On the Connection Between Braid Monodromies, Fundamental Groups, and Special Pencils of Plane Curves

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TRENZAS: Jornada Temática Interdisciplinar de la RET

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- *Topological*: Braid Monodromy, Fundamental Group, Alexander Polynomial.
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- *Algebraic*: Existence of pencils containing $\mathcal{C}$.

A New Look at a Classical Example

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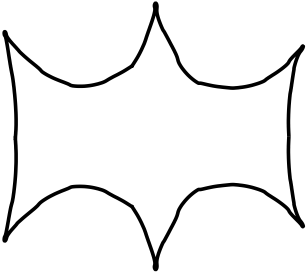

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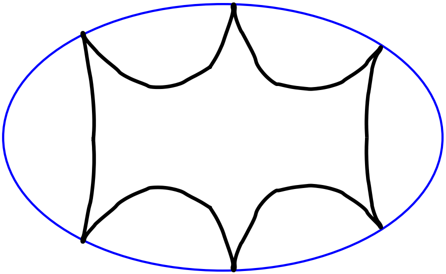

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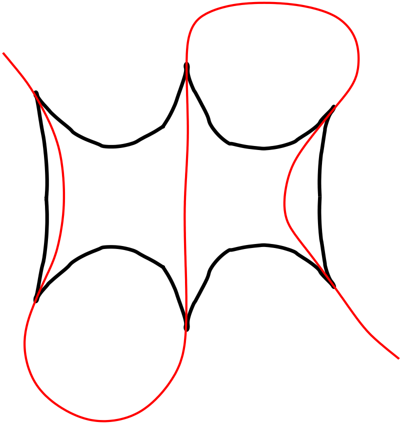

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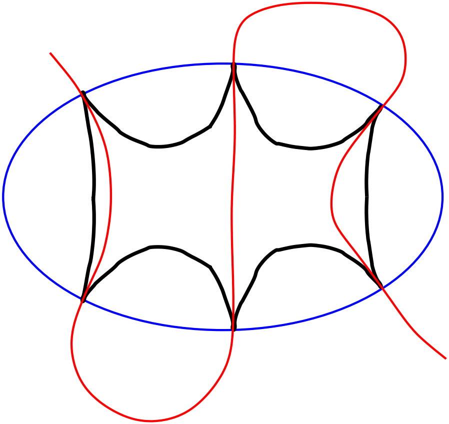

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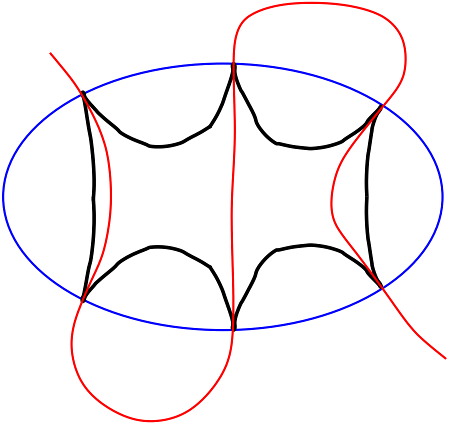

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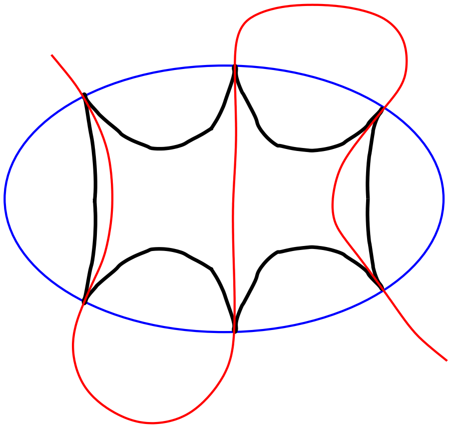

- $\pi_1(X) = \mathbb{Z}_2 * \mathbb{Z}_3$ and $\Delta_{\mathcal{C}}(t) = t^2 - t + 1$.
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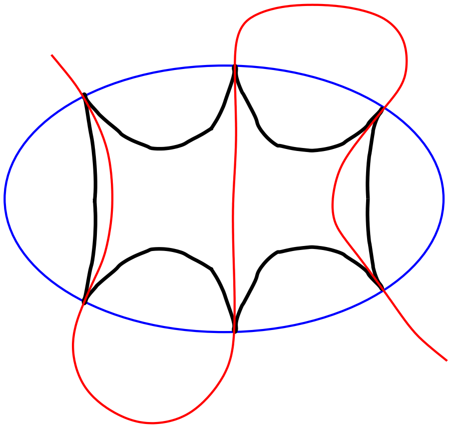

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- F belongs to the pencil generated by (h_2^3, h_3^2).

Geometric basis

$\bar{\mathcal{C}} = \mathcal{C}_0 \cup \mathcal{C}_1 \cup ... \cup \mathcal{C}_r$, $d_i = \deg \mathcal{C}_i$
$\mathcal{C}_0$ transversal line.
$\mathbb{C}^2 := \mathbb{P}^2 \setminus \mathcal{C}_0$, $\mathcal{C} := \bar{\mathcal{C}} \cap \mathbb{C}^2$
$\pi : \mathbb{C}^2 \setminus \mathcal{C} \to \mathbb{P}^1 \setminus Z_n$
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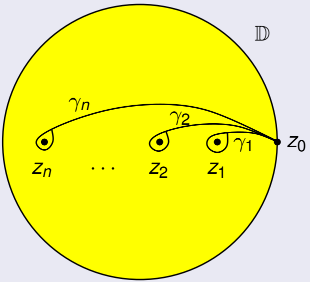
Definition

Geometric basis:



$$\gamma_n \gamma_{n-1} \cdots \gamma_1 = \partial \mathbb{D}$$

Braid Monodromy Representation

Definition

Consider the braid monodromy action:

$$\rho : \pi_1(\mathbb{D} \setminus Z_n, z_0) \longrightarrow Diff^+(F_{z_0}) \cong \mathbb{B}_d.$$

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$$(\rho\gamma_1, ..., \rho\gamma_n) \in \mathbb{B}_d^n$$

is the *Braid Monodromy Representation* of $\mathcal{C}$ relative to (π, Γ, z_0).

Braid Monodromy Representation

Remark

- $\rho(\gamma_n)\rho(\gamma_{n-1})\cdots\rho(\gamma_2)\rho(\gamma_1) = \Delta_d^2 = (\sigma_1\cdots\sigma_{d-1})^d$
  Braid Monodromy Factorization.

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- Both actions commute ($\mathbb{B}_n\times\mathbb{B}_d$). *Hurwitz Moves*.

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Goal

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We will put together several ingredients, among which, the *Van Kampen Theorem* is key.

Let $\pi : X \longrightarrow M$ be a locally trivial fibration with section $s : M \longrightarrow X$. Consider $p \in M$ and $x_0 \in F_p$.

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Theorem

$\pi_1(X, x_0) = \pi_1(F_p, x_0) \rtimes \pi_1(M, p)$, where the action of $\pi_1(M, p)$ on $\pi_1(F_p, x_0)$ is given by the monodromy of π.

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Meridians around the same irreducible components of B are conjugate in $\pi_1(M \setminus B)$. Moreover, the conjugacy class of a meridian coincides with the set of homotopy classes of meridians around the same irreducible component.

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Proposition

The inclusion $M \setminus B \hookrightarrow M$ induces a surjective morphism, whose kernel is given by the smallest normal subgroup of $\pi_1(M \setminus B)$ containing meridians of all the irreducible components of B.

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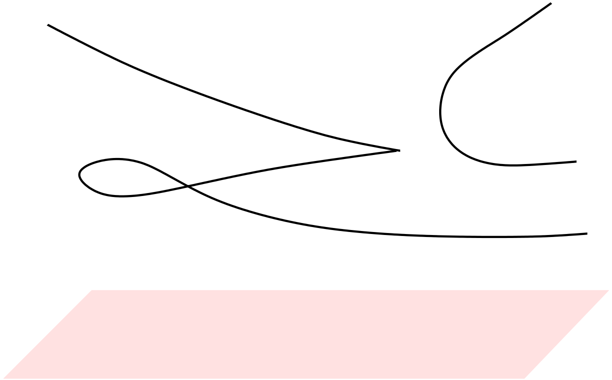

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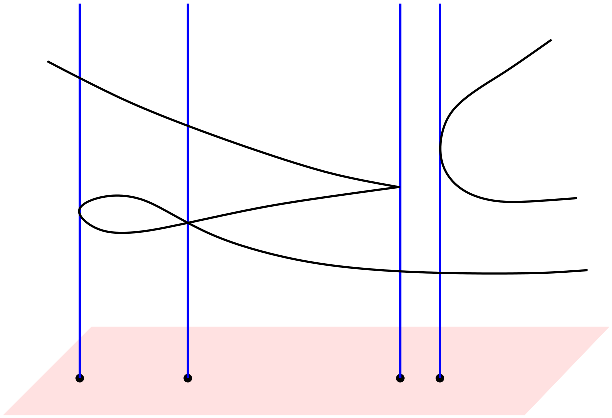

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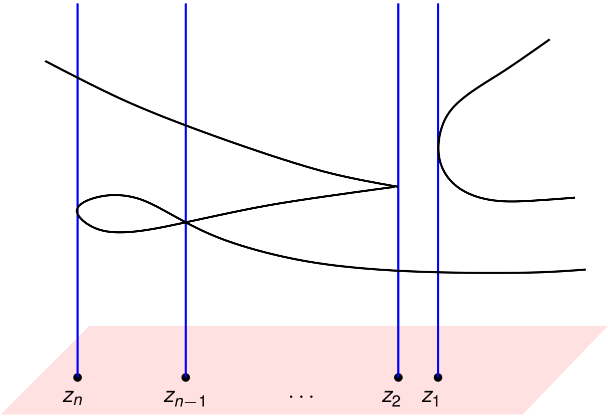

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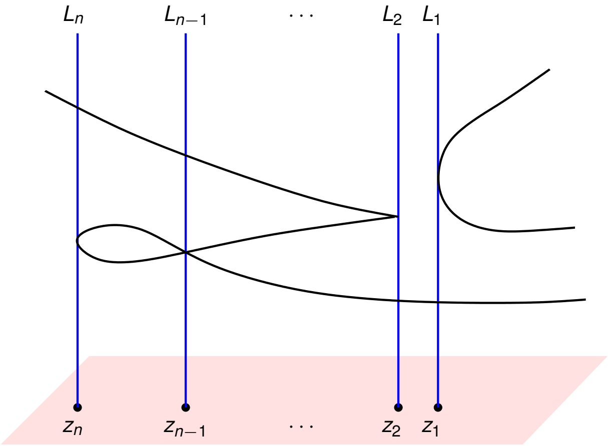

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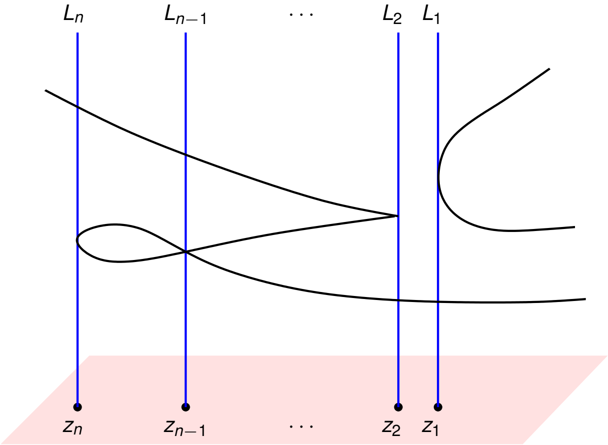

Remark (1)

Let $X = \mathbb{P}^2 \setminus (\mathcal{C} \cup L)$, then $\pi|_X : X \to \mathbb{P}^1 \setminus Z_n$ is a locally trivial fibration.

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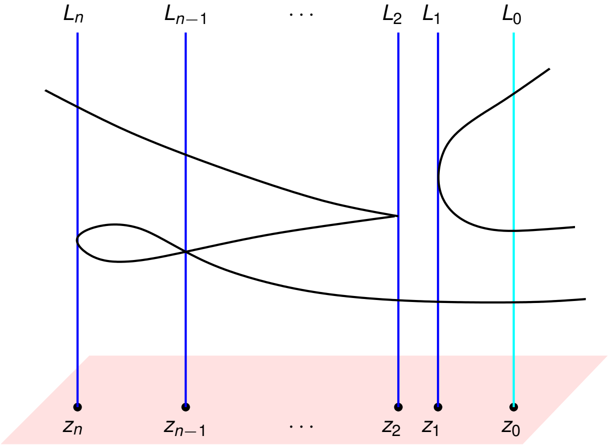

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Let $X = \mathbb{P}^2 \setminus (\mathcal{C} \cup L)$, then $\pi|_X : X \to \mathbb{P}^1 \setminus Z_n$ is a locally trivial fibration. Moreover, its fiber is $\mathbb{P}^1 \setminus Z_d$, where $d := \deg \mathcal{C}$.

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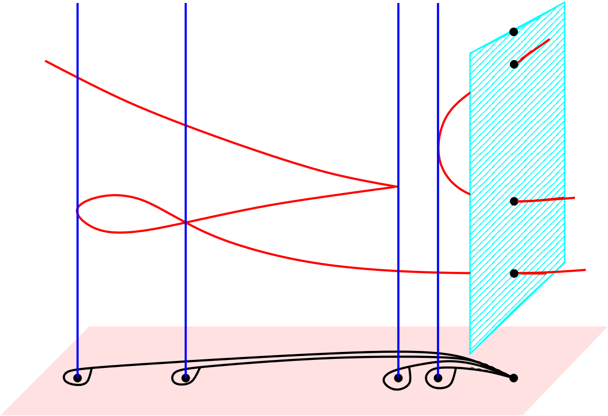

Remark (2)

By (2.3), $\pi_1(X, x_0) = \pi_1(F_{z_0}, x_0) \rtimes \pi_1(\mathbb{P}^1 \setminus Z_n, z_0)$. Action is given by the monodromy of $\pi_1(\mathbb{P}^1 \setminus Z_n, z_0)$ on $\pi_1(F_{z_0}, x_0)$ as follows :.

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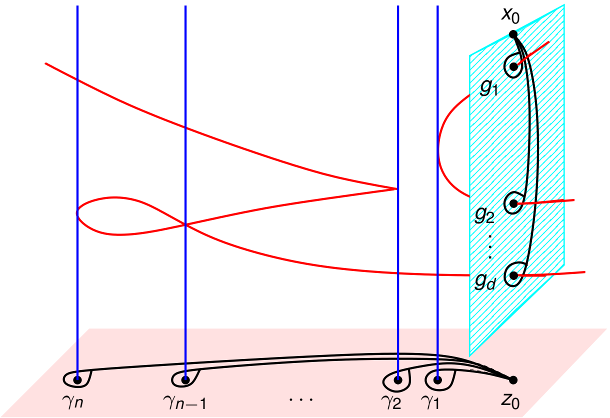

Remark (3)

Note that $\pi_1(F_{z_0}, x_0) = \langle g_1, ..., g_d : g_d g_{d-1} \cdots g_1 = 1 \rangle$ and $\pi_1(\mathbb{P}^1 \setminus Z_n, z_0) = \langle \gamma_1, ..., \gamma_n : \gamma_n \cdots \gamma_1 = 1 \rangle$.

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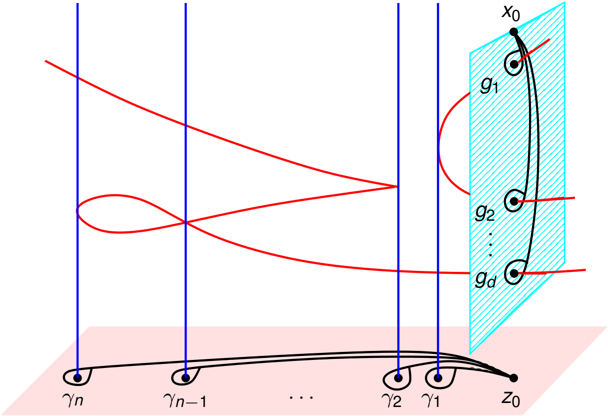

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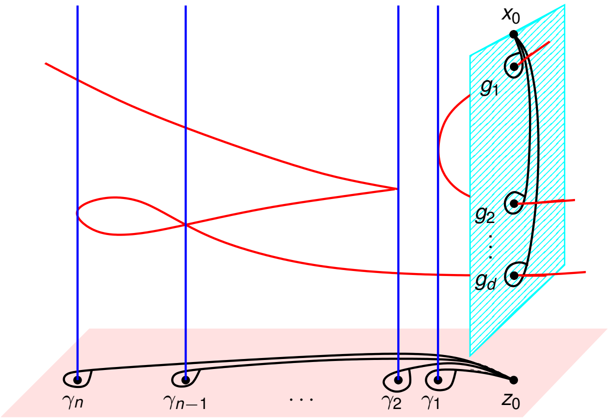

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Remark

- Let $\mathcal{C} = \mathcal{C}_1 \cup ... \cup \mathcal{C}_r$ the decomposition of $\mathcal{C}$ in its irreducible components, then

$$H_1(\mathbb{P}^2 \setminus \mathcal{C}) = \mathbb{Z}^{r-1} \oplus \mathbb{Z}/(d_1, ..., d_r),$$

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- Let $\mathbb{Q}[\mathbb{Z}] = \mathbb{Q}[t, t^{-1}]$ denote the group ring over $\mathbb{Q}$.
- $$K_\varepsilon := \ker \varepsilon \quad \text{and} \quad K'_\varepsilon := [K_\varepsilon, K_\varepsilon]$$
- $K_\varepsilon / K'_\varepsilon$ can be viewed as a module over the group ring $\mathbb{Z}[\mathbb{Z}]$.

Alexander Polynomials of a curve

We shall consider reducible, not necessarily reduced curves in $\mathbb{P}^2$.

$$\mathcal{C} := \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_r$$

$$(\varepsilon_1, \ldots, \varepsilon_r), \varepsilon_i \in \mathbb{Z} \text{ multiplicities.}$$

- Let $\mathcal{C}_0$ be a line transversal to $\mathcal{C}$.
- Recall that $H_1(\mathbb{P}^2 \setminus \mathcal{C}_0 \cup \mathcal{C}) = \mathbb{Z}^r$.
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Definition ([5])

The Alexander polynomial $\Delta_{\mathcal{C},\varepsilon}(t)$ of G relative to surjection $\varepsilon : G \to \mathbb{Z}$ is the order of the torsion of the $\mathbb{Q}[\mathbb{Z}] = \mathbb{Q}[t, t^{-1}]$-module $K_\varepsilon / K'_\varepsilon \otimes \mathbb{Q}$.

Alexander Polynomials of a curve

Theorem ([6],-)

The Alexander polynomial of $\mathcal{C}$ w.r.t. ε is the first invariant of the colored Burau representation matrix of the braid monodromy of $\mathcal{C}$ w.r.t. ε divided by $(1 - t_1^{\varepsilon_1} \cdots t_r^{\varepsilon_r})$.

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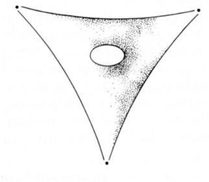
Colored Burau Representation:

$$\sigma_1 \mapsto \begin{pmatrix} -t_i^{\varepsilon_i} & 1 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ & \dots & & \dots & \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} \quad \sigma_i \mapsto \begin{pmatrix} \begin{matrix} 1 & \dots & 0 \\ & \dots & \\ 0 & \dots & 1 \end{matrix} & 0 & 0 \\ 0 & \begin{matrix} 1 & 0 & 0 \\ t_i^{\varepsilon_i} & -t_i^{\varepsilon_i} & 1 \\ 0 & 0 & 1 \end{matrix} & 0 \\ 0 & 0 & \begin{matrix} 1 & \dots & 0 \\ & \dots & \\ 0 & \dots & 1 \end{matrix} \end{pmatrix}$$

Definition (Orbifold)

An *orbifold* curve $S_{\bar{m}}$ is a Riemann surface S with a function $\bar{m} : S \to \mathbb{N}$ whose value is 1 outside a finite number of points. A point $P \in S$ for which $\bar{m}(P) > 1$ is called an *orbifold point*.



Orbifolds and Orbifold Fundamental Groups

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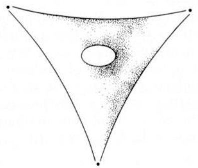

Definition (Orbifold Fundamental Group)

For an orbifold $S_{\bar{m}}$, let $P_1, \ldots, P_n$ be the orbifold points, $m_j := \bar{m}(P_j) > 1$. Then, the *orbifold fundamental group* of $S_{\bar{m}}$ is

$$\pi_1^{\mathrm{orb}}(S_{\bar{m}}) := \pi_1(S \setminus \{P_1, \ldots, P_n\})/\langle \mu_j^{m_j} = 1 \rangle,$$

where μ_j is a meridian of P_j. We will denote $S_{\bar{m}}$ simply by $S_{m_1,\ldots,m_n}$.

Orbifold Morphisms

Definition

A dominant algebraic morphism $\varphi : X \longrightarrow S$ defines an *orbifold morphism* $X \longrightarrow S_{\bar{m}}$ if for all $P \in S$, the divisor $\varphi^*(P)$ is a $\bar{m}(P)$-multiple.

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Proposition ([1, Proposition 1.5])

Let $\rho : X \to S$ define an orbifold morphism $X \to S_{\bar{m}}$. Then φ induces a morphism $\varphi_* : \pi_1(X) \to \pi_1^{\mathrm{orb}}(S_{\bar{m}})$.
Moreover, if the generic fiber is connected, then φ_* is surjective.

Example

Suppose F fits in a functional equation of type

$$F_2 h_2^3 + F_3 h_3^2 + F = 0, \tag{1}$$

Then (1) *induces a pencil map* $\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ *given by* $\varphi([x : y : z]) = [h_2^3 : h_3^2]$. *Since* $\varphi|_{\mathbb{P}^2 \setminus \mathcal{C}}$ *has two multiple fibers (over* $[0 : 1]$, $[1 : 0]$*) one has an orbifold morphism* $\varphi_{2,3} : \mathbb{P}^2 \setminus \mathcal{C} \to \mathbb{P}^1_{2,3} \setminus \{[1 : -1]\}$. *In particular, if the quasi-toric relation is primitive, then by Proposition 3.4, there is an epimorphism*

$$\varphi_{2,3} : \pi_1(\mathbb{P}^2 \setminus \mathcal{C}) \to \pi_1^{\mathrm{orb}}(\mathbb{P}^1_{2,3} \setminus \{[1 : -1]\}) = \mathbb{Z}_2 * \mathbb{Z}_3.$$

The Classical Example

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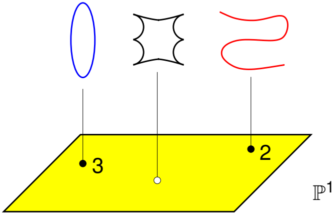

Another Application

Corollary

The number of multiple members in a pencil of plane curves (with no base components) is at most two.

Functional Relation $F_1 h_1^p + F_2 h_2^q + F_3 h_3^r = 0$

Definition

A curve $\mathcal{C} := \{F = 0\}$ satisfies a *quasi-toric relation* of type (p, q, r) if there exist homogeneous polynomials $h_1, h_2, h_3 \in \mathbb{C}[x, y, z]$ such that

$$h_1^p F_1 + h_2^q F_2 + h_3^r F_3 = 0,$$

where F_1, F_2, F_3 are homogeneous polynomials and $\{F_1 F_2 F_3 = 0\} = \mathcal{C}$.

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Theorem ([3])

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3. *The polynomial F fits in a quasi-toric relation of type $(3,3,3)$ (resp. $(2,3,6)$).*

Moreover, the set of quasi-toric relations of type $(3,3,3)$ (resp. $(2,3,6)$) has a group structure, whose rank is twice the multiplicity of ξ as a root of $\Delta_{\mathcal{C},\varepsilon}(t)$.

Examples

Example

Since the 6-cuspidal sextic $\mathcal{C}_{6,6}$ *from Example 4 is such that:* $\Delta_{\mathcal{C}_{6,6}}(t) = (t^2 - t + 1)$*, the decomposition* $F = f_2^3 + f_3^2$ *is essentially unique.*

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$F = (y^3 - z^3)(z^3 - x^3)(x^3 - y^3)$, $\mathcal{C} := \{F = 0\}$, *then* $\Delta_{\mathcal{C}}(t) = (t^2 + t + 1)^2(t - 1)^8$.

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However, there should exist another relation independent from (2)*, namely*

$$\ell_1^3 F_1 + \ell_2^3 F_2 + \ell_3^3 F_3 = 0, \tag{3}$$

where

$$F_i = (y - \omega_3^i z)(z - \omega_3^{i+1} x)(x - \omega_3^{i+2} y), \quad i = 1, 2, 3,$$

ω_3 *is a third-root of unity, and*

$$\begin{aligned}
\ell_1 &= (\omega_3 - \omega_3^2)x + (\omega_3 - \omega_3^2)y + (\omega_3^2 - 1)z, \\
\ell_2 &= (\omega_3 - \omega_3^2)z + (\omega_3 - \omega_3^2)x + (\omega_3^2 - 1)y, \\
\ell_3 &= (\omega_3 - \omega_3^2)y + (\omega_3 - \omega_3^2)z + (\omega_3^2 - 1)x.
\end{aligned}$$

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Braid Action