$ is the branch locus of π and will be denoted by Δ_π or $\Delta(X/Y)$. By the purity of the branch locus [133], if Y is smooth then Δ_π is an algebraic subset of pure codimension 1.

When we apply the algebraic theory of branched coverings to the study of Zariski pairs, we consider their associated analytic spaces. Here we summarize some results from algebraic geometry and analytic geometry which will be needed later.

Let Y be a normal algebraic variety over $\mathbb{C}$. We denote by $Y^{\rm an}$ its associated analytic space. The following statements are key in relating branched coverings of Y with those of $Y^{\rm an}$.

Theorem 3.2. *Let Y be a proper normal variety over $\mathbb{C}$. Let $\mathcal{X}$ be a normal complex analytic space and let $f : \mathcal{X} \to Y^{\rm an}$ be a proper* *morphism with finite fiber. Then there exists a unique normal variety X (up to isomorphism over Y) and a finite morphism $\pi : X \to Y$ such $X^{\mathrm{an}} \cong \mathcal{X}$ and $\pi^{\mathrm{an}} = f$ (up to isomorphism between X^{an} and $\mathcal{X}$).*

For a proof, see [52, EXPOSÉ XII, Corollaire 4.6]. The following theorem can also be found in [52, EXPOSÉ XII] or [51, Theorem 5.4].

Theorem of Grauert-Remmert 3.3. *Let $\mathcal{Y}$ be a normal analytic space and let B be a closed analytic subset of codimension 1. Let $\pi_o : U \to \mathcal{Y} \setminus B$ be an étale finite covering of $\mathcal{Y} \setminus B$. Then there exist a normal analytic space $\mathcal{X}$ containing U and a finite surjective morphism $\pi : \mathcal{X} \to \mathcal{Y}$ such that $\pi^{-1}(\mathcal{Y} \setminus B) = U$ and $\pi|_U = \pi_o$. Moreover $\mathcal{X}$ is unique up to isomorphism over $\mathcal{Y}$.*

Notation 3.4. Let $\pi : X \to Y$ be a G-covering of a smooth projective variety Y. Let B be a reduced divisor on Y whose irreducible decomposition will be denoted by $B = B_1 + \cdots + B_r$. Given a morphism $\sigma : X \to Y$ between smooth projective varieties and a divisor D in Y, $\sigma^*(D)$ (resp. $\sigma^{-1}(D)$) will denote its inverse image as a divisor (resp. as a set). Its strict transform will be denoted by $\sigma_q^{-1}(D)$.

Definition 3.5. A covering π is said to be *branched at* $e_1B_1 + \cdots + e_rB_r$ ($e_i \geq 2$) if

- $\Delta_\pi = B$ and
- the ramification index along the smooth part of B_i^{an} is e_i. Namely, for any smooth point $y \in B_i^{\mathrm{an}}$ and $x \in (\pi^{\mathrm{an}})^{-1}(y)$, there exist neighborhoods U_x and V_y, respectively, such that π^{an} is locally given by

$$(z_1, \ldots, z_n) \mapsto (w_1, \ldots, w_n) = (z_1^{e_i}, \ldots, z_n),$$

  where $(z_1, \ldots, z_n)$ and $(w_1, \ldots, w_n)$ denote local coordinates on U_x and U_y, respectively, such that $x = (0, \ldots, 0), y = (0, \ldots, 0)$ and $B_i^{\mathrm{an}} \cap V_y$ is given by $w_1 = 0$.

Let γ_i be a meridian around B_i as in Figure 1, and $[\gamma_i]$ denote its class in $\pi_1(Y^{\mathrm{an}} \setminus B^{\mathrm{an}}, p_o)$.

Proposition 3.6. *Let Y be a smooth projective variety and let $B = B_1 + \cdots + B_r$ be the decomposition into irreducible components of a reduced divisor B on Y. If there exists a G-covering $\pi : X \to Y$ branched at $e_1B_1 + \cdots + e_rB_r$, then there exists a normal subgroup H_π of $\pi_1(Y^{\mathrm{an}} \setminus B^{\mathrm{an}}, p_o)$ such that:*

(*i*) $[\gamma_i]^{e_i} \in H_\pi, [\gamma_i]^k \notin H_\pi, (1 \leq k \leq e_i - 1)$, *and*
(*ii*) $\pi_1(Y^{\mathrm{an}} \setminus B^{\mathrm{an}}, p_o)/H_\pi \cong G$.

Conversely, if there exists a normal subgroup H of $\pi_1(Y^{\mathrm{an}} \setminus B^{\mathrm{an}}, p_o)$ satisfying the above two conditions for H_π, then there exists a G-covering $\pi_H : X_H \to Y$ branched at $e_1 B_1 + \cdots + e_r B_r$.

Proof. Since G acts on X such that $Y = X/G$ ([114]), G also acts on X^{an} ([52, §3]) transitively on each fiber $(\pi^{\mathrm{an}})^{-1}(y)$, $y \in Y$. Hence $X^{\mathrm{an}}/G = Y^{\mathrm{an}}$. Since $X^{\mathrm{an}} \setminus (\pi^{\mathrm{an}})^{-1}(B^{\mathrm{an}}) \to Y^{\mathrm{an}} \setminus B^{\mathrm{an}}$ is étale, our statement easily follows from the standard theory of covering spaces.

Conversely, let H be the normal subgroup in the statement. Let $\mathcal{X}'_H$ be an étale covering of $Y^{\mathrm{an}} \setminus B^{\mathrm{an}}$ corresponding to H. By Theorem 3.3, there exists a normal analytic space $\mathcal{X}_H$ and a finite morphism $\pi_H^{\mathrm{an}} : \mathcal{X}_H \to Y^{\mathrm{an}}$ extending the covering morphism $\mathcal{X}'_H \to Y^{an} \setminus B^{an}$. Since $G \cong \pi_1(Y^{\mathrm{an}} \setminus B^{\mathrm{an}}, p_o)/H$ acts on $\mathcal{X}'_H$ so that $\mathcal{X}'_H/G = Y^{\mathrm{an}} \setminus B^{\mathrm{an}}$, G also acts on $\mathcal{X}_H$ ([52, Proposition 5.3]). Hence one has a morphism $\mathcal{X}_H/H \to Y$, which is finite and an isomorphism on $Y^{\mathrm{an}} \setminus B^{\mathrm{an}}$. By Zariski's main theorem (e.g., see [122, Theorem 1.11]), $\mathcal{X}_H/G \cong Y$.

By Theorem 3.2, there exists a normal variety X_H and a finite morphism $\pi_H : X_H \to Y$. Since G acts on $\mathcal{X}_H$, it also acts on X_H over Y. This implies that $G \subset \mathrm{Aut}_{\mathbb{C}(Y)}(\mathbb{C}(X))$. Since $\deg \pi_H = \#G$, X_H is a G-covering of Y and thus the statement on the ramification index follows from how we extend $\mathcal{X}'_H$ to $\mathcal{X}_H$ along B_i. Q.E.D.

Remark 3.7. We recall some facts on Galois theory of Galois coverings. Let Y be a normal algebraic variety. Let K be a finite extension of $\mathbb{C}(Y)$ and let X_K be the normalization of Y in K called the *"K-normalization of Y"*. There exists a canonical finite surjective morphism $\pi_K : X_K \to Y$. Hence X_K is a covering of Y with $\mathbb{C}(X_K) = K$. If K is a Galois extension, then $\pi_K : X_K \to Y$ is a Galois covering. Conversely, note that any covering $\pi : X \to Y$ defines a finite field extension $\mathbb{C}(X)$ of $\mathbb{C}(Y)$.

Let G be a finite group and let H be a normal subgroup. Consider a Galois extension K of $\mathbb{C}(Y)$ with $\mathrm{Gal}(K/\mathbb{C}(Y)) \cong G$.

Let $\pi : X \to Y$ be a G-covering corresponding the extension $K/\mathbb{C}(Y)$. Let K^H be the fixed field by H. The field K^H is also a Galois extension of $\mathbb{C}(Y)$ with $\mathrm{Gal}(K^H/\mathbb{C}(Y)) \cong G/H$. Let $D_H(X/Y)$ be the K^H-normalization of $\mathbb{C}(Y)$. Since K/K^H is an H-extension and $K^H/\mathbb{C}(Y)$ is a G/H-extension, X_K is an H-covering of $D_H(X/Y)$ and $D_H(X/Y)$ is a G/H-covering of Y. The corresponding covering morphisms will be denoted by

$$\beta_{1,H}(\pi) : D_H(X/Y) \to Y, \quad \text{and} \quad \beta_{2,H}(\pi) : X \to D_H(X/Y). \tag{22}$$

Note that $\pi = \beta_{1,H}(\pi) \circ \beta_{2,H}(\pi)$.

3.2. Dihedral coverings

Let D_{2n} denote the dihedral group of order $2n$. The following presentation of D_{2n} will be extensively used throughout this section $\langle \sigma, \tau \mid \sigma^2 = \tau^n = (\sigma\tau)^2 = 1 \rangle$.

Remark 3.8. Since we consider non-Abelian branched coverings, we will always assume that $\#D_{2n} \geq 6$.

Following the notation introduced in §3.1, we will study the case $G = D_{2n}$ and $H = \langle \tau \rangle$. Since there is no ambiguity for H, we will use notations $D(X/Y)$, $\beta_1(\pi)$ and $\beta_2(\pi)$ for simplicity. The notion of *generic* and *non-generic* D_{2n}-coverings will be key in our arguments.

Definition 3.9. A D_{2n}-covering $\pi : X \to Y$ of a smooth variety Y is said to be *generic* if $\Delta_\pi = \Delta_{\beta_1(\pi)}$, otherwise π is said to be *non-generic*.

Note that, if $\pi : X \to Y$ is a D_{2n}-covering, then $D(X/Y)$ is a double covering of Y and X is an n-cyclic covering of $D(X/Y)$, whose morphisms will be denoted by $\beta_1(\pi)$ and $\beta_2(\pi)$ respectively, as in (22).

Remark 3.10. In what follows, Y will be assumed to be smooth and simply connected. Also note that $\sim$ will denote linear equivalence of divisors.

Let us start with a sufficient condition for the existence of D_{2n}-coverings.

Proposition 3.11. *Let Z be a smooth double covering of Y with covering morphism $f : Z \to Y$ and D be an effective divisor on Z such that:*

- (i) *D and $\sigma^* D$ have no common component, where σ denotes the covering transformation,*
- (ii) *if $D = \sum_{i=1}^h a_i D_i$ is the irreducible decomposition, then for all $i = 1, \ldots, h$, $0 < a_i$ and $\gcd\{a_1, \ldots, a_h, n\} = 1$, and*
- (iii) *there exists a line bundle $\mathcal{L}$ on Z such that $D - \sigma^* D \sim n\mathcal{L}$.*

Then there exists a D_{2n}-covering X of Y such that

- (a) *$D(X/Y) = Z$,*
- (b) *the branch locus $\Delta_{\beta_2(\pi)}$ of $\beta_2(\pi)$ is contained in* $\mathrm{Supp}(D + \sigma^* D)$*, i.e., $\Delta(X/Y) \subset \Delta_{\beta_1(\pi)} \cup f(\mathrm{Supp}(D))$ and*
- (c) *if $D_i \subset \Delta_{\beta_2(\pi)}$, then the ramification index along D_i is $\dfrac{n}{\gcd(n, a_i)}$.*

Proof. For n odd, our statement is a special case of [117, Proposition 1.1], except for part (c), which follows from the proof of [117, Proposition 0.4]. For n even, a similar proof to that of [117, Proposition 1.1] also works by [117, Remark 3.1, Proposition 0.6]. Q.E.D.

As for a necessary condition for the existence of D_{2n}-coverings, one has the following:

Proposition 3.12 ([112, §2, §3]). *Let $\pi : X \to Y$ be a D_{2n}-covering such that $D(X/Y)$ is smooth. Let us denote by σ the covering transformation of $\beta_1(\pi)$. Then there exist (possibly empty) effective divisors, D_1 and D_2, and a line bundle $\mathcal{L}$ on $D(X/Y)$ satisfying the following conditions:*

- (i) *D_1 and $\sigma^* D_1$ have no common components. Moreover, if we denote its irreducible decomposition by $\sum_j a_j D_{1,j}$, then $0 \leq a_j < n$.*
- (ii) *If $D_2 \neq \emptyset$, then n is even and D_2 is a reduced divisor such that there exists a divisor B_2 on Y satisfying $D_2 = f^* B_2$.*
- (iii) *$D_1 + \frac{n}{2} D_2 - \sigma^* D_1 \sim n\mathcal{L}$.*
- (iv) *$\Delta(X/D(X/Y)) = \mathrm{Supp}(D_1 + \sigma^* D_1 + D_2)$. The ramification index along $D_{1,j}$ (resp. an irreducible component of D_2) is $\dfrac{n}{\gcd(a_i, n)}$ (resp. 2).*

Corollary 3.13. *Let D be an irreducible component of $\beta_1(\pi)(\Delta_{\beta_2(\pi)})$. If the ramification index of $\beta_2(\pi)$ along $\beta_1(\pi)^{-1}(D)$ is > 2, then the divisor $\beta_1(\pi)^* D$ is of the form $D' + \sigma^* D'$ for some irreducible divisor D' on $D(X/Y)$ such that $D' \neq \sigma^* D'$.*

3.3. Zariski's example and D_6-coverings

Let us review Zariski's example of sextics with six cusps using D_6-coverings as in Zariski's original proof. Our purpose is to give a detailed proof in modern language of (Z5). Let us start with the following Lemma.

Lemma 3.14. *Let B be a sextic with 6 cusps such that a D_6-covering $\pi : S \to \mathbb{P}^2$ with $\Delta(S/\mathbb{P}^2) = B$ exists. Then the following statements hold:*

- (i) *$\beta_1(\pi) : D(S/\mathbb{P}^2) \to \mathbb{P}^2$ is a double covering branched at $2B$.*
- (ii) *The branch locus of $\beta_2(\pi) : S \to D(S/\mathbb{P}^2)$ is contained in $\mathrm{Sing}(D(S/\mathbb{P}^2))$ and S is smooth.*

Proof. For (i), since $\mathbb{P}^2$ is simply connected one has $\Delta_{\beta_1(\pi)} \neq \emptyset$. This means $\Delta_{\beta_1(\pi)} = B$. For (ii), we first show that $\Delta_{\beta_2(\pi)} \subset \mathrm{Sing}(D(S/\mathbb{P}^2))$ and $\Delta_{\beta_2(\pi)} \neq \emptyset$. Since D_6 has no element of order 6, $\beta_2(\pi)$ cannot be branched along $\beta_1(\pi)^{-1}(B)$. This means that $\Delta_{\beta_2(\pi)} \subset \mathrm{Sing}(D(S/\mathbb{P}^2))$. Let $\mu : Z \to D(S/\mathbb{P}^2)$ be the minimal resolution. Since $K_Z \sim 0$ and the irregularity $q = 0$ by a general theory of double coverings (see [58, §2]) one has that Z is a $K3$-surface. In particular, Z is simply connected. If $\beta_2(\pi) : S \to D(S/\mathbb{P}^2)$ is unramified, then $Z \times_{D(S/\mathbb{P}^2)} S$ gives an étale cyclic triple covering of Z, but this is impossible. Hence $\Delta_{\beta_2(\pi)} \neq \emptyset$. By [118, Lemma 8.8], $\Delta_{\beta_2(\pi)} = \mathrm{Sing}(D(S/\mathbb{P}^2))$. Finally, the local structure of π around a cusp of B (described in [111, §2, Example 3]) forces S to be smooth. Q.E.D.

Lemma 3.15. *Let B be a sextic with 6 cusps. If a D_6-covering $\pi : S \to \mathbb{P}^2$ branched at $2B$ exists, then*

- (i) *the quotient surface $X := S/\langle\sigma\rangle$ is smooth for any element of $\sigma \in D_6$ of order 2,*
- (ii) *$K_X \sim -\overline{\pi}^* l$, where $\overline{\pi} : X \to \mathbb{P}^2$ denotes the induced non-Galois triple covering and l denotes a line of $\mathbb{P}^2$, and*
- (iii) *X is a del-Pezzo surface of degree 3.*

Proof. For (i), due to the local structure of $\pi : S \to \mathbb{P}^2$ around each cusp of B ([112, Example 3, §2]), $\pi^* B$ is of the form $2(R_1 + R_2 + R_3)$, where R_i is a smooth divisor such that τ acts on the set $\{R_1, R_2, R_3\}$ transitively. One may assume that the fixed locus of σ is R_1, and this implies that X is smooth.
For (ii) and (iii), note that S is a $K3$-surface. Let us assume that the ramification locus of the quotient morphism $\alpha : S \to X$ is R_1. Since $R_1^2 = 6$ and R_1 is smooth, R_1 is numerically effective by [19, Proposition VIII 13]. Also note that $0 \sim K_S \sim \alpha^* K_X + R_1$, and hence $\alpha^* K_X \sim -R_1$. Thus $-K_X$ is numerically effective and $K_X^2 = 3$. This implies that X is a rational surface.
Now choose a general point x of $\mathbb{P}^2$. Let $\rho : \hat{X} \to X$ be the composition of the blowing-ups at the three points of $\overline{\pi}^{-1}(x)$. The pencil of lines through x on $\mathbb{P}^2$ induces an elliptic fibration $\varphi_x : \hat{X} \to \mathbb{P}^1$ and the three exceptional curves of ρ give sections of φ_x.

In order to complete the proof we need the following result.

Claim 3.16. $\varphi_x : \hat{X} \to \mathbb{P}^1$ *is relatively minimal.*

Proof of Claim. Since $K_{\hat{X}}^2 = 0$ and $\hat{X}$ is a rational surface, the topological Euler number of $\hat{X}$ is 12. This implies that φ_x is relatively minimal. Q.E.D.

By this Claim, $K_{\hat{X}} \sim -F$, F being a fiber of φ_x. By our construction of $\hat{X}$, $\rho^*(\overline{\pi}^* l) \sim F + E_1 + E_2 + E_3$, where E_i $(i = 1, 2, 3)$ denote the exceptional curves of ρ. Hence

$$\rho^*(\overline{\pi}^* l) \sim F + E_1 + E_2 + E_3 \sim -K_{\hat{X}} + E_1 + E_2 + E_3 \sim -\rho^*(K_X).$$

Therefore $\overline{\pi}^* l \sim -K_X$. In particular, $-K_X$ is ample and thus X is a del-Pezzo surface of degree 3. Q.E.D.

We are now in a position to prove the following:

Proposition 3.17. *Let B be a sextic with 6 cusps. Then there exists a D_6-covering branched at $2B$ if and only if B is given by an equation of the form $F^3 + G^2 = 0$, where $F(X_0, X_1, X_2)$ and $G(X_0, X_1, X_2)$ are homogeneous polynomials of degree 2 and 3, respectively.*

Proof. Suppose that B is given by the equation $F^3 + G^2 = 0$ as above. Consider the cubic surface X in $\mathbb{P}^3$ given by

$$X : X_3^3 + 3F(X_0, X_1, X_2)X_3 + 2G(X_0, X_1, X_2) = 0,$$

where $[X_0 : X_1 : X_2 : X_3]$ denotes a homogeneous coordinate system of $\mathbb{P}^3$. Let $P = [0 : 0 : 0 : 1]$ and let $pr_P : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ be the projection centered at P. The restriction pr_P to X gives a non-Galois triple covering $pr_{P|X} : X \to \mathbb{P}^2$. By its defining equation, $\Delta(X/\mathbb{P}^2) = B$. The Galois closure K of $\mathbb{C}(X)$ is a D_6-extension of $\mathbb{C}(\mathbb{P}^2)$ and the K-normalization S of $\mathbb{P}^2$ is a D_6-covering $\pi : S \to \mathbb{P}^2$. By [111, Lemma 1.4], $\Delta(S/\mathbb{P}^2) = \Delta(X/\mathbb{P}^2) = B$ and by Lemma 3.14, π is branched at $2B$. These arguments follow Zariski's original idea.

The converse is the less detailed part in [130]. Suppose that there exists a D_6-covering $\pi : S \to \mathbb{P}^2$ branched at $2B$. Let $\overline{\pi} : X \to \mathbb{P}^2$ be a non-Galois triple covering as in Lemma 3.15. By [111, Lemma 1.4], $\Delta(S/\mathbb{P}^2) = \Delta(X/\mathbb{P}^2) = B$ and X is a del-Pezzo surface of degree 3 according to Lemma 3.15. Hence X is embedded as a cubic hypersurface in $\mathbb{P}^3$ and its embedding is given by $\phi_{|-K_X|}$. Moreover, since $K_X \sim -\overline{\pi}^* l$ by Lemma 3.15, one has the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{\phi_{|-K_X|}} & \mathbb{P}^3 \setminus \{P_0\} \\ {\scriptstyle\overline{\pi}}\big\downarrow & & \big\downarrow{\scriptstyle pr} \\ \mathbb{P}^2 & = & \mathbb{P}^2, \end{array}$$

where pr denotes the projection centered at a suitable point $P_0 \in \mathbb{P}^3 \setminus \phi_{|-K_X|}(X)$. By choosing homogeneous coordinates $[X_0 : X_1 : X_2 : X_3]$

appropriately, one may assume that $P_0 = [0 : 0 : 0 : 1]$. This implies that pr is given by

$$[X_0 : X_1 : X_2 : X_3] \mapsto [X_0 : X_1 : X_2],$$

and that $\phi_{|-K_X|}(X)$ is given by the equation

$$X_3^3 + g_1(X_0, X_1, X_2)X_3^2 + g_2(X_0, X_1, X_2)X_3 + g_3(X_0, X_1, X_2) = 0,$$

where $g_i(X_0, X_1, X_2)$ are homogeneous polynomials of degree i. Now the defining equation of B is given by the discriminant of the above cubic equation which is

$$-4\left(g_2 - \frac{1}{3}g_1^2\right)^3 - 27\left(-\frac{1}{3}g_1g_2 + g_3 + \frac{2}{27}g_1^3\right)^2.$$

Q.E.D.

Remark 3.18. Any sextic given by an equation of the form $F^3 + G^2 = 0$, where F and G are homogeneous polynomials of degrees 2 and 3, respectively, is called a $(2,3)$-*torus sextic*. Zariski pairs of sextics given by $(2,3)$-torus and non-torus sextics are extensively studied by Oka [95, 96]. He uses Alexander polynomials to distinguish the topology of the complements. It may be interesting to revisit his proofs using the geometry of cubic surfaces.

3.4. Generic D_{2n}-coverings and Zariski pairs

In this section, an application of generic D_{2n}-coverings for the study of Zariski pairs will be shown. This method was used in [12, 15, 113, 115, 116]. Let Σ be a smooth projective surface and let B be a reduced divisor on Σ.

Remark 3.19. Thoughout this section, Σ is assumed to be simply connected.

Our purpose is to answer the following question:

Question 3.20. *Are there necessary and sufficient algebraic conditions on B for the existence of generic D_{2n}-coverings with $\Delta_\pi = B$?*

Suppose that Question 3.20 has a positive answer and let (P) be such a condition. The existence of a pair (B_1, B_2) of reduced divisors on Σ such that B_1 satisfies (P), while B_2 does not, implies that $(\Sigma, B_1) \not\approx (\Sigma, B_2)$. Hence if $\Sigma = \mathbb{P}^2$ and the combinatorial data of B_1 and B_2 are the same, (B_1, B_2) is a Zariski pair.

Now let us consider Question 3.20 in the case of n odd. The existence of a double covering $f' : Z' \to \Sigma$ with $\Delta_{f'} = B$ will always be assumed. Let

$$\begin{array}{ccc} Z' & \xleftarrow[\mu]{} & Z \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \Sigma & \xleftarrow[\rho]{} & W. \end{array}$$

denote the canonical resolution of Z' (see [58] for a definition).

Lemma 3.21. *Let $\pi : S \to \Sigma$ be a generic D_{2n}-covering with $\Delta_\pi = B$ (n is not necessarily odd in this lemma). Then*

(i) $D(S/\Sigma) \cong Z'$ *over* Σ *and*

(ii) $\Delta_{\beta_2(\pi)} \subset \mathrm{Sing}(D(S/\Sigma))$.

Proof. By hypothesis, the branch locus of $\beta_1(\pi) : D(S/\Sigma) \to \Sigma$ is B. Since Σ is simply connected, any double covering of Σ is determined by its branch locus up to isomorphism over Σ. This implies (i). The statement (ii) is immediate by hypothesis. Q.E.D.

Suppose that a generic D_{2n}-covering $\pi : S \to \Sigma$ with $\Delta_\pi = B$ exists. Let $\tilde{S}$ be the $\mathbb{C}(S)$-normalization of W and let $\tilde{\mu} : \tilde{S} \to S$ be an induced morphism. The induced covering morphism from $\tilde{S}$ to W will be denoted by $\tilde{\pi}$. Note that $\tilde{S}$ is again a D_{2n}-covering and one may assume that $D(\tilde{S}/W) = Z$, since $\mathbb{C}(Z') = \mathbb{C}(Z)$ and the $\mathbb{C}(S)$-normalization of Z is also $\tilde{S}$. Thus one has the following commutative diagram:

$$\begin{array}{ccc} S & \xleftarrow[\tilde{\mu}]{} & \tilde{S} \\ {\scriptstyle \beta_2(\pi)}\downarrow & & \downarrow{\scriptstyle \beta_2(\tilde{\pi})} \\ Z' & \xleftarrow[\mu]{} & Z \\ {\scriptstyle f'=\beta_1(\pi)}\downarrow & & \downarrow{\scriptstyle f=\beta_1(\tilde{\pi})} \\ \Sigma & \xleftarrow[\rho]{} & W. \end{array}$$

By Lemma 3.21(ii), $\beta_2(\tilde{\pi}) : \tilde{S} \to Z$ is an n-cyclic covering whose branch locus is contained in the exceptional locus of μ.

Conversely, suppose that there exists an n-cyclic covering $g : X \to Z$ such that:

(D1) Δ_g is contained in the exceptional locus of μ and

(D2) the composition $f \circ g : X \to W$ gives rise to a D_{2n}-covering.

The Stein factorization of $\rho \circ f \circ g : X \to \Sigma$ gives a generic D_{2n}-covering of Σ with $\Delta(X/\Sigma) = B$. Thus Question 3.20 is reduced to the following:

Question 3.22. *Find a sufficient and necessary condition for the existence of an n-cyclic covering $g : X \to Z$ satisfying* (D1) *and* (D2) *above.*

As usual, let us denote by σ the covering transformation of the double covering f. Propositions 3.11 and 3.12 provide a partial answer to Question 3.22 as follows:

Proposition 3.23. *Assume that Z is simply connected. A generic D_{2p}-covering (p odd prime) $\pi : S \to \Sigma$ with $\Delta_\pi = B$ exists if and only if there exist a non-empty effective divisor D and a line bundle $\mathcal{L}$ on Z satisfying the following conditions:*

- (*i*) *D and $\sigma^* D$ have no common components. Moreover, if $D = \sum_i a_i D_i$ denotes its irreducible decomposition, then* $\gcd(a_i, p) = 1$.
- (*ii*) $\operatorname{Supp}(D + \sigma^* D)$ *is contained in the exceptional set of μ.*
- (*iii*) *$D - \sigma^* D \sim p\mathcal{L}$.*

Proof. As noted above, a generic D_{2n}-covering $\pi : S \to \Sigma$ with $\Delta_\pi = B$ exists if and only if there exists another D_{2n}-covering $\tilde{\pi} : \tilde{S} \to W$ satisfying conditions (D1)-(D2) above. Applying Propositions 3.11 and 3.12 to $\beta_2(\tilde{\pi}) = f : Z \to W$ the statement follows, since the branch locus of $\beta_2(\tilde{\pi})$ is non-empty and is contained in the exceptional set of μ.
Q.E.D.

From now on, we assume that singularities of B are at most simple singularities (see [20] for simple singularities). In this case, another version of Proposition 3.23 can be stated. In order to explain it, we need some preparation.

Let us keep the assumption in Proposition 3.23. Since Z is simply connected, $H^2(Z, \mathbb{Z})$ is a unimodular lattice with respect to the intersection pairing. Let $\mathrm{NS}(Z)$ be the Néron-Severi group of Z. It is a sublattice of $H^2(Z, \mathbb{Z})$ such that $H^2(Z, \mathbb{Z}) / \mathrm{NS}(Z)$ is torsion-free. Since Z is simply connected, the Picard group $\mathrm{Pic}(Z)$ coincides with $\mathrm{NS}(Z)$. For $x \in \operatorname{Sing}(Z')$, R_x denotes the subgroup of $\mathrm{NS}(Z)$ generated by the irreducible components of the exceptional set arising from x. The lattice R_x is a negative definite sublattice of $\mathrm{NS}(Z)$. One can define the following sublattice of $\mathrm{NS}(Z)$:

$$T := \bigoplus_{x \in \operatorname{Sing}(Z')} R_x.$$

The type of x	G_{R_x}
$\mathbb{A}_n$	$\mathbb{Z}/(n+1)\mathbb{Z}$
$\mathbb{D}_n$ $n \equiv 1 \bmod 2$	$\mathbb{Z}/4\mathbb{Z}$
$\mathbb{D}_n$ $n \equiv 0 \bmod 2$	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$
$\mathbb{E}_6$	$\mathbb{Z}/3\mathbb{Z}$
$\mathbb{E}_7$	$\mathbb{Z}/2\mathbb{Z}$
$\mathbb{E}_8$	$\{0\}$

Table 2.

Given a lattice L, its dual lattice will be denoted by $L^\vee$ and its quotient modulo L by $G_L := L^\vee/L$. Associated with T one has $G_T \cong \bigoplus_{x \in \mathrm{Sing}(Z')} G_{R_x}$. For a rational double point x, the results of Table 2 are well known.

One can consider both R_x and $R_x^\vee$ as subgroups of $R_x \otimes \mathbb{Q}$ and give a $\mathbb{Q}$-divisor which produces a generator of G_{R_x} in the cases of $x = \mathbb{E}_6$ and $\mathbb{A}_n$. This will come in handy for later use. For this purpose, let us label the irreducible components of the exceptional divisors for singularities of type $\mathbb{A}_n$ and $\mathbb{E}_6$ as in Figure 10. Note that $\sigma^*\Theta_k = \Theta_{n+1-k}$ if x is of type $\mathbb{A}_n$, and $\sigma^*\Theta_1 = \Theta_6$, $\sigma^*\Theta_2 = \Theta_5$ if x is of type $\mathbb{E}_6$.

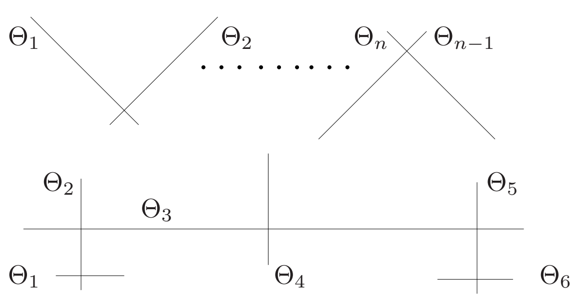

Fig. 10.

Lemma 3.24. *G_{R_x} is generated by the class of $\mathbb{Q}$-divisors $\dfrac{D_x}{n+1}$ (resp. $\dfrac{D_x}{3}$) for x of type $\mathbb{A}_n$ (resp. x of type $\mathbb{E}_6$), where*

$$D_x = \begin{cases} \displaystyle\sum_{k=1}^{\frac{n}{2}} (n+1-k)(\Theta_k - \Theta_{n+1-k}) & \text{if } x \text{ is of type } \mathbb{A}_n,\ n \text{ even} \\ \sum_{k=1}^{\frac{n-1}{2}} (n+1-k)(\Theta_k - \Theta_{n-k}) + \frac{n+1}{2}\Theta_{\frac{n+1}{2}} & \text{if } x \text{ is of type } \mathbb{A}_n,\ n \text{ odd,} \end{cases}$$

and

$$D_x = (\Theta_1 - \Theta_5) + 2(\Theta_2 - \Theta_6) \quad \text{if } x \text{ is of type } \mathbb{E}_6.$$

Proof. Our statement easily follows by considering the inverse of the intersection matrix of R_x. Q.E.D.

Let us now concentrate on the torsion part $(\mathrm{NS}(Z)/T)_{\mathrm{tor}}$ of $\mathrm{NS}(Z)/T$.

Lemma 3.25.

$$(\mathrm{NS}(Z)/T)_{\mathrm{tor}} \cong T^{\perp\perp}/T,$$

where $\bullet^{\perp}$ denotes the orthogonal complement in $H^2(Z,\mathbb{Z})$.

Proof. Since $H^2(Z,\mathbb{Z})/\mathrm{NS}(Z)$ is torsion free, one has $\mathrm{NS}(Z)^{\perp\perp} = \mathrm{NS}(Z)$. This implies that $T^{\perp\perp} \subset \mathrm{NS}(Z)$. Since $(\mathrm{NS}(Z)/T))_{\mathrm{tor}} \subset T^{\perp\perp}/T$, the result follows. Q.E.D.

Let ν be the homomorphism $T^{\perp\perp} \to T^{\vee} \to G_T$. Let L be an element in $T^{\perp\perp}$ such that its image in G_T has order p (p odd prime). Since $G_T \cong \bigoplus_{x\in\mathrm{Sing}(Z')} G_{R_x}$, one has

$$\nu(L) = (\gamma_x)_{x\in\mathrm{Sing}(Z')} \in \bigoplus_{x\in\mathrm{Sing}(Z')} G_{R_x}.$$

Note that $\gamma_x = 0$ unless x is a singular point of type $\mathbb{A}_n$ ($n+1 \equiv 0 \bmod p$) or type $\mathbb{E}_6$ (the latter case happens only when $p = 3$).

Lemma 3.26. *Assume that $\gamma_x \neq 0$. If x is a singular point of type $\mathbb{A}_n$ ($n+1 \equiv 0 \bmod p$), then*

$$\gamma_x = \text{ the class of } \tfrac{k_x}{p} D_x \qquad 0 < k_x \leq p-1,$$

and if x is a singular point of type $\mathbb{E}_6$, then $p = 3$ and

$$\gamma_x = \text{ the class of } \tfrac{k_x}{3} D_x \qquad k_x = 1, 2,$$

where D_x denotes the divisor in Lemma 3.24.

Proof. Since R_x is a cyclic group generated by the class described in Lemma 3.24, the result follows. Q.E.D.

Now we are in a position to state another version of Proposition 3.23.

Theorem 3.27. *A generic D_{2p}-covering (p odd prime) of Σ with $\Delta_\pi = B$ exists if and only if $\mathrm{NS}(Z)/T$ has a p-torsion element.*

Proof. Suppose that there exists a generic D_{2p}-covering with branch locus B. Then by Proposition 3.23, there exists a non-empty divisor D and a line bundle $\mathcal{L}$ satisfying the three conditions. One can show that $\mathcal{L}$ gives rise to a p-torsion element of $\mathrm{NS}(Z)/T$. By the condition (iii) of Proposition 3.23, $p\mathcal{L} \in T$. Hence it is enough to show that $\mathcal{L} \notin T$. Suppose that $\mathcal{L} \in T$. Hence $\mathcal{L} \sim \sum_{x\in \mathrm{Sing}(Z')} \sum_i m_{i,x}\Theta_{i,x}$, where $\Theta_{i,x}$ denotes the irreducible components of the exceptional set arising from x. By plugging this relation into the one given in condition (iii), one obtains a non-trivial linear relation among $\Theta_{i,x}$'s. This leads to contradiction.

For the converse, let us suppose that $\mathrm{NS}(Z)/T$ has a p-torsion element. By Lemma 3.25, there exists an element L_1 in $T^{\perp\perp}$ such that whose class in $T^{\perp\perp}/T$ has torsion p. By Lemma 3.26,

$$L_1 \sim_{\mathbb{Q}} \sum_{x\in \mathrm{Sing}(Z')} \frac{k_x}{p} D_x \bmod T,$$

where $\sim_{\mathbb{Q}}$ denotes $\mathbb{Q}$-linear equivalence of divisors. This implies that there exists an element L_2 in T such that

$$L_1 + L_2 \sim_{\mathbb{Q}} \sum_{x\in \mathrm{Sing}(Z'),\gamma_x\neq 0} \frac{k_x}{p} D_x.$$

Let us define a divisor D on Z as follows:

$$D = \sum_{x\in \mathrm{Sing}(Z')} D_x^+,$$

where D_x^+ is defined as:

- $D_x^+ = 0$, if $\gamma_x = 0$.
- $D_x^+ = k_x \left(\sum_{i=1}^{\frac{n}{2}} (n+1-i)\Theta_i\right)$, if $\gamma_x \neq 0$ and x is of type $\mathbb{A}_n$ (n even).
- $D_x^+ = k_x \left(\sum_{i=1}^{\frac{n-1}{2}} (n+1-i)\Theta_i\right)$, if $\gamma_x \neq 0$ and x is of type $\mathbb{A}_n$ (n odd).
- $D_x^+ = k_x(\Theta_1 + 2\Theta_2)$, if $\gamma_x \neq 0$ and x is of type $\mathbb{E}_6$.

By the definition of D,

$$D - \sigma^* D \sim p(L_1 + L_2 - L_3),$$

where

$$L_3 = \sum_{x=\mathbb{A}_{n_x}(n_x \equiv 1 \bmod 2), \gamma_x \neq 0} \frac{k_x(n_x+1)}{2p} \Theta_{\frac{n_x+1}{2}}.$$

Moreover, the greatest common divisor of the coefficients of the irreducible components of D and p is 1. Thus the pair $(D, \mathcal{L})$, $\mathcal{L} = L_1 + L_2 - L_3$, satisfies the conditions in Proposition 3.23. Q.E.D.

Example 3.28. Let us consider Zariski's example from the viewpoint of Theorem 3.27, where $\Sigma = \mathbb{P}^2$ and B is a sextic with 6 cusps. In this case, the double covering $f' : Z' \to \mathbb{P}^2$ with $\Delta_{f'} = B$ has 6 $\mathbb{A}_2$-singularities x_i $(i = 1, \ldots, 6)$ and its canonical resolution Z is a $K3$-surface. Hence Z is simply connected. Let us denote by $\Theta_{i,j}$ the exceptional curves arising at x_i $(j = 1, 2)$. By Proposition 3.17, a generic D_6-covering $\pi : S \to \mathbb{P}^2$ of $\mathbb{P}^2$ with $\Delta_\pi = B$ exists if and only if B is a $(2,3)$-torus curve.

If B is a $(2,3)$-torus curve, there exists a conic $\mathcal{Q}$ passing through the 6 cusps. One can show that $\mathcal{Q}$ gives rise to a 3-torsion element in $\mathrm{NS}(Z)/T$. Let $\rho_q^{-1}\mathcal{Q}$ be the proper transform of $\mathcal{Q}$ in W. Then $f^*(\rho_q^{-1}\mathcal{Q})$ is of the form $\mathcal{Q}^+ + \mathcal{Q}^-$. After relabeling $\Theta_{i,j}$ if necessary, we may assume that $\mathcal{Q}^+ \cdot \Theta_{i,1} = 1, \mathcal{Q}^- \cdot \Theta_{i,2} = 0$ $(i = 1, \ldots, 6)$.

Claim 3.29. $3\mathcal{Q}^+ \sim 3\tilde{f}^* l - \sum_{i=1}^{6}(2\Theta_{i,1} + \Theta_{i,2})$*, where* l *denotes a generic line in* $\mathbb{P}^2$ *and* $\tilde{f} = \rho \circ f$*.*

Proof of Claim. Let us consider

$$D := 3\mathcal{Q}^+ - 3\tilde{f}^* l + \sum_{i=1}^{6}(2\Theta_{i,1} + \Theta_{i,2}).$$

One can see that $(\tilde{f}^* l) \cdot D = 0$ and $D^2 = 0$. By the Hodge index theorem, one has that $D \approx 0$. Since Z is simply connected, $D \sim 0$. Q.E.D.

By the Claim, note that the class of $\mathcal{Q}^+ - \tilde{f}^* l$ in $\mathrm{NS}(Z)/T$ gives a 3-torsion element. On the other hand, if B is not a $(2,3)$-torus curve, no D_6-covering branched at $2B$ exists. Hence, $\mathrm{NS}(Z)/T$ has no 3-torsion.

Remark 3.30.

(i) When using Theorem 3.27, we often replace T by $M \oplus T$, where M is a sublattice of $\mathrm{NS}(Z)$ orthogonal to T and such that $p \not| \operatorname{disc} M$, $\operatorname{disc} \bullet$ being the discriminant of a lattice $\bullet$. For example, $M = \tilde{f}^* \mathrm{NS}(\Sigma)$ in case $p \not| \operatorname{disc} \mathrm{NS}(\Sigma)$.

(ii) By Theorem 3.27, the problem of the existence of generic D_{2p}-coverings of Σ with $\Delta_\pi = B$ is reduced to that of primitive and non-primitive embeddings of T into $\mathrm{NS}(Z)$. In the case when $\Sigma = \mathbb{P}^2$ and B is a sextic with at most simple singularities, Z is a $K3$-surface. In this case, using Nikulin's lattice theory and the surjectivity of the period map, more detailed results than the existence of generic D_{2p}-coverings are obtained in [39].

3.5. Non-generic D_{2n}-coverings and Zariski k-plets

In this section, we will consider non-generic D_{2n}-coverings and their application to Zariski k-plets.

Let $\mathbf{B} = \mathbf{B}_1 + \mathbf{B}_2$ be a reduced divisor such that:

(i) there exists a double covering $f' : Z' \to \Sigma$ with $\Delta_{f'} = \mathbf{B}_1$, and

(ii) $\mathbf{B}_2$ is irreducible.

Let

$$\begin{array}{ccc} Z' & \xleftarrow[\mu]{} & Z \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \Sigma & \xleftarrow[\rho]{} & W \end{array}$$

be the canonical resolution of Z'.

Proposition 3.31. *Suppose that Σ is simply connected and the divisor $f^*(\rho_q^{-1}\mathbf{B}_2)$ consists of two distinct irreducible components $\mathbf{B}_2^+$ and $\mathbf{B}_2^-$. Assume also that there exist both an effective divisor D and a line bundle $\mathcal{L}$ on Z satisfying:*

(i) $D = \mathbf{B}_2^+ + D'$*. D' and σ^*D' have no common components,*

(ii) $\mathrm{Supp}(D' + \sigma^*D')$ *is contained in the exceptional set of μ, and*

(iii) $D - \sigma^*D \sim n\mathcal{L}$.

Then there exists a D_{2n}-covering $\pi : S \to \Sigma$ branched at $2\mathbf{B}_1 + n\mathbf{B}_2$ such that $\Delta_{\beta_1(\pi)} = \mathbf{B}_1$.

Proof. By Proposition 3.11, there exists a D_{2n}-covering $\tilde{\pi} : \tilde{S} \to W$ such that $\beta_1(\tilde{\pi}) = f$, $D(\tilde{S}/W) = Z$, $\mathbf{B}_2^+ \cup \mathbf{B}_2^- \subset \Delta_{\beta_2(\pi)} \subset \mathrm{Supp}(D + \sigma^*D)$ and whose ramification index along $\mathbf{B}_2^{\pm}$ is n. Since the irreducible components of D' are in the exceptional set of μ, the Stein factorization of $\rho \circ \tilde{\pi}$ gives the desired D_{2n}-covering. Q.E.D.

Proposition 3.32. *Under the notation above, if a D_{2n}-covering branched at $2\mathbf{B}_1 + n\mathbf{B}_2$ with $\Delta_{\beta_1(\pi)} = \mathbf{B}_1$ exists, then the following holds:*

(i) $f^*(\rho_q^{-1}\mathbf{B}_2)$ *consists of two irreducible components,* $\mathbf{B}_2^{\pm}$*,*

(ii) *there exist effective divisors* D_1 *and* D_2*, and a line bundle* $\mathcal{L}$ *on* Z *such that*

- $\mathrm{Supp}(D_1 + \sigma^* D_1 + D_2)$ *is contained in the exceptional set of* μ*,*
- D_1 *and* $\sigma^* D_1$ *have no common components,*
- *if* $D_2 \neq \emptyset$*, then* n *is even,* D_2 *is reduced, and* $D' = \sigma^* D'$ *for each irreducible component* D' *of* D_2*, and*
- $(\mathbf{B}_2^+ + D_1 + \frac{n}{2} D_2) - (\mathbf{B}_2^- + \sigma^* D_1) \sim n\mathcal{L}$*.*

Proof. Let us denote by $\pi : S \to \Sigma$ the D_{2n}-covering given by hypothesis. Let $\tilde{S}$ be the $\mathbb{C}(S)$-normalization of W. The induced morphism $\tilde{\pi} : \tilde{S} \to W$ is a D_{2n}-covering with $D(\tilde{S}/W) = Z$ and $\Delta_{\beta_2(\tilde{\pi})}$ is contained in the union of $\mathrm{Supp}(f^*(\rho_q^{-1}\mathbf{B}_2))$ with the exceptional subset of μ. Since $f^*(\rho_q^{-1}\mathbf{B}_2)$ is a part of $\Delta_{\beta_2(\tilde{\pi})}$, by Corollary 3.13, $f^*(\rho_q^{-1}\mathbf{B}_2)$ is of the form $\mathbf{B}_2^+ + \mathbf{B}_2^-$, which implies part ($i$). For part ($ii$) let $\tilde{D}_1$, $\tilde{D}_2$, and $\tilde{\mathcal{L}}$ be the two effective divisors and the line bundle on Z respectively, given by Proposition 3.12 applied to $\tilde{\pi} : \tilde{S} \to W$. Then by hypothesis, $\tilde{D}_1$ is of the form $a\mathbf{B}^+ + D_1'$. Moreover, $\mathrm{Supp}(D_1' + \sigma^* D_1' + \tilde{D}_2)$ is contained in the exceptional set of μ. By the assumption on the ramification index along $\mathbf{B}_2$, one has that $\gcd(a, n) = 1$ and there exists an integer a' $(0 < a < n)$ such that $aa' \equiv 1 \bmod n$. Note that a' is odd if n is even, therefore

$$a'(\tilde{D}_1 + \frac{n}{2}\tilde{D}_2) = \mathbf{B}_2^+ + aD_1' + \frac{n}{2}\tilde{D}_2 + nM,$$

for some effective divisor M. Hence,

$$(\mathbf{B}_2^+ + a'D_1' + \frac{n}{2}\tilde{D}_2) + (\mathbf{B}_2^- + a'\sigma^* D_1') \sim n(a'\mathcal{L} + \sigma^* M - M).$$

The result follows by considering $D_1 := a'D_1'$, $D_2 := \tilde{D}_2$ and $\mathcal{L} := a'\mathcal{L} + \sigma^* M - M$. Q.E.D.

As an application of Propositions 3.31 and 3.32 one has the following:

Theorem 3.33. *Let* $B_1 + B_{2,j}$ $(j = 1, \ldots, k)$ *be reduced divisors on* Σ *satisfying:*

- B_1 *is smooth,*
- $B_{2,j}$ $(j = 1, \ldots, k)$ *are irreducible and not homeomorphic to* B_1*,*
- $B_1 + B_{2,j}$ $(j = 1, \ldots, k)$ *have the same combinatorial data,*
- *there exists a double covering* $f : Z \to \Sigma$ *with* $\Delta_f = B_1$ *such that*

  – *Z is simply connected and*
  – *$f^*B_{2,j}$ is of the form $B^+_{2,j} + B^-_{2,j}$, $B^+_{2,j} \neq B^-_{2,j}$,*
- *there exist distinct positive integers n_j $(j = 1, \dots, k)$ and non-trivial line bundles $\mathcal{L}_1, \dots, \mathcal{L}_k$ such that*
  – *$B^+_{2,j} - B^-_{2,j} \sim n_j\mathcal{L}_j$, and*
  – *no line bundle M_j satisfies $L_j \sim d_jM_j$ for any j and $d_j \geq 2$.*

Hence $(\Sigma, B_1 + B_{2,i}) \not\approx (\Sigma, B_1 + B_{2,j})$.

Proof. By Proposition 3.31, there exists a non-generic D_{2n_j}-covering $\pi_j : S_j \to \Sigma$ branched at $2B_1 + n_jB_{2,j}$ with $\Delta_{\beta_1(\pi_j)} = B_1$ for each j. Since $B_{2,j}$ $(j = 1, \dots, k)$ are not homeomorphic to B_1, there does not exist any homeomorphism $f : \mathbb{P}^2 \to \mathbb{P}^2$ such that $f(B_1) = B_{2,j}$ and $f(B_{2,j'}) = B_1$ for any j, j'. Hence, in order to prove this statement, it is enough to show that there exists no D_{2n_l}-covering $\pi'_l : S'_l \to \Sigma$ branched at $2B_1 + n_lB_{2,j}$ with $\Delta_{\beta_1(\pi'_l)} = B_1$ if $l \neq j$. If such a covering existed, then by Proposition 3.32, there should exist a line bundle $\mathcal{L}'$ such that $B^+_2 - B^-_2 \sim n_l\mathcal{L}'$. On the other hand, $B^+_2 - B^-_2 \sim n_j\mathcal{L}_j$. Let $d = \gcd(n_l, n_j)$ and set $n_j = n'_jd$, $n_l = n'_ld$. Thus $n'_l\mathcal{L}' \sim n'_j\mathcal{L}_j$, as Z is simply connected. Choose an integer b so that $n'_jb = mn'_l + 1$. Thus $\mathcal{L}_j \sim n'_l(b\mathcal{L}' - m\mathcal{L}_j)$ and $n'_l > 1$, which contradicts the hypothesis. Q.E.D.

Theorem 3.33 serves as the main tool to find the Zariski k-plet given in [16], where Zariski k-plets are explicitly obtained for any k. We recall that Zariski k-plets for any k were also obtained by V.S Kulikov in [68] in a more theoretical way. He proves the Chisini conjecture in many cases, i.e., if a curve $\mathcal{C}$ is the branch locus of a generic projection, then $\mathcal{C}$ determines the monodromy of the associated covering. In that case, $\mathcal{C}$ is an irreducible curve having only ordinary nodes and cusps as singularities, and its numerical invariants are determined by the numerical invariants of the surface. F. Catanese [26, 27] had shown that there exist moduli spaces of surfaces with given numerical invariants but different topologies and that is how the theoretical existence of Zariski k-plets was proved.

In what follows we will sketch an explicit construction of a Zariski k-plet.

Example 3.34. Let C_0 be a smooth conic on $\mathbb{P}^2$ and let $f : Z \to \mathbb{P}^2$ be a double covering with $\Delta_f = C_0$. It is well known that $Z = \mathbb{P}^1 \times \mathbb{P}^1$, the covering transformation σ exchanges the two rulings on Z, and $\operatorname{Pic}(Z) \cong \mathbb{Z} \oplus \mathbb{Z}$. Hence a class in $\operatorname{Pic}(Z)$ can be described by a pair of integers. Note that $\sigma^*(a, b) = (b, a)$ and $D_0 := (f^*C_0)_{\text{red}} \sim (1, 1)$.

Let $g_n(t) \in \mathbb{C}(t)$ be a rational function of degree n. Given any $(a,b) \in \mathrm{Pic}(Z)$ one can define a morphism $\eta_{a,b}$ from $\mathbb{P}^1$ to $\mathbb{P}^1 \times \mathbb{P}^1$ by

$$t \mapsto (g_a(t), g_b(t)),$$

where t denotes a non-homogeneous coordinate of $\mathbb{P}^1$. If $g_a(t)$ and $g_b(t)$ are generic, then the image $D_{a,b} := \eta_{a,b}(\mathbb{P}^1)$ satisfies the following properties:

- $D_{a,b}(\sim (a,b))$ is a rational curve with $ab-(a+b)+1$ distinct nodes.
- $D_{a,b}$ and $\sigma^* D_{a,b}$ meet at a^2+b^2 distinct points, $a+b$ of which are on D_0.

These two properties imply that $f(D_{a,b})$ is a rational curve of degree $a+b$ tangent to C_0 at $a+b$ distinct points and with $\binom{a+b-1}{2}$ distinct nodes.

Let us fix an integer $m \geq 4$. Take $\lfloor \frac{m}{2} \rfloor$ distinct pairs of integers: $(m-j, j)$, $j = 1, \ldots, \lfloor \frac{m}{2} \rfloor$, where $\lfloor \bullet \rfloor$ denotes the greatest integer not exceeding $\bullet$, and consider a nodal rational curve $D_{m-j,j}$ as above. Consider $B_1 = C_0$, $B_{2,j} = f(D_{m-j,j})$ $(j = 1, \ldots, \lfloor \frac{m}{2} \rfloor)$. Since $D_{m-j,j} - \sigma^* D_{m-j,j} \sim (m-2j, 2j-m) = (m-2j)(1,-1)$ and $m-2j$ $(j = 1, \ldots, \lfloor \frac{m}{2} \rfloor)$ are all different, $(B_1+B_{2,1}, \ldots, B_1+B_{2,\lfloor \frac{m}{2} \rfloor})$ is a Zariski $\lfloor \frac{m}{2} \rfloor$-plet by Theorem 3.33.

In Example 3.34, when m is odd, one has a stronger statement. In this case, the fundamental groups themselves (disregarding the peripheral information) distinguish the Zariski pair.

Proposition 3.35. *Let $(B_1 + B_{2,1}, \ldots, B_1 + B_{2,\lfloor \frac{m}{2} \rfloor})$ be as in Example* 3.34*. If m is odd, then*

$$\pi_1(\mathbb{P}^2 \setminus (B_1 + B_{2,i}), p_o) \not\cong \pi_1(\mathbb{P}^2 \setminus (B_1 + B_{2,j}), p_o),$$

for any $i < j$.

Let us start by proving the following lemma.

Lemma 3.36. *Let $B_1 + B_{2,j}$ be as in Proposition* 3.35*. If a D_{2n}-covering (n odd) $\pi : S \to \mathbb{P}^2$ with $\Delta_\pi \subset B_1 + B_{2,j}$ exists, then*

(*i*) $\Delta_\pi = B_1 + B_{2,j}$*,* $D(S/\mathbb{P}^2) = Z$ *and* $\beta_1(\pi) = f$*, and*
(*ii*) π *is branched at* $2B_1 + nB_{2,j}$*.*

Proof. Since $\mathbb{P}^2$ is simply connected, $\Delta_{\beta_1(\pi)} \neq \emptyset$. Also note that the branch locus of a double covering is a reduced curve of even degree, $\Delta_{\beta_1(\pi)} = B_1$. This implies that $D(S/\mathbb{P}^2) = Z$ and $\beta_1(\pi) = f$. Since Z

is also simply connected, $\Delta_{\beta_2(\pi)} \neq \emptyset$ and thus $\Delta_{\beta_2(\pi)} = f^*(B_{2,j})$, which proves (i).

In order to prove (ii) it is enough to show that the ramification index along $B_{2,j}$ is n. Since n is odd, by [119], there exists a rational function $\theta \in \mathbb{C}(S)$ such that:

- $\mathbb{C}(S) = \mathbb{C}(\mathbb{P}^2)(\theta)$ and
- the action of $D_{2n} = \langle \sigma, \tau \mid \sigma^2 = \tau^n = (\sigma\tau)^2 = 1\rangle$ on θ is given by

$$\theta^\sigma = \frac{1}{\theta}, \qquad \theta^\tau = \zeta_n \theta, \quad \zeta_n = \exp\left(\frac{2\pi\sqrt{-1}}{n}\right).$$

Considering $\varphi := \theta^n$, one has that $\varphi \in \mathbb{C}(Z)$. Since $\mathbb{C}(S) = \mathbb{C}(Z)(\sqrt[n]{\varphi})$, one may assume that the divisor (φ) of φ is of the form

$$(\varphi) = (aD_{m-j,j} + nD') - (a\sigma^* D_{m-j,j} + nD'')$$

for some effective divisors D' and D''. We claim that $\gcd(a,n) = 1$. If $\gcd(a,n) = d > 1$, then one has

$$\left(\frac{a}{d} D_{m-j,j} + \frac{n}{d} D'\right) - \left(\frac{a}{d}\sigma^* D_{m-j,j} + \frac{n}{d} D''\right) \sim 0,$$

for Z is simply connected. Hence there exists $\psi \in \mathbb{C}(Z)$ such that $\varphi = \psi^d$. This means that the polynomial $x^n - \varphi = x^n - \psi^d$ is reducible in $\mathbb{C}(Z)[x]$, which contradicts $\mathbb{C}(Z) = \mathbb{C}(Z)(\sqrt[n]{\varphi})$. Therefore $\gcd(a,n) = 1$ and hence $\beta_2(\pi)$ is branched at $n(D_{m-j,j} + \sigma^* D_{m-j,j})$. Q.E.D.

Proof of Proposition 3.35. Choose any $i < j$. Since a $D_{2(m-2i)}$-covering branched at $2B_1 + (m-2i)B_{2,i}$ exists, there is a surjective homomorphism $\pi_1(\mathbb{P}^2 \setminus (B_1 + B_{2,i}), p_o) \to D_{2(m-2i)}$. If $\pi_1(\mathbb{P}^2 \setminus (B_1 + B_{2,j}), p_o) \cong \pi_1(\mathbb{P}^2 \setminus (B_1 + B_{2,j}), p_o)$, then there also exists a surjective homomorphism $\pi_1(\mathbb{P}^2 \setminus (B_1 + B_{2,j}), p_o) \to D_{2(m-2i)}$. Therefore a $D_{2(m-2i)}$-covering $\pi : S \to \mathbb{P}^2$ with $\Delta_\pi \subset B_1 + B_{2,j}$ has to exist. By Lemma 3.36, $D(S/\mathbb{P}^2) = Z$, $\beta_1(\pi) = f$, and π is branched at $2B_1 + (m-2i)B_{2,j}$. Hence

$$D_{m-j,j} - \sigma^* D_{m-j,j} \sim (m-2j, -m+2j) \sim (m-2i)\mathcal{L}$$

for some $\mathcal{L} \in \operatorname{Pic}(Z)$, which is not possible. Therefore

$$\pi_1(\mathbb{P}^2 \setminus (B_1 + B_{2,i}), p_o) \not\cong \pi_1(\mathbb{P}^2 \setminus (B_1 + B_{2,j}), p_o),$$

for any $i < j$. Q.E.D.

§4. The Zariski pair of Namba and Tsuchihashi

4.1. Description of the combinatorial stratum

In this section, an elementary construction of the first Zariski pair of an arrangement of conics ([88]) will be given. In this paper Namba and M. Tsuchihashi show the existence of two arrangements of smooth conics both having the following combinatorics: if $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4$ denote four smooth conics, then $\mathcal{C}_1 \pitchfork \mathcal{C}_2$, $\mathcal{C}_3 \pitchfork \mathcal{C}_4$, and $\mathcal{C}_i, \mathcal{C}_j$ are bitangent if $i = 1, 2$, $j = 3, 4$. They computed the fundamental groups of two such arrangements and proved that they are not isomorphic.

From a more geometrical point of view, our interest is to describe the irreducible components of the combinatorial strata of such curves in terms of position of singularities. In order to do so, other combinatorial strata of curves will be defined along the way.

Let $\mathcal{M} \subset \mathbb{P}_8$ be the combinatorial stratum described above. The ordered version of $\mathcal{M}$ will be denoted by $\widetilde{\mathcal{M}} \subset (\mathbb{P}_2)^4$. Following the notations in [88], $\mathcal{A} \subset \mathbb{P}_6$ will be the family of all curves which decompose in three smooth conics $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ whose combinatorics results from $\mathcal{M} \subset \mathbb{P}_8$ by removing any conic. Finally, $\widetilde{\mathcal{A}} \subset (\mathbb{P}_2)^3$ will denote the ordered stratum associated with $\mathcal{A} \subset \mathbb{P}_6$, that is, triples $(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$ such that $\mathcal{C}_1 \pitchfork \mathcal{C}_2$ and $\mathcal{C}_i, \mathcal{C}_3$ are bitangent $(i = 1, 2)$.

Notation 4.1. Given a projective space $\mathbb{P}$ and $A \subset \mathbb{P}$ we will denote $\Sigma(A)$ the smallest projective subspace of $\mathbb{P}$ containing A.

Lemma 4.2. *The families $\mathcal{A}$ and $\widetilde{\mathcal{A}}$ are irreducible as algebraic varieties (and thus, connected).*

Proof. By a natural mapping $\widetilde{\mathcal{A}} \twoheadrightarrow \mathcal{A}$ given by $(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3) \mapsto \mathcal{C}_1 + \mathcal{C}_2 + \mathcal{C}_3$, it is enough to prove the statement for $\widetilde{\mathcal{A}}$.

Let $\widetilde{\mathcal{A}}_1$ be the subset of $(\mathbb{P}_1)^2 \times \mathbb{P}_2$ such that $(L_1, L_2, \mathcal{C}_3) \in \widetilde{\mathcal{A}}_1$ if and only if $\mathcal{C}_3$ is smooth and $L_1 + L_2 + \mathcal{C}_3$ has only ordinary double points. Given $(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3) \in \widetilde{\mathcal{A}}$, one can consider the lines L_i joining the bitangent intersection points of $\mathcal{C}_i$ and $\mathcal{C}_3$, $i = 1, 2$. This defines a natural mapping $\widetilde{\mathcal{A}} \to \widetilde{\mathcal{A}}_1$. It is straightforward to show that $\widetilde{\mathcal{A}}_1$ is irreducible. Therefore, it only remains to show that this mapping is surjective with irreducible fibers. Note that given $(L_1, L_2, \mathcal{C}_3) \in \widetilde{\mathcal{A}}_1$, its fiber is a Zariski open subset of $\Sigma(\mathcal{C}_3, 2L_1) \times \Sigma(\mathcal{C}_3, 2L_2) \times \{\mathcal{C}_3\}$. Q.E.D.

Definition 4.3. Let $S \subset \mathbb{P}_2$ be a pencil of conics. A point $P \in \mathbb{P}^2$ is said to be *associated with* S if P is a singular point of a member of S (recall that a multiple curve is singular).

Lemma 4.4. *Let $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ be conics (not in a pencil) such that $P \notin \mathcal{C}_i$, $i = 1, 2, 3$, is associated with both $\Sigma(\mathcal{C}_1, \mathcal{C}_2)$ and $\Sigma(\mathcal{C}_2, \mathcal{C}_3)$. Then, P is associated with $\Sigma(\mathcal{C}_1, \mathcal{C}_3)$. Moreover, if $\mathcal{D}_{12}$ and $\mathcal{D}_{23}$ are singular conics in $\Sigma(\mathcal{C}_1, \mathcal{C}_2)$ and $\Sigma(\mathcal{C}_2, \mathcal{C}_3)$, respectively, such that P is a double point of both, then there is a singular conic $\mathcal{D}_{13} \in \Sigma(\mathcal{C}_1, \mathcal{C}_3)$, containing P as a double point, such that $\mathcal{D}_{13} \in \Sigma(\mathcal{D}_{12}, \mathcal{D}_{23})$.*

Proof. If C_i (resp. D_{jk}) denotes an equation for $\mathcal{C}_i$ (resp. $\mathcal{D}_{jk}$), then there exist constants such that

$$\begin{aligned} \alpha_1 C_1 + \alpha_2 C_2 =& D_{12} \\ \beta_2 C_2 + \beta_3 C_3 =& D_{23}. \end{aligned}$$

Since $P \notin \mathcal{C}_2$ we have $\alpha_1\alpha_2\beta_1\beta_2 \neq 0$. Then $\beta_2\alpha_1 C_1 - \alpha_2\beta_3 C_3 = \beta_2 D_{12} - \alpha_2 D_{23}$. Since P is a double point of $\mathcal{D}_{12}$ and $\mathcal{D}_{23}$, the result follows. Q.E.D.

Let $\mathcal{B} \subset \mathbb{P}_4$ be the family of curves which decompose into two transversal smooth conics and $\widetilde{\mathcal{B}} \subset (\mathbb{P}_2)^2$ its ordered version. The pencil spanned by each element in $\mathcal{B}$ has exactly three associated points, which according to Lemma 4.4 are the double points of the three singular conics in the pencil generated by the two transversal smooth conics. Let us define the following combinatorial stratum

$$\mathcal{P} := \{(\mathcal{C}_1 + \mathcal{C}_2, P) \in \mathbb{P}_4 \times \mathbb{P}^2 \mid \mathcal{C}_1 + \mathcal{C}_2 \in \mathcal{B}, P \text{ associated with } \Sigma(\mathcal{C}_1, \mathcal{C}_2)\}$$

and denote by $\widetilde{\mathcal{P}} \subset (\mathbb{P}_2)^2 \times \mathbb{P}^2$ its ordered version.

Proposition 4.5. *Let $(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3) \in \widetilde{\mathcal{A}}$ and consider L_i the line joining the tangency points of $\mathcal{C}_i$ and $\mathcal{C}_3$, $i = 1, 2$. Then, $P := L_1 \cap L_2$ is associated with $\Sigma(\mathcal{C}_1, \mathcal{C}_2)$. Also, if $\mathcal{D}_{12}$ is the reducible conic of $\Sigma(\mathcal{C}_1, \mathcal{C}_2)$ containing P, then $\mathcal{D}_{12} \in \Sigma(2L_1, 2L_2)$.*

Moreover, given $(\mathcal{C}_1, \mathcal{C}_2, P) \in \widetilde{\mathcal{P}}$, there exists an irreducible quasiprojective subvariety U of $\mathbb{P}_2$ such that $\mathcal{C}_3 \in U$ if and only if $(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3) \in \widetilde{\mathcal{A}}$ and P is obtained as above.

Proof. Let us fix $(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3) \in \widetilde{\mathcal{A}}$ and let us consider L_1, L_2, P as in the statement. Note that P is associated with $\Sigma(\mathcal{C}_1, 2L_1) = \Sigma(\mathcal{C}_1, \mathcal{C}_3)$ and also with $\Sigma(\mathcal{C}_2, 2L_2) = \Sigma(\mathcal{C}_2, \mathcal{C}_3)$. Then, by Lemma 4.4, P is associated with $\Sigma(\mathcal{C}_1, \mathcal{C}_2)$. Note also that $2L_1$ and $2L_2$ are the reducible conics of the *Moreover* part of Lemma 4.4.

For the last statement, let $\mathcal{D}_{12}$ be the reducible conic of $\Sigma(\mathcal{C}_1, \mathcal{C}_2)$ containing P and fix a line L_1 through P transversal to $\mathcal{C}_1$. In the pencil $\Sigma(2L_1, D_{12})$ there is another double line $2L_2$. For *generic* L_1, L_2 is transversal to $\mathcal{C}_2$. The projective subspace

$$S := \Sigma(\mathcal{C}_1, \mathcal{C}_2, 2L_1, 2L_2) = \Sigma(\mathcal{C}_1, \mathcal{D}_{12}, 2L_1, 2L_2) \subset \mathbb{P}_2$$

is of dimension 2 and contains the pencils $\Sigma(2L_i, \mathcal{C}_i)$ which are lines in S and thus intersect at a conic $\mathcal{C}_3$. Q.E.D.

Definition 4.6. The point P in Proposition 4.5 is said to be *associated* with $(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$.

Corollary 4.7. *The spaces $\widetilde{\mathcal{P}}$ and $\mathcal{P}$ are irreducible.*

Note that the natural projection $\widetilde{\mathcal{M}} \twoheadrightarrow \mathcal{M}$ is 4:1. For simplicity if $\widetilde{\mathcal{C}} := (\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4) \in \widetilde{\mathcal{M}}$, then $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{C}_4$ will denote its projection. Viceversa, we will add a tilde for any element in the fiber of $\mathcal{C}$.

Corollary 4.8. *The spaces $\widetilde{\mathcal{M}}$ (resp. $\mathcal{M}$) have two irreducible (and connected) components $\widetilde{\mathcal{M}}_+, \widetilde{\mathcal{M}}_-$ (resp. $\mathcal{M}_+, \mathcal{M}_-$).*

Moreover, curves in $\mathcal{M}_+$ and $\mathcal{M}_-$ can be distinguished as follows: given $\mathcal{C} \in \mathcal{M}$, $\widetilde{\mathcal{C}} := (\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4) \in \widetilde{\mathcal{M}}$, and P_i the point associated with $(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_i)$, $i = 3, 4$, then $\mathcal{C} \in \mathcal{M}_+$ (resp. $\mathcal{M}_-$) if and only if $P_3 = P_4$ (resp. $P_3 \neq P_4$).

There is another geometrical property which distinguishes the component $\mathcal{M}_+$ from $\mathcal{M}_-$.

Theorem 4.9. *Let $\mathcal{C} \in \mathcal{M}$. Then $\mathcal{C} \in \mathcal{M}_+$ if and only if there exists a conic passing through its eight tacnodes.*

Proof. Let us fix some notation. Given $i \in \{1, 2\}$, $j \in \{3, 4\}$, we will consider $\mathcal{C}_i \cap \mathcal{C}_j := \{P_{ij}, Q_{ij}\}$ and denote by L_{ij} the line joining P_{ij} and Q_{ij}. Let us also define $P_j := L_{1j} \cap L_{2j}$, $j = 3, 4$, which are both points associated with $\Sigma(\mathcal{C}_1, \mathcal{C}_2)$ according to Proposition 4.5. Let us denote by $\mathcal{D}_j$ the conic in $\Sigma(\mathcal{C}_1, \mathcal{C}_2)$ containing P_j as a double point. Note that $\mathcal{D}_j \in \Sigma(2L_{1j}, 2L_{2j})$ by Proposition 4.5.

Let us consider the pencils of conics $\Lambda_i := \Sigma(\mathcal{C}_i, L_{i3} + L_{i4})$, $i = 1, 2$, and let $S := \Sigma(\Lambda_1, \Lambda_2)$. The desired conic should belong to $\Lambda_1 \cap \Lambda_2$, and it exists (uniquely) if and only if $\dim S = 2$. Also note that $S = \Sigma(\mathcal{C}_1, \mathcal{D}_3, L_{13} + L_{14}, L_{23} + L_{24})$. Since $P := P_3 \notin \mathcal{C}_1$ and $P \in \mathcal{D}_3 \cap L_{13} \cap L_{23}$, one has:

$$S_P := \{\mathcal{C} \in S \mid P \in \mathcal{C}\} = \Sigma(\mathcal{D}_3, L_{13} + L_{14}, L_{23} + L_{24}) \subsetneq S.$$

Let us suppose that the conic exists, i.e., $\dim S = 2$ and $\dim S_P = 1$. Since P is a base point of the pencil S_P and a double point for one element in S_P, either it is a double point for *any* element of the pencil, or the tangent line at P of the *general* member is constant. The second possibility cannot happen since $L_{13} \neq L_{23}$. Since P is a double point, $P_3 \in L_{14} \cap L_{24}$, i.e. $P_3 = P_4$.

Let us assume now that $P = P_3 = P_4$, in particular $\mathcal{D}_3 = \mathcal{D}_4$. Let us choose coordinates such that $P := [0 : 0 : 1]$, $\mathcal{D}_3 : xy = 0$, and $L_{13} : x - y = 0$. Since $\mathcal{D}_3 \in \Sigma(2L_{13}, 2L_{23})$, it is easily seen that $L_{23} : x + y = 0$. Analogously, one can prove that there exists $\alpha \in \mathbb{C} \setminus \{0, \pm 1\}$ such that $L_{14} : x - \alpha y = 0$ and $L_{24} : x + \alpha y = 0$. An easy computation concludes that $\mathcal{D}_3 \in \Sigma(L_{13} + L_{14}, L_{23} + L_{24})$, i.e., $\dim S_P = 1$ and $\dim S = 2$. Q.E.D.

For the computation of Alexander polynomials and characteristic varieties it is useful to calculate the space of curves of a given degree passing through some points. Let $\mathcal{C} \in \mathcal{M}$ and let $P_1, \ldots, P_8$ be the eight tacnodes in $\mathcal{C}$. Let us consider

$$\mathcal{K}_{k,8} := \{\mathcal{D} \in \mathbb{P}_k \mid P_1, \ldots, P_8 \in \mathcal{D}\},$$
$$\mathcal{K}_{k,6} := \{\mathcal{D} \in \mathbb{P}_k \mid P_1, \ldots, P_6 \in \mathcal{D}\}.$$

In principle there are several choices for $\mathcal{K}_{k,6}$, but the kind of results we will obtain for $\mathcal{K}_{k,6}$ do not depend on the choice of such points. Let us consider the mappings

$$\sigma_{k,8} : H^0(\mathbb{P}^2; \mathcal{O}(k)) \to \mathbb{C}^8$$
$$\sigma_{k,6} : H^0(\mathbb{P}^2; \mathcal{O}(k)) \to \mathbb{C}^6$$

defined as in the exact sequence (18) in §2.2. Then $\mathcal{K}_{k,8}$ (resp. $\mathcal{K}_{k,6}$) is the projective space of $\ker \sigma_{k,8}$ (resp. $\ker \sigma_{k,6}$).

Proposition 4.10. *If* $\mathcal{C} \in \mathcal{M}_+$ *then* $\dim \mathcal{K}_{3,8} = 2$, *whereas if* $\mathcal{C} \in \mathcal{M}_-$ *then* $\dim \mathcal{K}_{3,8} = 1$.

Proof. Assume first that $\mathcal{C} \in \mathcal{M}_+$. Since there is a conic $\mathcal{Q}$ passing through the eight points, $\mathcal{Q}$ is a base component of $\mathcal{K}_{3,8}$, i.e., any element of $\mathcal{K}_{3,8}$ is of the form $\mathcal{Q} + L$, $L \in \mathbb{P}_1$. This is trivial if $\mathcal{Q}$ is smooth. If it is singular, then $\mathcal{Q} = L_1 + L_2$. It is easily seen that no line can contain five of these points; then both L_1 and L_2 are also base components. On the other hand, $\mathcal{Q} + L \in \mathcal{K}_{3,8}$, for all $L \in \mathbb{P}_1$. Then, $\dim \mathcal{K}_{3,8} = \dim \mathbb{P}_1 = 2$.

By Theorem 4.9, it is enough to prove that $\dim \mathcal{K}_{3,8} \geq 2$ forces the eight points to be on a conic. Two cases will be considered:

Case. *No three points among* $P_1, \ldots, P_8$ *are aligned.*

Let us consider a conic $\mathcal{Q}$ passing through $P_1, \ldots, P_5$. The hypothesis added for this case implies that $\mathcal{Q}$ is irreducible. Take $P_9, P_{10} \in \mathcal{Q}$

different from the other eight points. Since $\dim \mathcal{K}_{3,8} \geq 2$, the elements of $\mathcal{K}_{3,8}$ passing through the two extra points P_9 and P_{10} form a subspace of dimension at least 0. Hence this space is not empty and $\mathcal{Q}$ is a base component. Since the other three points cannot be aligned, one may suppose that, say $P_6 \in \mathcal{Q}$. One can now repeat the above argument on the elements of $\mathcal{K}_{3,8}$ passing through one extra point $P_9 \in \mathcal{Q}$. The given subspace has dimension at least 1 and $\mathcal{Q}$ as a base component. Therefore, say $P_7 \in \mathcal{Q}$, and thus, $\mathcal{Q}$ is a base component of $\mathcal{K}_{3,8}$. Since the space of lines passing through P_8 is of dimension 1 and $\dim \mathcal{K}_{3,8} \geq 2$, $\mathcal{Q}$ passes through the eight points $P_1, \ldots, P_8$.

Case. *P_1, P_2, P_3 are aligned.*

We will first prove that this case forces four points to be aligned. Let L be the line through P_1, P_2, P_3 and suppose that no other P_i is in L. Choosing a generic extra point in L as above, one can deduce that L is a base component of a subspace of $\mathcal{K}_{3,8}$ of dimension at least 1. Therefore there exists a pencil of conics passing through $P_4, \ldots, P_8$, which can only happen if at least four of these points are aligned.

After reordering, one can assume that $P_1, \ldots, P_4 \in L$. Then L is a base component of $\mathcal{K}_{3,8}$ and hence there is a linear family of conics through four points of dimension at least 2. Choosing an extra generic point on $\mathbb{P}^2$ one obtains as a subspace a pencil of conics through five points, hence four points are aligned. Note that the extra point can be chosen in an open Zariski set of $\mathbb{P}^2$, thus the four aligned points must be $P_5, \ldots, P_8$. Therefore there is also a conic passing through $P_1, \ldots, P_8$. Q.E.D.

Proposition 4.11. *If $\mathcal{C} \in \mathcal{M}$ then $\dim \mathcal{K}_{3,6} = 3$.*

Proof. It is enough to show that $\dim \mathcal{K}_{3,6} > 3$ leads to contradiction. As in Proposition 4.10 two cases will also be considered:

Case. *No three points among $P_1, \ldots, P_6$ are aligned.*

Let us consider the irreducible conic $\mathcal{Q}$ passing through $P_1, ..., P_5$. Consider two extra points $P_9, P_{10} \in \mathcal{Q}$ and two extra points $P_{11}, P_{12} \notin \mathcal{Q}$. The subspace of cubics in $\mathcal{K}_{3,6}$ passing through $P_9, P_{10}, P_{11}, P_{12}$ is not empty and contains $\mathcal{Q}$ as a base component. Since P_{11}, P_{12} can be chosen in such a way that P_6, P_{11}, P_{12} are not on a line, this implies that $P_6 \in \mathcal{Q}$. Repeating the argument with $P_9 \in \mathcal{Q}$ and three extra points $P_{10}, P_{11}, P_{12} \notin \mathcal{Q}$ one obtains again a non-empty subspace of $\mathcal{K}_{3,6}$ containing $\mathcal{Q}$ as a base component. Since P_{10}, P_{11}, P_{12} need not be on a line, a contradiction results.

Case. *P_1, P_2, P_3 are aligned.*

Let L_1 be the line joining P_1, P_2, P_3 and assume that no other point P_i $(i = 4, 5, 6)$ belongs to L_1. Considering $P_9 \in L$ and $P_{10}, P_{11} \notin L$ one obtains a pencil in $\mathcal{K}_{3,6}$ having L_1 as a base component. Hence the five points $P_4, P_5, P_6, P_{10}, P_{11}$ belong to a pencil of conics. Since P_{10}, P_{11} can be chosen so that the line joining them does not contain any other P_i, one concludes that P_4, P_5, P_6 are also aligned. Let L_2 be such a line. Since we assumed that no four points are aligned, $P_9 \in L_1 \cap L_2$ is an extra point. The subspace of curves in $\mathcal{K}_{3,6}$ passing through $P_1, P_2, P_3, P_4, P_5, P_6, P_9$ has dimension at least 3 and has $L_1 \cup L_2$ as a base component. This leads to contradiction since $\dim \mathbb{P}_1 = 2$.

Therefore four points, say P_1, P_2, P_3, P_4, belong to a line L, which automatically becomes a base component of $\mathcal{K}_{3,6}$. Note that neither P_5 nor P_6 can belong to L since $\operatorname{mult}_{P_i}(L, \mathcal{C}) \geq 2$ and $\deg \mathcal{C} = 8$. One can now choose four extra points $P_9, P_{10}, P_{11}, P_{12} \notin L$ such that $P_5, P_6, P_9, P_{10}, P_{11}, P_{12}$ do not belong to a conic. Since $\dim \mathcal{K}_{3,6} > 3$, the subspace of curves in $\mathcal{K}_{3,6}$ passing through $P_1, ..., P_6, P_9, ..., P_{12}$ is not empty and has L as base component. This contradicts the choice of $P_9, P_{10}, P_{11}, P_{12}$. Q.E.D.

Proposition 4.12. *If* $\mathcal{C} \in \mathcal{M}$ *then* $\dim \mathcal{K}_{4,8} = 6$.

Proof. It is enough to prove that $\sigma_{4,8}$ is surjective. If $\mathcal{C} \in \mathcal{M}_-$, since $\sigma_{3,8}$ is surjective, so is $\sigma_{4,8}$. If $\mathcal{C} \in \mathcal{M}_+$, then let $\sum_{j=1}^{8} a_i x_i = 0$ be the equation of the image of $\sigma_{3,8}$ and let us suppose that $a_8 \neq 0$. It is enough to find a quartic curve passing through $P_1, \ldots, P_7$ and not through P_8; we can order this points such that P_8 is not in the line through P_5, P_6. In order to do so, one can choose a conic through $P_1, \ldots, P_4$, the line through P_5, P_6, and a generic line through P_7.

Q.E.D.

4.2. Computation of characteristic varieties

We will first describe the geometrical method to compute some components of the characteristic varieties of a curve $\mathcal{C} \in \mathcal{M}$ as proposed by Libgober in [75] −see also a brief sketch of it on page 48. After that, a similar geometrical argument allows for a computation of their Alexander polynomial as proposed in [80, 43, 2].

According to Example 2.24(2) a tacnode (that is, an $\mathbb{A}_3$-singularity) has associated with it the sequence of ideals of quasiadjunction shown in Figure 11.

We follow the notation introduced on page 48. Let $\bar{\xi} := (\xi_1, \xi_2, \xi_3, \xi_4) \in (\mathbb{C}^*)^4$ be a torsion point such that $\ell_{\bar{X}} = 2(X_1 + X_2 + X_3 + X_4) \in \mathbb{N}$,

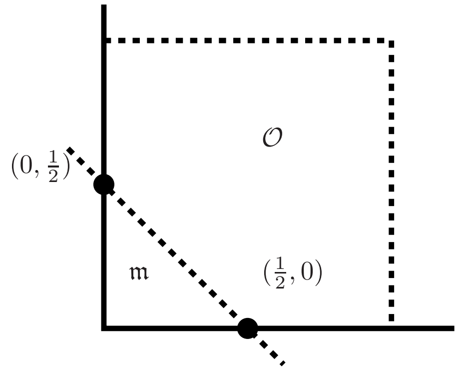

Fig. 11. Quasiadjunction ideals for tacnodes

where $X_i = \frac{\log \xi_i}{2\pi\sqrt{-1}} \in (0,1)$. Then $\bar{\xi} \in \operatorname{Char}_1^*(\mathcal{C})$ if and only if

$$\sigma_{\bar{X}} : H^0(\mathbb{P}^2, \mathcal{O}(5 - \ell_{\bar{X}})) \to \bigoplus_{P \in \operatorname{Sing} \mathcal{C}} \mathcal{O}_{\mathbb{P}^2,P}/(\mathcal{A}_{\mathcal{C}}^{\bar{X}})_P =: V_{\bar{X}}$$

is not surjective. Let P be a tacnode of $\mathcal{C}_i + \mathcal{C}_j$. Following Example 2.24(2), $(\mathcal{A}_{\mathcal{C}}^{\bar{X}})_P \neq \mathcal{O}_{\mathbb{P}^2,P}$ if and only if $2(X_i + X_j) \leq 1$, in which case $(\mathcal{A}_{\mathcal{C}}^{\bar{X}})_P = \mathfrak{m}$ the maximal ideal.

Note that we can restrict ourselves to the case $\ell_{\bar{X}} \leq 5$. Also note that, using [75], at least one of these equations is satisfied:

$$2(X_i + X_j) = 1, \quad i = 1, 2, \quad j = 3, 4. \tag{23}$$

Without loss of generality, it can be assumed that $2(X_1 + X_3) = 1$. In that case $2(X_2 + X_4) = \ell_{\bar{X}} - 1$. The case $\ell_{\bar{X}} = 5$ is not possible since $X_2, X_4 < 1$. If $\ell_{\bar{X}} = 1$, then $X_2 = X_4 = 0$, which corresponds to coordinate components.

For $\ell_{\bar{X}} = 4$, one has $2(X_2 + X_4) = 3$ and $\dim H^0(\mathbb{P}^2, \mathcal{O}(1)) = 3$. Since only non-surjective $\sigma_{\bar{X}}$ matter, an extra equation $2(X_i + X_j) \leq 1$ must be satisfied for some appropriate indices (i, j). Without loss of generality, it can be assumed that $2(X_1 + X_4) \leq 1$, and $2(X_2 + X_3) \geq 3$, which has no solution in the open hypercube $(0, 1)^4$.

The same arguments apply to $\ell_{\bar{X}} = 3$ but in this case one has $\dim H^0(\mathbb{P}^2, \mathcal{O}(2)) = 6$, $X_2 + X_4 = 1$, $2(X_1 + X_4) \leq 1$, and $X_2 + X_3 \geq 1$. There are solutions of the system in $(0, 1)^4$. In this case $\dim V_{\bar{X}} = 4$ and it is easily seen that $\sigma_{\bar{X}}$ is surjective.

We finish with the case $\ell_{\bar{X}} = 2$, where $2(X_2 + X_4) = 1$. In this situation $2(X_1+X_4) = 1+2(X_4-X_3)$ and $2(X_2+X_3) = 1-2(X_4-X_3)$, hence, either $X_3 = X_4$, or say $2(X_1 + X_4) > 1$ and $2(X_2 + X_3) < 1$. In the latter case $\dim V_{\bar{X}} = 6$ and, by Proposition 4.11, $\sigma_{3,6} = \sigma_{\bar{X}}$ is surjective.

Therefore *all* of the equations in (23) are satisfied, in which case, $\sigma_{\bar{X}} = \sigma_{3,8}$ and its cokernel has dimension 1 (resp. 0) for curves in $\mathcal{M}_+$ (resp. $\mathcal{M}_-$) by Proposition 4.10.

Thus for $\mathcal{M}_+$ there is superabundance for solutions $\bar{X} \in (0,1)^4$ of $X_1 + X_2 + X_3 + X_4 = 1$ and (23) i.e. for

$$\{(X_1, X_1, 1/2 - X_1, 1/2 - X_1) \mid X_1 \in (0,1)\},$$

whose exponential is $\{(t, t, -t^{-1}, -t^{-1}) \mid t \in \mathbb{C}^*\}$. We have proved the following.

Proposition 4.13.

$$\operatorname{Char}_1^*(\mathcal{C}) = \begin{cases} \{(t, t, -t^{-1}, -t^{-1}) \mid t \in \mathbb{C}^*\} & \text{if } \mathcal{C} \in \mathcal{M}_+ \\ \emptyset & \text{if } \mathcal{C} \in \mathcal{M}_-. \end{cases}$$

Remark 4.14. As mentioned in Remark 2.17(2), since the number of positive dimensional components of $\operatorname{Char}_1^*(\mathcal{C})$ is different in these cases, the fundamental groups are non-isomorphic.

Note that, according to Theorem 2.26 one can calculate the roots of the Alexander polynomial of $\mathcal{C}$ as follows

$$Z(\Delta_{\mathcal{C}}(t)) \setminus \{1\} = \begin{cases} \pm\sqrt{-1} & \text{if } \mathcal{C} \in \mathcal{M}_+ \\ \emptyset & \text{if } \mathcal{C} \in \mathcal{M}_- \end{cases}.$$

We will consider $\mathcal{A}_k := \mathcal{A}_{\mathcal{C}}^{(\frac{1-k}{d},\dots,\frac{1-k}{d})}$ (according to the definition of ideal sheaf of quasi-adjunction) and referred to this ideal as an *Alexander ideal sheaf* of $\mathcal{C}$. One can also obtain $\Delta_{\mathcal{C}}(t)$ geometrically using the exponents of the singularities as follows.

Theorem 4.15 ([72, 2])**.** *The Alexander polynomial of $\mathcal{C}$ can be written as the product*

$$\Delta_{\mathcal{C}}(t) = (t-1)^{r-1} \prod_{k=1}^{d-1} \Delta_k^{b_k}(t),$$

where $\Delta_k = \left(t - \exp(\frac{2\pi k\sqrt{-1}}{d})\right)\left(t - \exp(\frac{-2\pi k\sqrt{-1}}{d})\right)$ *and b_k is the defect of the Alexander ideal sheaf $\mathcal{A}_k(k-3)$.*

Each exponent $\frac{k}{8}$, $k = 4, 5, 6, 7$ has an Alexander ideal $\mathcal{A}_k(k-3)$ associated with it. The quotient sheaf $\mathcal{O}(k-3)/\mathcal{A}_k(k-3)$ is supported at the eight tacnodes of $\mathcal{C}$ as follows

$$(\mathcal{A}_k)_P = \begin{cases} \mathcal{O}_P & \text{if } k = 4, 5 \\ \mathfrak{m}_P & \text{if } k = 6, 7, \end{cases}$$

where $\mathfrak{m}_P$ is the maximal ideal at P −we refer to [43, 80, 2] for the details. Hence $h^1(\mathcal{A}_4(1)) = h^1(\mathcal{A}_5(2)) = 0$, and thus the only two interesting cases are the exponents $\frac{6}{8}$ and $\frac{7}{8}$. For the first exponent, by Proposition 4.10, $h^0(\mathcal{A}_6(3)) = \dim\ker\sigma_{3,8} = \begin{cases} 3 & \text{if } \mathcal{C} \in \mathcal{M}_+ \\ 2 & \text{if } \mathcal{C} \in \mathcal{M}_- \end{cases}$, hence

$$h^1(\mathcal{A}_6(3)) = h^0(\mathcal{A}_6(3)) - 2 = \begin{cases} 1 & \text{if } \mathcal{C} \in \mathcal{M}_+ \\ 0 & \text{if } \mathcal{C} \in \mathcal{M}_- \end{cases}.$$

For the second exponent, by Proposition 4.12, $h^0(\mathcal{A}_7(4)) = \dim\ker\sigma_{4,8} = 7$, hence $h^1(\mathcal{A}_7(4)) = h^0(\mathcal{A}_7(4)) - 7 = 0$. By Theorem 4.15 one has

$$\Delta_{\mathcal{C}}(t) = \begin{cases} (t-1)^3(t^2+1) & \text{if } \mathcal{C} \in \mathcal{M}_+ \\ (t-1)^3 & \text{if } \mathcal{C} \in \mathcal{M}_- \end{cases}.$$

4.3. A tower of D_{2n}-coverings

In this subsection, we will explain another way to study this example by using D_{2n}-coverings. Let us start by introducing the notion of a tower of dihedral coverings.

Definition 4.16. Let Y be a normal projective variety. A sequence $\{\pi_i : X_i \to Y\}_{i\in I}$ of Galois coverings is called a *tower of dihedral coverings* if it satisfies the following conditions:

(*i*) $\pi_i : X_i \to Y$ is a D_{2n_i}-coverings ($n_i \geq 3$) for each i.

(*ii*) If $n_i | n_j$, then there exists a morphism $\eta_{ji} : X_j \to X_i$ such that $\pi_j = \pi_i \circ \eta_{ji}$.

Here is an example of a tower of dihedral coverings, which we need later.

Example 4.17. Let $\varphi_n : \mathbb{P}^1 \to \mathbb{P}^1$ ($n \geq 3$) be the family of morphisms given by

$$t \mapsto s = \frac{1}{2}\left(t^n + \frac{1}{t^n}\right),$$

where s, t are non-homogeneous coordinates. It is easy to see that φ_n is a D_{2n}-covering branched at $2[1 : \pm 1] + n[0 : 1]$, where $[s_0 : s_1]$ are homogeneous coordinates with $s = \frac{s_1}{s_0}$. In fact, D_{2n} acts on the source $\mathbb{P}^1$ in such a way that $t^\sigma = t^{-1}$, $t^\tau = \zeta_n t$, $\zeta_n = \exp(\frac{2\pi\sqrt{-1}}{n})$. The set of dihedral coverings $\{\varphi_n : \mathbb{P}^1 \to \mathbb{P}^1\}_{n\geq 3}$ is a tower of dihedral coverings.

Proposition 4.18. *Let $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{C}_4 \in \mathcal{M}$. Then the following two statements are equivalent:*

(1) *There exists a tower of dihedral coverings $\{\pi_n : X_n \to \mathbb{P}^2\}_{n\in\mathbb{N}}$ such that:*
   (a) $\mathrm{Gal}(X_n/\mathbb{P}^2) \cong D_{2p_n}$, *$p_n$ odd prime,*
   (b) *π_n is branched at $2(\mathcal{C}_1 + \mathcal{C}_2) + p_n(\mathcal{C}_3 + \mathcal{C}_4)$, and*
   (c) *$p_n \neq p_m$ for any $n \neq m$.*

(2) *There exists a conic through the 8 tacnodes of $\mathcal{C}$, i.e., $\mathcal{C} \in \mathcal{M}_+$.*

Note that Proposition 4.18 implies that a pair $(\mathcal{C}_+, \mathcal{C}_-)$ $(\mathcal{C}_+ \in \mathcal{M}_+, \mathcal{C}_- \in \mathcal{M}_-)$ is a Zariski pair. We need several steps to prove Proposition 4.18.

Lemma 4.19. *Let D_1 and D_2 be reduced plane curves of degree $2m$ such that all intersection points between D_1 and D_2 give rise to tacnodes in $D_1 + D_2$ (i.e., D_1 is tangent to D_2 at $2m^2$ distinct points). Let $\Lambda = \{\lambda_1 D_1 + \lambda_2 D_2\}_{[\lambda_1,\lambda_2]\in\mathbb{P}^1}$ be the pencil of curves spanned by D_1 and D_2. If there exists a unique reduced plane curve, E, of degree m passing through all the $2m^2$ intersection points of D_1 and D_2, then $2E \in \Lambda$.*

Proof. Since E meets D_1 at $2m^2$ distinct points, E is smooth at each intersection point and meets D_1 transversely. Choose a general point x on E. Let C_x be a member of Λ passing through x. Then C_x meets E at $2m^2 + 1$ points. Hence E is contained in C_x. Write $C_x = E + E'$. Then E' is a curve of degree m. Since the base points of Λ consists of $D_1 \cap D_2$ with multiplicity 2 at each base point, E' also passes through all the intersection points of $D_1 \cap D_2$. This implies $E' = E$.

Lemma 4.20. *Let $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{C}_4 \in \mathcal{M}$. If there exists a conic passing through 8 tacnodes, then there exists a D_{2m}-covering of $\mathbb{P}^2$ branched at $2(\mathcal{C}_1 + \mathcal{C}_2) + m(\mathcal{C}_3 + \mathcal{C}_4)$ for any $m \geq 3$.*

Proof. Let $\Lambda := \Sigma(\mathcal{C}_1 + \mathcal{C}_2, \mathcal{C}_3 + \mathcal{C}_4)$ be the pencil of plane curves spanned by $\mathcal{C}_1 + \mathcal{C}_2$ and $\mathcal{C}_3 + \mathcal{C}_4$. Since the conic $\mathcal{Q}$ in the assumption passes through the eight (double) base points of Λ, we have $2\mathcal{Q} \in \Lambda$ by Lemma 4.19. Let C_i (resp. Q) be defining homogeneous polynomials for $\mathcal{C}_i$ (resp. $\mathcal{Q}$). Hence we may assume that $C_3C_4 = C_1C_2 - Q^2$. Let $\Phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ be the rational function given by

$$[z_0 : z_1 : z_2] \mapsto [C_3C_4 : -(C_1C_2 + Q^2) : C_3C_4].$$

Let $\varphi_m : \mathbb{P}^1 \to \mathbb{P}^1$ be the D_{2m}-covering described in Example 4.17. Let $\rho : \hat{\mathbb{P}}^2 \to \mathbb{P}^2$ be the resolution of indeterminacy of Φ and put $\hat{\Phi} = \rho \circ \Phi$. Let X' be the fibered product $\hat{\mathbb{P}}^2 \times_{\mathbb{P}^1} \mathbb{P}^1$ by $\hat{\Phi}$ and φ_m. Also let X_m be the $\mathbb{C}(X')$-normalization of $\mathbb{P}^2$. Then X_m is a D_{2m}-covering branched at $2(\mathcal{C}_1 + \mathcal{C}_2) + m(\mathcal{C}_3 + \mathcal{C}_4)$. Q.E.D.

Proof of Proposition 4.18 ($\Leftarrow$). Let $\pi_m : X_m \to \mathbb{P}^2$ be the D_{2m}-covering as in Lemma 4.20. Then $\{\pi_m : X_m \to \mathbb{P}^2\}_{m\geq 3}$ is a tower of dihedral coverings such that $\mathrm{Gal}(X_m/\mathbb{P}^2) \cong D_{2m}$. Also π_m is branched at $2(\mathcal{C}_1 + \mathcal{C}_2) + m(\mathcal{C}_3 + \mathcal{C}_4)$. In particular, one obtains the desired D_{2p_n}-coverings for odd primes p_n $(n = 1, 2, 3, ...)$. Q.E.D.

Finally, our purpose until the end of this section is to prove the converse. Let $f' : Z' \to \mathbb{P}^2$ be a double covering branched at $\mathcal{C}_1 + \mathcal{D}_2$, and let $\mu : Z \to Z'$ be the canonical resolution of Z'. Let

$$\begin{tikzcd} Z' & Z \arrow[l, "\mu"'] \arrow[d, "f"] \\ \mathbb{P}^2 \arrow[from=u, "f'"'] & \hat{\mathbb{P}}^2 \arrow[l, "\rho"'] \end{tikzcd}$$

denote the diagram for the canonical resolution. In our case, the morphism $\rho : \hat{\mathbb{P}}^2 \to \mathbb{P}^2$ is a composition of 4 blowing-ups at the 4 nodes of $\mathcal{C}_1 + \mathcal{C}_2$. Let $f : Z \to \hat{\mathbb{P}}^2$ be the induced double covering. Note that Z is simply connected, as it is a rational surface. Suppose that there exists a D_{2p_n}-covering $\pi_n : X_n \to \mathbb{P}^2$ branched at $2(\mathcal{C}_1 + \mathcal{C}_2) + p_n(\mathcal{C}_3 + \mathcal{C}_4)$. Then $D(X_n/\mathbb{P}^2) = Z'$ and $\beta_1(\pi_n) = f'$. Let $\tilde{X}_n$ be the $\mathbb{C}(X_n)$-normalization of $\hat{\mathbb{P}}^2$. $\tilde{X}_n$ is a D_{2p_n}-covering of $\hat{\mathbb{P}}^2$ such that $D(\tilde{X}_n/\hat{\mathbb{P}}^2) = Z$ and $\beta_1(\tilde{\pi}_n) = f$, $\tilde{\pi}_n$ being the covering morphism. Summing up, one obtains the following commutative diagram:

$$\begin{tikzcd} X_n \arrow[d] & \tilde{X}_n \arrow[l] \arrow[d] \\ Z' \arrow[d] & Z \arrow[l, "\mu"'] \arrow[d] \\ \mathbb{P}^2 & \hat{\mathbb{P}}^2. \arrow[l, "\rho"'] \end{tikzcd}$$

Since the irreducible components of the exceptional divisor μ are invariant under the covering transform of f, they are not contained in $\Delta_{\beta_2(\tilde{\pi}_n)}$ (Corollary 3.13). Hence $\Delta_f = \rho_q^{-1}(\mathcal{C}_1 + \mathcal{C}_2)$, and $\Delta_{\beta_2(\tilde{\pi}_n)} = (\rho \circ f)^*(\mathcal{C}_3 + \mathcal{C}_4)$. Let us denote $(\rho \circ f)^*\mathcal{C}_3 = \mathcal{C}_3^+ + \mathcal{C}_3^-$ and $(\rho \circ f)^*\mathcal{C}_4 = \mathcal{C}_4^+ + \mathcal{C}_4^-$. One has the following.

Lemma 4.21. *There exists an integer k with $0 < k \leq \frac{p_n - 1}{2}$ such that either $(\mathcal{C}_3^+ + k\mathcal{C}_4^+) - (\mathcal{C}_3^- + k\mathcal{C}_4^-)$ or $(\mathcal{C}_3^+ + k\mathcal{C}_4^-) - (\mathcal{C}_3^+ + k\mathcal{C}_4^+)$ is p_n-divisible in* $\mathrm{Pic}(Z)$.

Proof. By Proposition 3.12, either $(a_3\mathcal{C}_3^+ + a_4\mathcal{C}_4^+) - (a_3\mathcal{C}_3^- + a_4\mathcal{C}_4^-)$ or $(a_3\mathcal{C}_3^+ + a_4\mathcal{C}_4^-) - (a_3\mathcal{C}_3^- + a_4\mathcal{C}_4^+)$, where a_i $(i = 3, 4)$ are integers with $0 < a_i < p_n$ $(i = 3, 4)$, is p_n-divisible. The latter case is reduced to the first case by considering

$$\big(a_3\mathcal{C}_3^+ + (p_n - a_4)\mathcal{C}_4^+\big) - \big(a_3\mathcal{C}_3^- + (p_n - a_4)\mathcal{C}_4^-\big) + p_n(\mathcal{C}_4^- - \mathcal{C}_4^+).$$

So one may assume that

$$(a_3\mathcal{C}_3^+ + a_4\mathcal{C}_4^+) - (a_3\mathcal{C}_3^- + a_4\mathcal{C}_4^-) \sim p_n L$$

for some $L \in \mathrm{Pic}(Z)$. Choose an integer b $(0 < b < p_n)$ so that $a_3 b \equiv 1 \bmod p_n$. Let us define

$$\begin{aligned} D_1 &:= b(a_3\mathcal{C}_3^+ + a_4\mathcal{C}_4^+) - p_n\left(\left\lfloor \frac{a_3 b}{p_n} \right\rfloor \mathcal{C}_3^+ + \left\lfloor \frac{a_4 b}{p_n} \right\rfloor \mathcal{C}_4^+\right), \\ &= \mathcal{C}_3^+ + \left(a_4 b - p_n \left\lfloor \frac{a_4 b}{p_n} \right\rfloor \mathcal{C}_4^+\right). \end{aligned}$$

If $0 < a_4 b - \left\lfloor \frac{a_4 b}{p_n} \right\rfloor \leq \frac{p_n - 1}{2}$, then define $k = a_4 b - \left\lfloor \frac{a_4 b}{p_n} \right\rfloor$ so that the divisor $(\mathcal{C}_3^+ + k\mathcal{C}_4^+) - (\mathcal{C}_3^- + k\mathcal{C}_4^-)$ is p_n-divisible.

If $a_4 b - \left\lfloor \frac{a_4 b}{p_n} \right\rfloor > \frac{p_n - 1}{2}$, then define $k = p_n + \left\lfloor \frac{a_4 b}{p_n} \right\rfloor - a_4 b$ so that the divisor $(\mathcal{C}_3^+ + k\mathcal{C}_4^-) - (\mathcal{C}_3^- + k\mathcal{C}_4^+)$ is p_n-divisible. Q.E.D.

Lemma 4.22. *The self-intersection numbers of $(\mathcal{C}_3^+ + k\mathcal{C}_4^+) - (\mathcal{C}_3^- + k\mathcal{C}_4^-)$ and $(\mathcal{C}_3^+ + k\mathcal{C}_4^-) - (\mathcal{C}_3^- + k\mathcal{C}_4^+)$ are either $-8(k-1)^2$, $-8(k+1)^2$ or $-8(k^2+1)$.*

Proof. Since

$$8 = ((\rho \circ f)^*\mathcal{C}_3)^2 = (\mathcal{C}_3^+)^2 + (\mathcal{C}_3^-)^2 + 2\mathcal{C}_3^+ \cdot \mathcal{C}_3^-,\ \mathcal{C}_3^+ \cdot \mathcal{C}_3^- = 4,\ (\mathcal{C}_3^+)^2 = (\mathcal{C}_3^-)^2,$$

one has $(\mathcal{C}_3^+)^2 = (\mathcal{C}_3^-)^2 = 0$. Similarly $(\mathcal{C}_4^+)^2 = (\mathcal{C}_4^-)^2 = 0$. Hence $(\mathcal{C}_i^+ - \mathcal{C}_i^-)^2 = -8 \quad (i = 3, 4)$. Also, since

$$8 = ((\rho \circ f)^*\mathcal{C}_3) \cdot ((\rho \circ f)^*\mathcal{C}_4) = \mathcal{C}_3^+ \cdot \mathcal{C}_4^+ + \mathcal{C}_3^+ \cdot \mathcal{C}_4^- + \mathcal{C}_3^- \cdot \mathcal{C}_4^+ + \mathcal{C}_3^- \cdot \mathcal{C}_4^-$$

and

$$\mathcal{C}_3^+ \cdot \mathcal{C}_4^+ = \mathcal{C}_3^- \cdot \mathcal{C}_4^-, \mathcal{C}_3^+ \cdot \mathcal{C}_4^- = \mathcal{C}_3^- \cdot \mathcal{C}_4^+,$$

one concludes that either

(a) $\mathcal{C}_3^+\mathcal{C}_4^+ = \mathcal{C}_3^-\mathcal{C}_4^- = 4$, $\mathcal{C}_3^+\mathcal{C}_4^- = \mathcal{C}_3^-\mathcal{C}_4^+ = 0$, or
(b) $\mathcal{C}_3^+\mathcal{C}_4^+ = \mathcal{C}_3^-\mathcal{C}_4^- = 4$, $\mathcal{C}_3^+\mathcal{C}_4^- = \mathcal{C}_3^-\mathcal{C}_4^+ = 0$, or
(c) $\mathcal{C}_3^+\mathcal{C}_4^+ = \mathcal{C}_3^-\mathcal{C}_4^- = 2$, $\mathcal{C}_3^+\mathcal{C}_4^- = \mathcal{C}_3^-\mathcal{C}_4^+ = 2$

holds. Thus

$$\left\{(\mathcal{C}_3^+ - \mathcal{C}_3^-) \pm k(\mathcal{C}_4^+ - \mathcal{C}_4^-)\right\}^2 = \begin{cases} -8(k \mp 1)^2 & \text{for the case (a),} \\ -8(k \pm 1)^2 & \text{for the case (b),} \\ -8(k^2+1) & \text{for the case (c).} \end{cases}$$

Q.E.D.

Lemma 4.23. *Let k be as above. Then $k = 1$.*

Proof. We will only consider the case when $(\mathcal{C}_3^+ - \mathcal{C}_3^-) + k(\mathcal{C}_4^+ - \mathcal{C}_4^-)$ is p_n-divisible (the other case is analogous). Suppose that

$$(\mathcal{C}_3^+ - \mathcal{C}_3^-) + k(\mathcal{C}_4^+ - \mathcal{C}_4^-) \sim p_n L,$$

for some $L \in \operatorname{Pic}(Z)$. After computing the self-intersection numbers of both sides, one deduces that either $-8(k-1)^2 = p_n^2 L^2$, or $-8(k+1)^2 = p_n^2 L^2$, or $-8(k^2+1) = p_n^2 L^2$ holds. Since p_n is odd, L^2 is an integer and $0 < k \leq \frac{p_n - 1}{2}$, which leads to $k = 1$ and $L^2 = 0$. Q.E.D.

Lemma 4.24. *Either*

$$(\mathcal{C}_3^+ + \mathcal{C}_4^+) - (\mathcal{C}_3^- + \mathcal{C}_4^-) \sim 0, \qquad \textit{or} \qquad (\mathcal{C}_3^+ + \mathcal{C}_4^-) - (\mathcal{C}_3^- + \mathcal{C}_4^+) \sim 0$$

holds.

Proof. By Lemma 4.23, either $(\mathcal{C}_3^+ + \mathcal{C}_4^+) - (\mathcal{C}_3^- + \mathcal{C}_4^-)$, or $(\mathcal{C}_3^+ + \mathcal{C}_4^-) - (\mathcal{C}_3^- + \mathcal{C}_4^+)$ is p_n-divisible. By the assumption in Proposition 4.18, at least one of them is p_n-divisible for infinitely many odd prime numbers p_n. Since Z is simply connected, $\operatorname{Pic}(Z)$ is a finitely generated free $\mathbb{Z}$-module. Therefore, either $(\mathcal{C}_3^+ + \mathcal{C}_4^+) - (\mathcal{C}_3^- + \mathcal{C}_4^-) \sim 0$, or $(\mathcal{C}_3^+ + \mathcal{C}_4^-) - (\mathcal{C}_3^- + \mathcal{C}_4^+) \sim 0$ holds. Q.E.D.

Proof of Proposition 4.18 ($\Rightarrow$). In what follows, only the case $(\mathcal{C}_3^+ - \mathcal{C}_3^-) + (\mathcal{C}_4^+ - \mathcal{C}_4^-) \sim 0$ will be considered (the other case is analogous). Let φ be a rational function on Z such that

$$(\varphi) = (\mathcal{C}_3^+ + \mathcal{C}_4^+) - (\mathcal{C}_3^- + \mathcal{C}_4^-).$$

Note that one can choose φ in such a way that $\varphi^{\sigma_f} = 1/\varphi$, where σ_f denote the involution of the double covering $f : Z \to \hat{\mathbb{P}}^2$. Let $K_n := \mathbb{C}(Z)(\sqrt[p_n]{\varphi})$ and let S_n be the K_n-normalization of $\mathbb{P}^2$.

One can easily see that $\rho_n : S_n \to \mathbb{P}^2$ is a D_{2p_n}-covering such that $D(S_n/\mathbb{P}^2) = Z'$ and ρ_n is branched at $2(\mathcal{C}_1 + \mathcal{C}_2) + p_n(\mathcal{C}_3 + \mathcal{C}_4)$. In fact, S_n is isomorphic to X_n, but we do not need it here. Set

$$u := \frac{\varphi + \dfrac{1}{\varphi}}{2}.$$

Since u is a σ_f-invariant rational function, $u \in \mathbb{C}(\mathbb{P}^2)$. Let $\Phi_n : S_n \dashrightarrow \mathbb{P}^1$ and $\overline{\Phi}_n : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ be the rational maps given by $\sqrt[p_n]{\varphi}$ and u. Note that Φ_n is D_{2p_n}-equivariant and the following diagram

$$\begin{array}{ccc} S_n & \overset{\Phi_n}{\dashrightarrow} & \mathbb{P}^1 \\ \rho_n \downarrow & & \downarrow \varphi_{p_n} \\ \mathbb{P}^2 & \underset{\overline{\Phi}_n}{\dashrightarrow} & \mathbb{P}^1, \end{array} \tag{24}$$

is commutative, where $\varphi_{p_n} : \mathbb{P}^1 \to \mathbb{P}^1$ is the D_{2p_n}-covering in Example 4.17. Note that S_n is birational to the fibered product $\mathbb{P}^2 \times_{\mathbb{P}^1} \mathbb{P}^1$ over $\mathbb{P}^2$.

Write $u = \frac{F_0}{F_\infty}$, where F_0 and F_∞ are homogeneous polynomials. The polar divisor of u is $\mathcal{C}_3^+ + \mathcal{C}_3^- + \mathcal{C}_4^+ + \mathcal{C}_4^-$. This implies that the plane curve given by $F_\infty = 0$ is $\mathcal{C}_3 + \mathcal{C}_4$ and $\deg F_\infty = 4$. Using the commutative diagram (24) and since φ_{p_n} is branched at $2[1 : \pm 1] + p_n[0 : 1]$, the curves $\mathcal{D}_1$ and $\mathcal{D}_2$ given by the equations $F_0 - F_\infty = 0$ and $F_0 + F_\infty = 0$, respectively, satisfy either (i) or (ii) below:

- (i) $\mathcal{D}_1 = \mathcal{C}_1 + \mathcal{C}_2$ and $\mathcal{D}_2$ is of the form $2\mathcal{Q}$ for some conic (or vice versa).
- (ii) $\mathcal{D}_1 = \mathcal{C}_1 + 2L_1$ for some line L_1 and $\mathcal{D}_2 = \mathcal{C}_2 + 2L_2$ for some line L_2.

If (ii) occurs, it implies that $\mathcal{C}_3 + \mathcal{C}_4 \in \Sigma(\mathcal{C}_1 + 2L_1, \mathcal{C}_2 + 2L_2)$. This means that $\mathcal{C}_3 + \mathcal{C}_4$ passes through $\mathcal{C}_1 \cap \mathcal{C}_2$, but this is impossible by the combinatorics of $\mathcal{C} \in \mathcal{M}$. Hence (i) must occur and $\mathcal{Q}$ is the unique conic passing through all the tacnodes of $\mathcal{C}$. Q.E.D.

Remark 4.25. The proof above shows that there exists a tower of dihedral coverings $\{\tilde{\pi}_n : \tilde{X}_n \to \mathbb{P}^2\}_{n \geq 1}$ such that $\mathrm{Gal}(\tilde{X}_n/\mathbb{P}^2) \cong D_{2p_n}$ (p_n odd prime), where $\tilde{\pi}_n$ is branched at $2(\mathcal{C}_1 + \mathcal{C}_2) + p_n(\mathcal{C}_3 + \mathcal{C}_4)$, if and only if there exists another tower of dihedral coverings $\{\pi_m : X_m \to$

$\mathbb{P}^2\}_{m\geq 3}$ such that $\mathrm{Gal}(X_m/\mathbb{P}^2) \cong D_{2m}$, where π_m is branched at $2(\mathcal{C}_1 + \mathcal{C}_2) + m(\mathcal{C}_3 + \mathcal{C}_4)$.

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