TOPOLOGY OF ARRANGEMENTS AND POSITION OF SINGULARITIES

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Contents

1. Definitions and settings

Definition 1.1. Let K be a field, $n \in \mathbb{N}$. A *hyperplane arrangement* is a finite collection of hyperplanes in one of the following cases:

(1) Linear hyperplanes of a K-vector space V of dimension n: *central arrangement.*
(2) Affine hyperplanes of a K-affine space E of dimension n: *affine arrangement.*
(3) Projective hyperplanes of a K-projective space $\mathbb{P}(V)$ of dimension n: *projective arrangement.*

These three concepts are closely related. Let us identify V and E with K^n. Note that central arrangement is a particular case of an affine arrangement. If we consider the standard embedding $K^n \hookrightarrow \mathbb{P}(K^{n+1}) =: \mathbb{P}^n(K)$, adding the hyperplane at infinity $H_\infty := \mathbb{P}^n(K) \setminus K^n$ to the collection we construct a projective arrangement from an affine arrangement. Finally, a projective arrangement in $\mathbb{P}^n(K)$ is essentially the same object as a central arrangement in K^{n+1}.

Definition 1.2. Let $\mathcal{A}$ be a hyperplane arrangement. The *complement* $M(\mathcal{A})$ of the arrangement is the complement of $\bigcup \mathcal{A}$ in the ambient space.

Remark 1.3. Let $\mathcal{A}$ be a central arrangement; it is obvious that $M(\mathcal{A})$ coincides as both central and affine arrangement. If $\mathcal{A}$ is an affine arrangement and $\mathcal{A}_\infty$ is the projective arrangement obtained adding H_∞, then $M(\mathcal{A}) = M(\mathcal{A}_\infty)$. Finally, if $\mathcal{A}$ is a non-empty projective arrangement and $\tilde{\mathcal{A}}$ is the corresponding central arrangement, then there is a natural identification $M(\tilde{\mathcal{A}}) \leftrightarrow M(\mathcal{A}) \times K^*$.

The combinatorics of an arrangement $\mathcal{A}$ is the poset $S(\mathcal{A})$ of all the intersections of elements in $\mathcal{A}$, with respect to reverse inclusion. The combinatorics catch the properties of $\mathcal{A}$ not depending on the actual equations of the hyperplanes. The main goal is to detect which properties of the arrangement depend only on the combinatorics.

In this notes $K = \mathbb{C}$ and the properties we are looking for are of topological type. Note that we may chose other fields with topological structure, specially $\mathbb{R}$ or the p-adic number field. We prefer the complex numbers since in this case the arrangement is codimension-2 topological subspace of the ambient space.

Example 1.4. Let $\mathcal{A}$ be a central arrangement in $\mathbb{C}^n$. Orlik and Solomon defined an graded algebra $A(\mathcal{A})$ which depends only on $S(\mathcal{A})$. They proved that $A(\mathcal{A})$ is isomorphic to $H^*(M(\mathcal{A}); \mathbb{Z})$ as $\mathbb{Z}$-algebra [23, 22].

This result opened an intensive research in order to find which topological invariants are combinatorial. In 1994, G. Rybnikov found two arrangements with the same combinatorics and such that the fundamental groups of their complements are non-isomorphic. This result was finally published in [24], see also [5]. There are other examples of arrangements with different topology and same combinatorics: in [4], two combinatorially-equivalent arrangements had different homeomorphism type for $(\mathbb{P}^2(\mathbb{C}), \mathcal{A})$. Though $\pi_1(M(\mathcal{A}))$ is not a combinatorial invariant, it is one of the most important topological invariants of the hyperplane arrangements. Zariski-Lefschetz theory shows that for computing fundamental groups we can restrict our attention to the case of line arrangements.

Theorem 1.5 (Zariski-Lefschetz [27])**.** *Let X be a quasi-projective smooth variety in $\mathbb{P}^2(\mathbb{C})$ of dimension n. Let H be a generic hyperplane. Then the morphism*

$\pi_j(X \cap H) \to \pi_j(X)$ induced by the injection is an isomorphism for $j < n-1$ and epimorphism for $j = n-1$.

Corollary 1.6. *The fundamental group of a hyperplane arrangement is also the fundamental group of a line arrangement.*

2. Zariski-van Kampen method and braid monodromy

For technical reasons it is better to consider affine arrangements instead of projective arrangements. Let $\mathcal{A} := \{\bar{L}_0, \bar{L}_1, \dots, \bar{L}_n\}$ be a line arrangement in $\mathbb{P}^2$. We fix a line $\bar{L}_\infty$ as line at infinity and consider $\mathbb{C}^2 = \mathbb{P}^2 \setminus \bar{L}_\infty$; we will denote $L_j := \bar{L}_j \cap \mathbb{C}^2$. There are several *natural* choices for $\bar{L}_\infty$.

($L_\infty 1$) We may choose $\bar{L}_\infty = \bar{L}_0$; in this case we will denote $\mathcal{A}^0 := \{L_1, \dots, L_n\}$ the associated affine line arrangement. Note that $M(\mathcal{A}) = M(\mathcal{A}^0)$.

($L_\infty 2$) Choose a generic $\bar{L}_\infty \pitchfork \bigcup \mathcal{A}$. In this case the associated line arrangement is denoted as $\mathcal{A}^\infty := \{L_0, L_1, \dots, L_n\}$. Of course, $M(\mathcal{A}) \neq M(\mathcal{A}^\infty)$ but:
  - (a) The topological type of $M(\mathcal{A}^\infty)$ does not depend on the particular choice of $\bar{L}_\infty$.
  - (b) As it will be proved later, the group $\pi_1(M(\mathcal{A}^\infty))$ is a central extension of $\pi_1(M(\mathcal{A}))$ by $\mathbb{Z}$.

Now, we will fix $\mathcal{A} := \{L_1, \dots, L_n\}$ an affine arrangement in $\mathbb{C}^2$. We consider the set of multiple points of the arrangement:

$$\mathcal{P} := \{P \in \mathbb{C}^2 \mid \exists i < j \text{ s.t. } P \in L_i \cap L_j\}.$$

For $P \in \mathcal{P}$, we denote $m_P := \#\{L \in \mathcal{A} \mid P \in L\}$. In order to apply Zariski-van Kampen method we need to consider the projection $\pi : \mathbb{C}^2 \to \mathbb{C}$, $(x, y) \mapsto x$. This projection depends on the choice of coordinates x, y.

Definition 2.1. The coordinates (and hence the projection) are said to be *generic* if no line L_j is vertical and if $(x, y_1), (x, y_2) \in \mathcal{P}$ then $y_1 = y_2$.

Remark 2.2. Later on we will also consider non-generic projections.

2.1. Fibered arrangements.

Definition 2.3. Let $\mathcal{A} = \{L_1, \dots, L_n\}$ be an affine arrangement in generic coordinates and let $\mathcal{P} = \{(x_1, y_1), \dots, (x_r, y_r)\}$. The *fibered arrangement* associated to $\mathcal{A}$ is

$$\mathcal{A}^\varphi := \mathcal{A} \cup \{V_{x_i} \mid i = 1, \dots, r\}, \quad V_t := \{x = t\}.$$

Proposition 2.4. *Let* $\mathcal{B} := \{x_1, \dots, x_r\}$. *The restriction* $\pi_| : M(\mathcal{A}^\varphi) \to \mathbb{C} \setminus \mathcal{B}$ *is a locally trivial fibration, the fiber is homeomorphic to* $F := \mathbb{C} \setminus \{n \text{ points}\}$. *In particular, the long exact homotopy sequence induces a short exact sequence*

$$1 = \pi_2(\mathbb{C} \setminus \mathcal{B}) \to \pi_1(F) \to \pi_1(M(\mathcal{A}^\varphi)) \to \pi_1(\mathbb{C} \setminus \mathcal{B}) \to 1.$$

Definition 2.5. Let X be a quasi-projective smooth variety, and let $A_1, \dots, A_r \subset X$ be irreducible hypersurfaces. Let $Y := X \setminus \bigcup_{j=1}^r A_i$ and $p \in Y$. A *meridian* of A_i in $\pi_1(Y; p)$ is obtained as follows:

- Fix a smooth point $p_i \in A_i$ of $\bigcup_{j=1}^r A_i$.
- Fix a *small* closed disk $\mathbb{D}_i$ centered at p_i transversal to A_i such that $\bigcup_{j=1}^r A_i \cap \mathbb{D}_i = \{p_i\}$.
- Let δ_i be the loop based at q_i which runs along $\partial\mathbb{D}_i$ counterclockwise.
- Let α_i be a path in Y from p to q_i.

Then, $\alpha_i \cdot \delta_i \cdot \alpha_i^{-1}$ is such a meridian. The conjugacy class of such a meridian does not depend on the above choices.

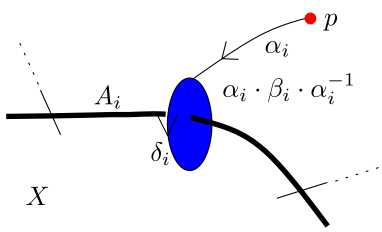

FIGURE 1. Meridian

Proposition 2.4 allows to produce a finite presentation of the group $\pi_1(M(\mathcal{A}^\varphi))$; we need to fix bases of the free groups in the extremities of the short exact sequence.

Definition 2.6. A *geometric basis* of the free group $\pi_1(\mathbb{C} \setminus \{t_1, \dots, t_r\}; t_0)$ is a basis of meridians $\mu_1, \dots, \mu_r$ (μ_i meridian of t_i) such that $(\mu_r \cdot \ldots \cdot \mu_1)^{-1}$ is a meridian of ∞.

It is useful to have a compact model of $M(\mathcal{A}^\varphi)$. Let $t_x \gg 0$ such that $\mathcal{B} \subset \mathring{\mathbb{D}}_{t_x}$. Consider also t_y such that $|y_j| \ll t_y$ and

$$\bigcup \mathcal{A} \cap (\mathbb{D}_{t_x} \times \mathbb{D}_{t_y}) \subset \partial\mathbb{D}_{t_x} \times \mathring{\mathbb{D}}_{t_y}.$$

The inclusion $(\mathbb{D}_{t_x} \times \mathbb{D}_{t_y}) \setminus \bigcup \mathcal{A}^\varphi \hookrightarrow M(\mathcal{A}^\varphi)$ is a homotopy equivalence. Let $p := (t_x, t_y)$ and denote $F := V_{t_x} \setminus \bigcup \mathcal{A}$.

Let us fix a geometric basis $\mu_1, \dots, \mu_n$ of the free group $\pi_1(F; p)$ and a geometric basis $\alpha_1, \dots, \alpha_r$ of $\pi_1(\mathbb{C} \setminus \mathcal{B}; t_x)$. Let us lift $\alpha_1, \dots, \alpha_r$ to $\tilde{\alpha}_1, \dots, \tilde{\alpha}_r$ in the line $y = t_y$.

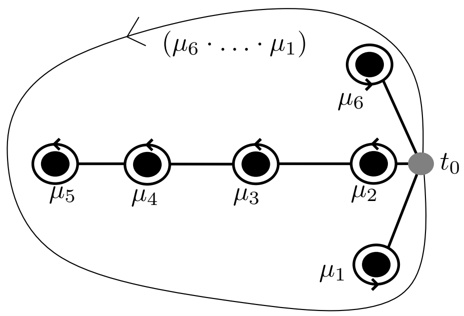

FIGURE 2. Geometric basis for $r = 6$

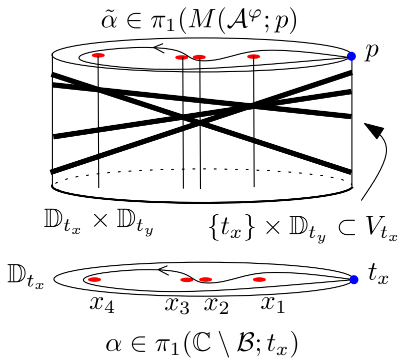

FIGURE 3. Polydisk model

Lemma 2.7. *The elements* $\mu_1, \ldots, \mu_n$ *and* $\tilde{\alpha}_1, \ldots, \tilde{\alpha}_r$ *generate* $\pi_1(M(\mathcal{A}^\varphi); p)$.

We state a general result (due to Fujita [14]) which will be necessary to compute $\pi_1(M(\mathcal{A}))$ from $\pi_1(M(\mathcal{A}^\varphi))$.

Lemma 2.8. *Let us consider the notation of* Proposition 2.4. *Consider the spaces* $Y := X \setminus \bigcup_{j=1}^r A_j$, $Z := X \setminus \bigcup_{j=s+1}^r A_j$, $1 \leq s \leq r$. *The inclusion* $Y \hookrightarrow Z$ *induces an epimorphism* $\pi_1(Y) \to \pi_1(Z)$. *Its kernel is generated by the meridians of* $A_1, \ldots, A_s$.

Idea of the Proof. First, we recall that $\pi_1 = \pi_1^{\mathcal{C}^\infty}$. The surjectivity follows from transversality of mappings $\mathbb{S}^1 \to Z$ with respect to A_j, $1 \leq j \leq s$. The description of the kernel follows from transversality of mappings $\mathbb{D}^2 \to Z$ with respect to A_j, $1 \leq j \leq s$. □

Proposition 2.9. *Let* $\mathcal{A}$ *be an affine line arrangement.*

(1) $\pi_1(M(\mathcal{A}); p)$ *is generated by* $\mu_1, \ldots, \mu_n$.
(2) $\mu_\infty := (\mu_n \cdot \ldots \cdot \mu_1)^{-1}$ *is a meridian of* $\bar{L}_\infty$ *in* $\pi_1(M(\mathcal{A}); p)$.
(3) *If* $\bar{L}_\infty \pitchfork \bigcup \mathcal{A}$ *then* μ_∞ *is central in* $\pi_1(M(\mathcal{A}); p)$.

Remark 2.10. Proposition 2.9(1) is a particular case of the surjectivity statement of Theorem 1.5. Proposition 2.9(3) is the centrality statement of $(L_\infty 2)$(b).

Proof. The statement (1) is a direct consequence of Lemmas 2.7 and 2.8. The statement (2) comes from the genericity condition; since the projection point $[0 : 1 : 0] \notin \bigcup \mathcal{A}$, then the result follows.

Let us prove the statement (3). Let $E := \mathbb{C}^2 \setminus (\mathring{\mathbb{D}}_{t_x} \times \mathring{\mathbb{D}}_{t_y})$ and set $\check{E} := E \setminus \bigcup \mathcal{A}$. By Lemma 2.8 the map $\pi_1(\check{E}; p) \twoheadrightarrow \pi_1(M(\mathcal{A}); p)$. The space $\check{E}$ is homeomorphic to $(\mathbb{C} \setminus \{n-1 \text{ points}\}) \times \mathbb{D}^*$; the meridian μ_∞ corresponds to the boundary of the factor $\mathbb{D}^*$. ☐

Remark 2.11. In orden to describe the group $\pi_1(M(\mathcal{A}^\varphi))$ (and hence $\pi_1(M(\mathcal{A}))$) we only need the conjugation action of $\tilde{\alpha}_j$ on $\pi_1(F; p) \subset \pi_1(M(\mathcal{A}); p)$.

2.2. **Braid action on free groups.**

Consider $\alpha \in \pi_1(\mathbb{C} \setminus \mathcal{B}; t_x) \equiv \pi_1(\mathbb{D}_{t_x} \setminus \mathcal{B}; t_x)$ represented by a closed loop $\alpha : [0, 1] \to \mathbb{D}_{t_x} \setminus \mathcal{B}$ as in Figure 3. We represent $(\alpha([0, 1]) \times \mathbb{D}_{t_y}) \setminus \bigcup \mathcal{A}$ in a cylinder where the top and bottom bases are identified, see Figure 4. The lift $\tilde{\alpha}$ is represented as an upward vertical path in Figure 4.

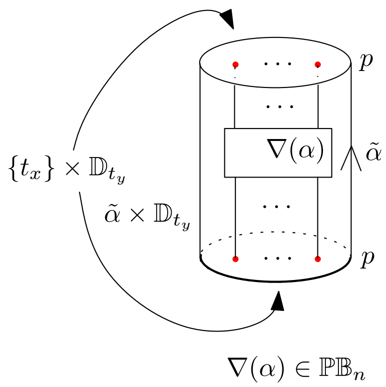

FIGURE 4. Action of the braid α

Let $\beta \in \pi_1(F; p)$ (F is represented by the identified bases of the cylinder in Figure 4). Let us consider the loop $\tilde{\alpha}^{-1} \cdot \beta \cdot \tilde{\alpha}$. This loop is homotopic to a loop in the top basis, i.e. it may be represented as an element in $\pi_1(F; p)$. This new loop depends on the trace of $\mathcal{A}$. A more precise description can be done using the homotopy equivalence between the complement of $\mathcal{A}$ in the cylinder and the punctured bases. In order to describe this conjugation we need to introduce the braid groups and their action on free groups.

Let $X_n := \{\mathbf{x} \in \mathbb{C}^n \mid x_i \neq x_j,\ 1 \leq i < j \leq n\}$ (the complement of the braid arrangement). The symmetric group Σ_n acts freely by entry-permutation on X_n. The quotient X_n/Σ_n is naturally identified with $Y_n := \{f(t) \in \mathbb{C}[t] \mid f \text{ monic without multiple roots}, \ \deg f = n\}$.

Definition 2.12. The fundamental group of X_n is the *braid group* $\mathbb{B}_n$ in n strands, while the fundamental group of Y_n is the *pure braid group* $\mathbb{PB}_n$ in n strands.

The group $\mathbb{B}_n$ admits the well-known Artin presentation

$$\mathbb{B}_n := \left\langle \sigma_1, \dots, \sigma_{n-1} \;\middle|\; \underset{1<i+1<j<n}{[\sigma_i, \sigma_j] = 1}, \quad \underset{1\leq i<n-1}{\sigma_i \cdot \sigma_{i+1} \cdot \sigma_i = \sigma_{i+1} \cdot \sigma_i \cdot \sigma_{i+1}} \right\rangle.$$

It is identified with the homotopy classes of n (non-intersecting) paths in $\mathbb{C}$ such that the sets of starting and ending points coincide. The pure braid group consists of classes where all paths are loops.

For the sake of simplicity let us fix the point $\mathbf{x} := (x_1, \dots, x_n)$, $x_i := -i$, as base point for the fundamental group. With this identification the generator σ_i is represented as in Figure 5. The group $\mathbb{B}_n$ acts naturally on the free group $\mathbb{F}_n$

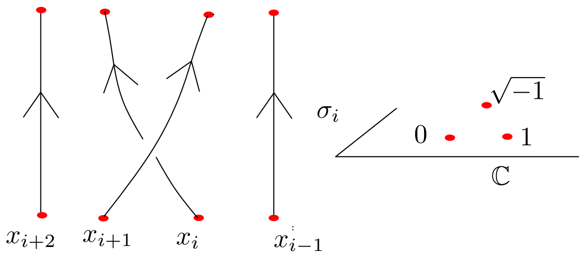

FIGURE 5.

generated by $\mu_1, \dots, \mu_n$. It is defined as follows:

$$\begin{array}{ccc} \mathbb{F}_n \times \mathbb{B}_n & \to & \mathbb{F}_n \\ (\mu, \tau) & \mapsto & \mu^\tau. \end{array}$$

where

$$\mu_i^{\sigma_j} := \begin{cases} \mu_{i+1} & \text{if } i = j \\ \mu_{i+1}\mu_i\mu_{i+1}^{-1} =: \mu_{i+1} * \mu_i & \text{if } i = j+1 \\ \mu_i & \text{if } i \neq j, j+1. \end{cases}$$

This action can be understood geometrically identifying $\mathbb{F}_n$ with the fundamental group $\pi_1(\mathbb{C} \setminus \{x_1, \dots, x_n\}; x_0)$, $x_0 := -(n+1)$, where $\mu_1, \dots, \mu_n$ is a geometric basis, see Figure 6.

The trace of $\mathcal{A}$ in this cylinder defines a braid $\nabla(\alpha)$. Moreover, a morphism $\nabla : \pi_1(\mathbb{C} \setminus \mathcal{B}; t_x) \to \mathbb{PB}_n \subset \mathbb{B}_n$ is defined where $\mathbb{B}_n$ is the braid group in n strands while $\mathbb{PB}_n$ is the pure braid group. With these arguments we can finally state Zariski-van Kampen Theorem.

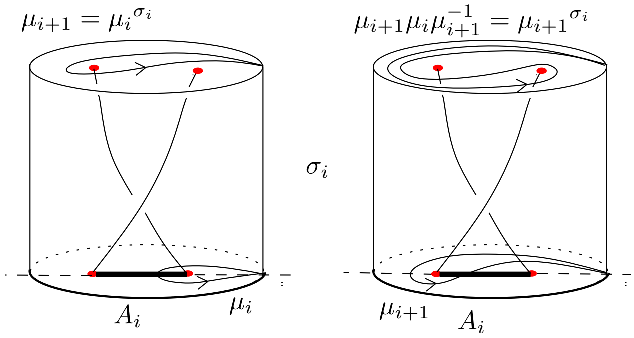

FIGURE 6.

Theorem 2.13. *The groups $\pi_1(M(\mathcal{A}^\varphi);p)$ and $\pi_1(M(\mathcal{A});p)$ admit the following finite presentations:*

(1) $\pi_1(M(\mathcal{A}^\varphi);p) = \langle \mu_1,\dots,\mu_n,\tilde{\alpha}_1,\dots,\tilde{\alpha}_r \mid \mu_i^{\tilde{\alpha}_j} = \mu_i^{\nabla(\alpha_j)}\rangle$.
(2) $\pi_1(M(\mathcal{A});p) = \langle \mu_1,\dots,\mu_n \mid \mu_i = \mu_i^{\nabla(\alpha_j)}\rangle$.

Example 2.14. Let $\mathcal{A}$ be an affine arrangement of n lines through $(0,0)\in\mathbb{C}^2$. In this case $\mathcal{B}=\{0\}$; hence $\pi_1(\mathbb{C}\setminus\mathcal{B}) = \langle \alpha \mid - \rangle$ where α meridian of 0. It is easily seen that $\nabla(\alpha)=\Delta_n^2$; this element is known as the *full-twist* and is a generator of the center of $\mathbb{B}_n$. A straightforward computation yields

$$\mu_i^{\Delta_n^2} = (\mu_n\cdot\ldots\cdot\mu_1)\cdot\mu_i\cdot(\mu_n\cdot\ldots\cdot\mu_1)^{-1}.$$

Hence $\pi_1(M(\mathcal{A})) = \langle \mu_1,\dots,\mu_n \mid [\mu_1,\dots,\mu_n]=1\rangle$, where $[\mu_1,\dots,\mu_n]=1$ is the set of relations:

$$[\mu_n\cdot\ldots\cdot\mu_1,\mu_i]=1,\quad 1\leq i<n.$$

Remark 2.15. For any arrangement $\mathcal{A}$, the behavior of Example 2.14 is the local behavior $\forall P\in\mathcal{P}$.

2.3. **Puiseux braid monodromy.**

Proposition 2.16. $\forall j\in\{1,\dots,r\}$, $\nabla(\alpha_j)=\tau_j^{-1}\cdot\Delta_{a_j,b_j}^2\cdot\tau_j$, *where* $1\leq a_j<b_j\leq n$ *and* Δ_{a_j,b_j}^2 *is the full-twist involving the* b_j-a_j+1 *strands from* a_j *to* b_j.

Proposition 2.17. *Let* $\mu_{i,j} := \mu_i^{\tau_j}$. *Then, the set of relations* $\mu_i=\mu_i^{\nabla(\alpha_j)}$ *for fixed* j, *can be replaced by* $[\mu_{1,j},\dots,\mu_{n,j}]=1$.

Proof. Let us fix $j\in\{i,\dots,r\}$. The set of relations $\mu_i=\mu_i^{\nabla(\alpha_j)}$, $i=1,\dots,n$ is equivalent to the set of relations $\mu=\mu^{\nabla(\alpha_j)}$ for all $\mu\in\pi_1(F)$. This is true since $\mu_1,\dots,\mu_r$ is a basis of $\pi_1(F)$.

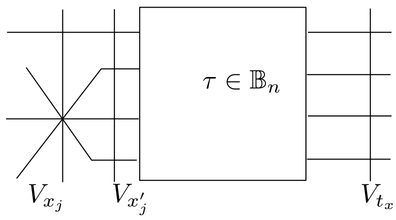

FIGURE 7.

Since the action of τ_j defines an automorphism of $\pi_1(F)$, it is also equivalent to the set of relations $\mu^{\tau_j} = (\mu^{\Delta^2_{a_j,b_j}})^{\tau_j}$ for all $\mu \in \pi_1(F)$. The, it is also equivalent to $\mu_i^{\tau_j} = (\mu_i^{\Delta^2_{a_j,b_j}})^{\tau_j}, a_j \leq i < b_j$, since for the other terms the relation is trivial.

Note that geometrically $\mu_{i,j}$ correspond to a basis of $\pi_1(F)$ in $V_{x'_j}$. $\square$

Example 2.18. In Figure 8, we see how the basis of the fiber changes before and after a point of multiplicity 5.

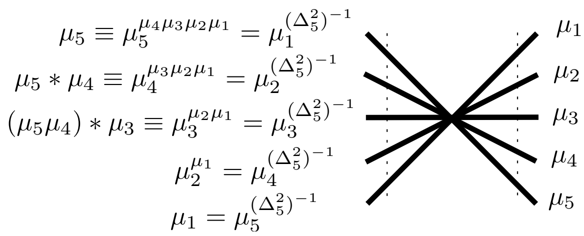

FIGURE 8.

Let us see how it applies to complexified real arrangements.

Example 2.19. Let us consider a complexified real arrangement. Its braid mon-

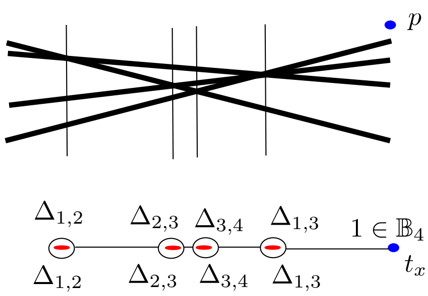

FIGURE 9.

odromy is defined by

$$\nabla(\alpha_1) = \Delta_{1,3}^2 \tag{2.1}$$

$$\nabla(\alpha_2) = \Delta_{1,3} * \sigma_3^2 \tag{2.2}$$

$$\nabla(\alpha_3) = (\Delta_{1,3}\sigma_3) * \sigma_2^2 \tag{2.3}$$

$$\nabla(\alpha_4) = (\Delta_{1,3}\sigma_3\sigma_2) * \sigma_1^2 \tag{2.4}$$

The computation is made from the real picture. How this affects to the fundamental group is shown in Figure 10.

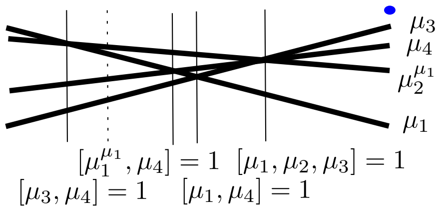

FIGURE 10.

The following result is well-know and a proof is derived from the above techniques.

Theorem 2.20. $(\mathbb{P}^2, \mathcal{A})$ *generic arrangement (all points in $\mathcal{P}$ are of multiplicity* 2*). Then,* $\pi_1(M(\mathcal{A}))$ *is abelian.*

Proof. Choose $\mathcal{A}$ with real equations and $\bar{L}_\infty \in \mathcal{A}$. From the real picture, all commutators of generators appear. There is an alternative proof: $\mathcal{A}$ can be chosen as a generic plane section of the coordinate arrangement in $\mathbb{P}^n$ and the complement is homeomorphic to $(\mathbb{C}^*)^n$. This is particularly interesting; since $(\mathbb{C}^*)^n$ is a $K(\pi, 1)$-space, the homology of the group is recovered. □

2.4. **Wiring diagram.**

Arvola [8] gave the following procedure to compute $\pi_1(M(\mathcal{A}))$ for an affine arrangement. Choose a normally embedded simple piecewise $\mathcal{C}^\infty$ arc $\Gamma : \mathbb{R} \to \mathbb{C}$ (identified with its image) such that $\{t_x\} \cup \mathcal{B} \subset \Gamma$ (t_x is the image of a big enough real number) and such that no vertex of Γ is in $\mathcal{B}$.

Definition 2.21. The *wiring space* is the pair $(\Gamma \times \mathbb{C}, \bigcup \mathcal{A} \cap (\Gamma \times \mathbb{C}))$.

The wiring space contains essentially all the topological information of the pair $(\mathbb{C}^2, \bigcup \mathcal{A})$. More precisely, if

$$\tilde{\Gamma} := \Gamma \cup \bigcup_{x_i \in \mathcal{B}} \mathbb{D}_\varepsilon(x_i)$$

then $(\mathbb{C}^2, \bigcup \mathcal{A})$ has the same homotopy type as $(\tilde{\Gamma}\times\mathbb{C}, \bigcup\mathcal{A}\cap(\tilde{\Gamma}\times\mathbb{C}))$.

Note that $\bigcup\mathcal{A}\cap(\Gamma\times\mathbb{C})$ is a union of *real lines*. Under genericity conditions we may choose a projection $\pi_\Gamma:\mathbb{C}\to\mathbb{R}$ such that for $\Pi_\Gamma := (1_\Gamma, \pi_\Gamma):\Gamma\times\mathbb{C}\to\Gamma\times\mathbb{R}\equiv\mathbb{R}^2$ we have that $\Pi_\Gamma(\bigcup\mathcal{A}\cap(\Gamma\times\mathbb{C}))$ is a union of n *lines* in $\mathbb{R}^2$ with two types of *crossing points*:

- The image of $\mathcal{P}$: *real crossings*.
- Some transversal double points called *virtual crossings*. As in knot theory we draw continuously the upward branch.

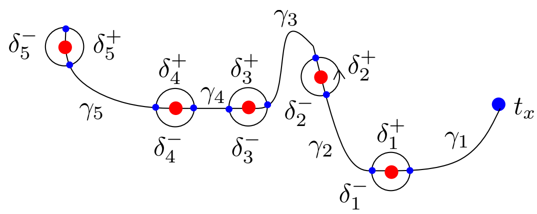

FIGURE 11. Γ and associated paths

Definition 2.22. The *wiring diagram* is the pair $(\mathbb{R}^2, \Pi_\Gamma(\bigcup\mathcal{A}\cap(\Gamma\times\mathbb{C})))$ with the information on virtual crossings.

Example 2.23. The real picture of a complexified real arrangement is a wiring diagram with no virtual crossing.

From $\tilde{\Gamma}$ we recover r arcs $\{\gamma_j\}_{j=1}^r$ (in Γ) and $2r$ cercle arcs $\{\delta_j^\pm\}_{j=1}^r$. From the wiring diagram we associate to γ_j a braid $\eta_j\in\mathbb{B}_n$ (coming from the virtual crossings); again from the diagram we associate to each $\delta_j^\pm$ the braid Δ_{a_j,b_j} (where a_j, b_j depends on the position of the corresponding multiple point). These braids determine the braid monodromy.

2.5. **Generic and non-generic braid monodromy.**

Definition 2.24. Let $\mathcal{A} := \{\bar{L}_1,\ldots,\bar{L}_n\}$ be a projective arrangement. The *generic braid monodromy* of $\mathcal{A}$ is the braid monodromy of $\mathcal{A}_\infty$: represented by an element $\nabla(\mathcal{A})\in(\mathbb{B}_n)^r$, $r := \#\mathcal{P}_\infty$.

Proposition 2.25. *The fundamental group $\pi_1(M(\mathcal{A}))$ is obtained as the quotient of $\pi_1(M(\mathcal{A}_\infty))$ by the normal subgroup generated by $\langle\mu_n\cdot\ldots\cdot\mu_1\rangle$.*

More properties can be deduced from generic braid monodromy.

Proposition 2.26 ([17])**.** *The braid monodromy $\nabla(\mathcal{A})$ determines the homotopy type of $M(\mathcal{A}_\infty)$.*

A braid monodromy $\nabla(\mathcal{A})$ is not well-defined. Simultaneous conjugation by an element of $\mathbb{B}_n$ yields an equivalent braid monodromy and the same happens if a Hurwitz action is applied. The Hurwitz action of the i^{th}-Artin generator of $\mathbb{B}_r$ is defined as:

$$(\tau_1,\dots,\tau_r)\mapsto(\tau_1,\dots,\tau_{i-1},\tau_{i+1},\tau_{i+1}*\tau_i,\tau_{i+2},\dots,\tau_r).$$

Remark 2.27. A generic braid monodromy $\nabla(\mathcal{A})$ is also useful to compute invariants via braid representations as it was proved by A. Libgober [18].

Sometimes it is either easier or more interesting to compute non-generic braid monodromies. A braid monodromy can be non-generic for various reasons:

(NG1) If $\bar{L}_\infty \not\pitchfork \mathcal{A}$: $(\tau_1,\dots,\tau_r)\to(\tau_1,\dots,\tau_r,(\tau_r\cdot\ldots\cdot\tau_1)^{-1}\Delta_n^2)$.

(NG2) If several multiple points are on the same vertical line then decompose the corresponding braids in pairwise commuting braids.

(NG3) If there are some vertical lines in $\mathcal{A}_\infty$, the first step is to compute the braid monodromy of the arrangement without vertical lines without changing the projection. How to obtain a generic braid monodromy is explained in [7]. Using wiring diagrams we can also obtained a generic braid monodromy. It is enough to turn slightly a wiring diagram and make non-parallel the vertical lines.

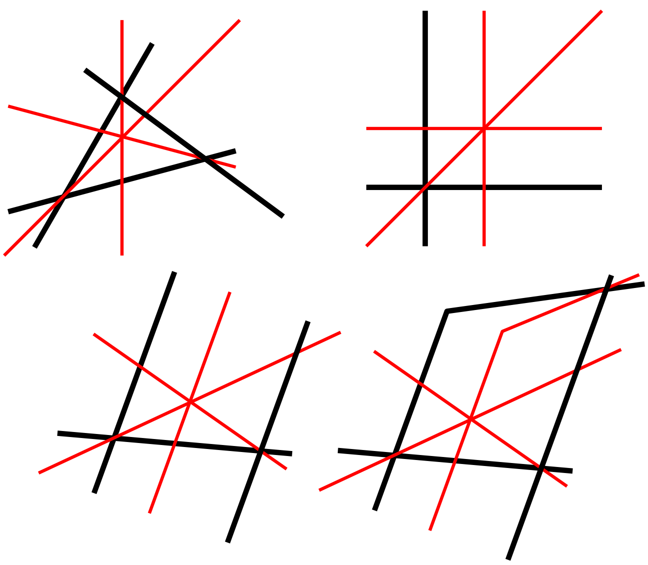

FIGURE 12. From non-generic to generic braid monodromy

3. Characteristic varieties and twisted cohomology

Let us consider $\mathcal{A} = \{\bar{L}_0, \bar{L}_1, \dots, \bar{L}_n\}$ a line arrangement in $\mathbb{P}^2$. In this section we will denote $\mathcal{P}_\mathcal{A} := \{\bar{L}_i \cap \bar{L}_j \mid 0 \leq i < j \leq n\}$. We may use $\mathcal{A}_0$ in order to compute $\pi_1(M(\mathcal{A}))$. In general, it is difficult to get properties from a presentation of $\pi_1(M(\mathcal{A}); p)$.

Using Alexander duality or Theorem 2.13 we can see that $H_1(M(\mathcal{A}); \mathbb{Z}) \cong \mathbb{Z}^n$. From now on, we use multiplicative notation for this homology group. A free generator system is given by $t_1, \dots, t_n$, $t_i \equiv \mu_i \bmod \pi_1(M(\mathcal{A}))'$. To keep the symmetry from the elements of the line arrangement, recall that $(\mu_n \cdot \dots \cdot \mu_1)^{-1}$ is a meridian of $\bar{L}_0$. Then $t_0 := (t_1 \cdot \ldots \cdot t_n)^{-1}$ is the homology class of a meridian of $\bar{L}_0$.

The abelianization map $\text{ab} : \pi_1(M(\mathcal{A}); p) \to H_1(M(\mathcal{A}); \mathbb{Z}) \cong \mathbb{Z}^n$ defines a covering $\rho : \tilde{M}(\mathcal{A}) \to M(\mathcal{A})$ which is called the *universal abelian covering.* Since the deck automorphism group of ρ is canonically identified with $H_1(M(\mathcal{A}); \mathbb{Z})$, we denote $t_1, \dots, t_n : \tilde{M}(\mathcal{A}) \to \tilde{M}(\mathcal{A})$ the generators of this deck automorphism group (for those elements canonically related with their homonyms).

The homology and cohomology groups $H_1(\tilde{M}(\mathcal{A}); \mathbb{C})$ and $H^1(\tilde{M}(\mathcal{A}); \mathbb{C})$ are canonically $\Lambda_\mathbb{C}$-modules, where $\Lambda_\mathbb{C} := \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$.

The space $M(\mathcal{A})$ has the homotopy type of a finite CW-complex $K(\mathcal{A})$ of dimension 2. Let us consider $C_*(\mathcal{A}) := C(K(\mathcal{A}); \mathbb{C})$ its chain complex, and consider also $C^*(\mathcal{A})$ the dual cochain complex.

Lifting the cells of $K(\mathcal{A})$, we obtain a CW-complex $\tilde{K}(\mathcal{A})$ having the same homotopy type as $\tilde{M}(\mathcal{A})$. We consider also the chain complex $\tilde{C}_*(\mathcal{A}) := C(\tilde{K}(\mathcal{A}); \mathbb{C})$ and its dual cochain complex $\tilde{C}^*(\mathcal{A})$. By construction, $\tilde{C}_*(\mathcal{A}), \tilde{C}^*(\mathcal{A})$ are free $\Lambda_\mathbb{C}$-modules. Moreover, $\dim_\mathbb{C} C_*(\mathcal{A}) = \operatorname{rank}_{\Lambda_\mathbb{C}} \tilde{C}_*(\mathcal{A})$, respecting the graduation.

3.1. **Twisted cohomology.**

The cells induce a graded (arbitrarily ordered) basis B of $C^*(\mathcal{A})$. With this basis, the complex

$$C^*(\mathcal{A}) : 0 \to C^0(\mathcal{A}) \xrightarrow{A_1} C^1(\mathcal{A}) \xrightarrow{A_2} C^2(\mathcal{A}) \to 0$$

is determined by matrices A_1, A_2 with $\mathbb{Z}$-coefficients.

In the same way, we can define a graded basis $\tilde{B}$ of $\tilde{C}_*(\mathcal{A})$ as free $\Lambda_\mathbb{C}$-module; each element of $\tilde{B}$ is an arbitrary lift of an element of B. Hence the complex

$$\tilde{C}^*(\mathcal{A}) : 0 \to \tilde{C}^0(\mathcal{A}) \xrightarrow{\tilde{A}_1} \tilde{C}^1(\mathcal{A}) \xrightarrow{\tilde{A}_2} \tilde{C}^2(\mathcal{A}) \to 0$$

is determined by matrices $\tilde{A}_1, \tilde{A}_2$ with $\Lambda_{\mathbb{C}}$-coefficients. Note that $\Lambda_{\mathbb{C}}$ is the ring of Laurent polynomials in $t_1, \ldots, t_n$ with complex coefficients. The evaluation of the matrix $\tilde{A}_i$ at $t_j = 1$, $1 \leq j \leq n$, yields A_i.

Definition 3.1. The *character torus* of $\mathcal{A}$ is defined as:

$$\mathbb{T}(\mathcal{A} := H^1(M(\mathcal{A}); \mathbb{C}^*) = \operatorname{Hom}(H_1(M(\mathcal{A}); \mathbb{Z}), \mathbb{C}^*) = \operatorname{Hom}(\pi_1(M(\mathcal{A})), \mathbb{C}^*).$$

A character $\xi \in \mathbb{T}(\mathcal{A})$ induces an evaluation map $\mathrm{ev}_\xi : \Lambda_{\mathbb{C}} \to \mathbb{C}$ which induces a $\Lambda_{\mathbb{C}}$-module structure on $\mathbb{C}$ denoted by $\mathbb{C}_\xi$. It also induces a local system of coefficients $\underline{\mathbb{C}_\xi}$, which has associated a cohomology group $H^1(M(\mathcal{A}); \underline{\mathbb{C}_\xi})$ which depends only on $\pi_1(M(\mathcal{A}))$.

Definition 3.2. The *twisted cohomology* $H^1(M(\mathcal{A}); \underline{\mathbb{C}_\xi})$ is obtained from the complex $\tilde{C}^*(\mathcal{A}) \otimes_{\Lambda_{\mathbb{C}}} \mathbb{C}_\xi$ which is obtained by the evaluation of $\tilde{A}_1, \tilde{A}_2$ using ev_ξ.

3.2. **Characteristic varieties.**

Definition 3.3. The *characteristic varieties* of $\mathcal{A}$ are: $\mathcal{V}_k(\mathcal{A}) := \{\xi \in \mathbb{T}(\mathcal{A}) \mid \dim_{\mathbb{C}} H^1(M(\mathcal{A}); \underline{\mathbb{C}_\xi}) \geq k\}$.

Remark 3.4. Note that the characteristic varieties of $\mathcal{A}$ depend only on $M(\mathcal{A})$; we may define the characteristic varieties of a CW-complex. Moreover, it depends only on the fundamental group, and it may be define for a group. For a finitely generated group, characteristic varieties are subspaces of a finite dimensional torus. It is not hard to see that they are algebraic varieties defined by equations with integer coefficients. Nevertheless, for the fundamental groups of quasi-projective varieties some restrictions on their properties exist.

Theorem 3.5 ([1, 20, 6])**.** *The irreducible components of $\mathcal{V}_k(\mathcal{A})$ are subtori translated by torsion elements.*

This theorem is a consequence of the following Arapura's result, which was refined later by Artal-Cogolludo-Matei.

Theorem 3.6 ([1, 6])**.** *If $\Sigma \subset \mathcal{V}_k(\mathcal{A})$ is an irreducible component one of the following (non-exclusive) situations happen:*

1. *There exists $\Phi : M(\mathcal{A}) \to X$ morphism onto an orbifold X and an irreducible component Σ_X of $\mathcal{V}_k(X)$ such that $\Sigma = \rho^*(\Sigma_X)$.*
2. *Σ is an isolated torsion point.*

Corollary 3.7. *It is enough to know $H^1(M(\mathcal{A}); \underline{\mathbb{C}_\xi})$ for ξ unitary (or torsion).*

We are going to combine this with Sakuma's formula [25]. Let $\xi \in \mathbb{T}(\mathcal{A})$ be a character of order h and let $\rho_\xi : M_\xi(\mathcal{A}) \to M(\mathcal{A})$ be the h-fold cyclic covering associated to ξ. We denote by $\tilde{\xi} : M_\xi(\mathcal{A}) \to M_\xi(\mathcal{A})$ the order-h deck automorphism of ρ_ξ. Let $\tilde{\xi}^* : H^\ell(M_\xi(\mathcal{A}); \mathbb{C}) \to H^\ell(M_\xi(\mathcal{A}); \mathbb{C})$ be the linear automorphism induced by $\tilde{\xi}$. We denote $H^\ell_\xi := \ker(\tilde{\xi}^* - \exp(\frac{2i\pi}{h}) \cdot 1_{H^\ell(M_\xi(\mathcal{A});\mathbb{C})})$

Proposition 3.8. $H^1_\xi \cong H^1(M(\mathcal{A}); \underline{\mathbb{C}}_\xi)$.

As a consequence of Corollary 3.7 and Proposition 3.8 we obtain the following result which guarantees the algebraic nature of characteristic varieties of arrangements (in fact, of quasi-projective manifolds).

Corollary 3.9. $\mathcal{V}_k(\mathcal{A})$ *depends on the Betti numbers of some quasi-projective smooth varieties.*

3.3. **Cohomology of projective and quasi-projective smooth varieties.**

We apply the general theory of quasi-projective varieties to our case. Let $M_\xi(\mathcal{A}) \subset X_\xi(\mathcal{A})$ be a smooth projective completion such that $D_\xi(\mathcal{A}) := X_\xi(\mathcal{A}) \setminus M_\xi(\mathcal{A})$ is a normal crossing divisor. We choose this completion such that ρ_ξ extends to a branched covering $\rho_\xi : X_\xi(\mathcal{A}) \to \mathbb{P}^2$. For this covering, $\rho_\xi^{-1}(\bigcup \mathcal{A}) = D_\xi(\mathcal{A})$. We denote by $\mathcal{D}_\xi(\mathcal{A})$ the set of irreducible components of $D_\xi(\mathcal{A})$.

Let us recall what Pure Hodge Theory implies, see for example [15]. There is a decomposition

$$H^1(X_\xi(\mathcal{A}); \mathbb{C}) \cong H^1(X_\xi(\mathcal{A}); \mathcal{O}_{X_\xi(\mathcal{A})}) \oplus H^0(X_\xi(\mathcal{A}); \Omega^1_{X_\xi(\mathcal{A})}) \tag{3.1}$$

such that

$$H^1(X_\xi(\mathcal{A}); \mathcal{O}_{X_\xi(\mathcal{A})}) \cong \overline{H^0(X_\xi(\mathcal{A}); \Omega^1_{X_\xi(\mathcal{A})})}.$$

Deligne's Mixed Hodge Theory for quasi-projective varieties [10] implies the following decomposition:

$$H^1(M_\xi(\mathcal{A}); \mathbb{C}) \cong H^1(X_\xi(\mathcal{A}); \mathcal{O}_{X_\xi(\mathcal{A})}) \oplus H^0(X_\xi(\mathcal{A}); \Omega^1_{X_\xi(\mathcal{A})} \log(D_\xi(\mathcal{A}))). \tag{3.2}$$

The first terms of the direct sum decompositions of (3.1) and (3.2). The second terms can be related using Poincaré residues. Let us consider the following short exact sequence of sheaves:

$$0 \to \Omega^1_{X_\xi(\mathcal{A})} \to \Omega^1_{X_\xi(\mathcal{A})} \log(D_\xi(\mathcal{A})) \to \bigoplus_{D \in \mathcal{D}_\xi(\mathcal{A})} i_* \mathcal{O}_D \to 0. \tag{3.3}$$

Let us consider two exact sequences associated to the first terms of the associated long exact sequence:

$$0 \to H^0(X_\xi; \Omega^1_{X_\xi}) \to H^0(X_\xi; \Omega^1_{X_\xi} \log(D_\xi)) \to H(\mathcal{A}) \to 0, \tag{3.4}$$

$$0 \to H(\mathcal{A}) \to \bigoplus_{D\subset D_\xi} H^0(D;\mathcal{O}_D) \to H^1(X_\xi;\Omega^1_{X_\xi}) \subset H^2(X_\xi;\mathbb{C}), \tag{3.5}$$

where $H(\mathcal{A})$ is the cokernel of the first map (and by exactness, the kernel of the second one).

We deduce the following formulæ:

(F1) $\dim H^1(X_\xi;\mathbb{C}) = 2\dim H^1(X_\xi;\mathcal{O}_{X_\xi})$.

(F2) $\dim H^1(M_\xi;\mathbb{C}) = 2\dim H^1(X_\xi;\mathcal{O}_{X_\xi}) + \dim\ker\left(\bigoplus_{D\in\mathcal{D}_\xi}\mathbb{C}\langle D\rangle \to H^2(X_\xi;\mathbb{C})\right)$.

(F3) $\dim H^1(M(\mathcal{A});\mathbb{C}_\xi) = \dim H^1(X_\xi;\mathcal{O}_{X_\xi})^\xi + \dim H^1(X_\xi;\mathcal{O}_{X_\xi})^{\bar\xi} + d_\xi$, where $d_\xi := \dim\ker\left(\bigoplus_{D\in\mathcal{D}_\xi}\mathbb{C}\langle D\rangle \to H^2(X_\xi;\mathbb{C})\right)^\xi$ and the superscript ξ means the eigenspace relative to ξ.

The two first terms in (F3) give depend only on X_ξ, namely its sum coincides with $\dim H^1(X_\xi;\mathbb{C})^\xi$, while the term d_ξ may depend on the quasi-projective variety $M(\mathcal{A})$.

Theorem 3.10 ([16])**.** *If ξ is fully ramified then $H^1(X_\xi;\mathbb{C})^\xi = H^1(M_\xi;\mathbb{C})^\xi$.*

4. Computation of the projective terms

The two first terms in (F3) give depend only on X_ξ and we will refer to them as the *projective terms*, since its sum coincides with $\dim H^1(X_\xi;\mathbb{C})^\xi$. The term d_ξ may depend on the quasi-projective variety $M(\mathcal{A})$ and it will be called the *quasi-projective term.*

Definition 4.1. A character $\xi \in \mathbb{T}(\mathcal{A})$ is *fully ramified* if $\xi(t_j) \neq 1$, $\forall j \in \{0,\dots,n\}$. For a general caracter we denote by $\mathcal{A}^\xi := \{\bar L_j \mid \xi(t_j)\neq 1\}$ the *ramification locus* and by $\mathcal{A}^\xi_0 := \{\bar L_j \mid \xi(t_j) = 1\}$ the *unramification locus.*

Theorem 4.2 ([16])**.** *If ξ is fully ramified then $H^1(X_\xi;\mathbb{C})^\xi = H^1(M_\xi;\mathbb{C})^\xi$.*

Hence, for fully ramified characters ξ, the projective terms determine at which characteristic varieties the character ξ belongs.

In [19], Libgober gives a procedure to compute $\dim H^1(X_\xi;\mathbb{C})^\xi$ which will be explained in a slightly different way than in the original sources where the general case of an algebraic plane curve is treated. The case of line arrangements is simpler.

Let us fix from now on a unitary character $\xi \in \mathbb{T}(\mathcal{A})$; in fact we only need to consider torsion characters but the exposition is essentially the same.

Definition 4.3. The *real representative* of ξ is given by $(r_0, r_1, \ldots, r_n) \in [0,1)^{n+1}$ such that $\xi(t_j) = \exp(2i\pi r_j)$. The *level* of ξ is $\ell(\xi) := r_0 + \cdots + r_n \in \mathbb{Z}$ (recall that $\prod_{i=0}^n t_i = 1$).

Definition 4.4. Let $P \in \mathcal{P}_{\mathcal{A}}$. Denote

$$r_P := \sum_{P \in L_j} r_j \quad \text{and} \quad s_P := \max\{0, \lfloor r_P \rfloor - 1\}.$$

The *ideal of quasiadjunction* of P with respect to ξ is $\mathcal{J}_{P,\xi} := \mathcal{M}_P^{s_P}$, where $\mathcal{M}_P$ is the maximal ideal of the local ring $\mathcal{O}_{\mathbb{P}^2,P}$.

Theorem 4.5 ([19]). *The dimension of* $\dim H^1(X_\xi; \mathcal{O}_{X_\xi})^\xi$ *equals*

$$\dim \operatorname{coker}\left(\sigma_\xi : H^0(\mathbb{P}^2; \mathcal{O}_{\mathbb{P}^2}(\ell(\xi)-3)) \to \bigoplus_{P \in \mathcal{P}} \mathcal{O}_{\mathbb{P}^2,P}/\mathcal{J}_{P,\xi}\right).$$

The map σ_ξ is defined as follows. Choose a line $\bar{L}_\infty$ disjoint to $\mathcal{P}_{\mathcal{A}}$. We identify this line as the line at infinity and we choose coordinates x, y for $\mathbb{C}^2$. Then

$$H^0(\mathbb{P}^2; \mathcal{O}_{\mathbb{P}^2}(k)) = \{f \in \mathbb{C}[x,y] \mid \deg f \leq k\}, \text{ for any } k.$$

The map σ_k consists of considering the germ of holomorphic function of a polynomial at any $P \in \mathcal{P}_{\mathcal{A}}$. Note that in particular, $\mathbb{P}(\ker \sigma_\xi)$ is identified the space of curves of degree $\ell(\xi)-3$ passing through P with multiplicity at least s_P, $\forall P \in \mathcal{P}_{\mathcal{A}}$.

Remark 4.6. If $m_P = 2$ then $\mathcal{J}_{P,\xi} = \mathcal{O}_{\mathbb{P}^2,P}$. We can restrict our attention to $\mathcal{P}_{>2,\mathcal{A}} := \{P \in \mathcal{P}_{\mathcal{A}} \mid m_P > 2\}$. For $P \in \mathcal{P}_{>2,\mathcal{A}}$, the bound $m_P - 1 > s_P$ holds.

The source and the target of σ_ξ are of combinatorial nature. On the other side, $\operatorname{coker} \sigma_\xi$ and $\ker \sigma_\xi$ are determined each other. Since $\ker \sigma_\xi$ consist of the space of curves of degree $\ell(\xi) - 3$ passing through $P \in \mathcal{P}_{>2}$ with multiplicity at least s_P. In particular, $\dim \ker \sigma_\xi$ is not *a priori* a combinatorial invariant and the same happens for the projective terms.

Example 4.7. Let $\mathcal{A} = \{\bar{L}_0, \ldots, \bar{L}_n\}$ be an arrangement with only one multiple point $P \in \bar{L}_i$. Consider a character ξ fully ramified of level ℓ. In this case, $s_P = \ell - 1$ and

$$\dim \operatorname{coker}\left(\sigma_\xi : H^0(\mathbb{P}^2; \mathcal{O}_{\mathbb{P}^2}(\ell - 3)) \to \mathcal{O}_{\mathbb{P}^2,P}/\mathcal{M}_P^{\ell-1}\right) = \ell - 1.$$

Note that $\ell(\bar{\xi}) = n + 1 - \ell$.

Example 4.8. Let us consider the Ceva arrangement consisting of the six lines passing through 4 points $P_1, \ldots, P_4$, in general position. Let us consider a character ξ; the images of the meridians of the lines are $t, s, u \in \mathbb{C}^*$, $tsu = 1$, as in Figure 13. We assume that $\ell(\xi) = 4$. Then, $s_{P_i} = 1$ and

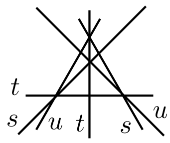

FIGURE 13. Ceva arrangement

$$\dim \operatorname{coker}\left(\sigma_\xi : H^0(\mathbb{P}^2; \mathcal{O}_{\mathbb{P}^2}(1)) \to \mathbb{C}^4\right) = 1.$$

Example 4.9. Let us consider the Pappus arrangement $\mathcal{A}$ given by

$$\begin{aligned} xz(4x - y + 2z) &= 0 \\ y(x - y - z)(2x + y + 2z) &= 0 \\ (x - y)(2x + y + z)(y + 2z) &= 0. \end{aligned}$$

The set $\mathcal{P}_\mathcal{A}$ consists of nine double points and nine triple points. Let us consider the character ξ defined by $\xi(\mu_i) := \exp(2i\pi\frac{2}{3})$. Its level is $\ell = 6$ and $s_P = 1$ for all triple points. Note that

$$\dim \operatorname{coker}\left(\sigma_\xi : H^0(\mathbb{P}^2; \mathcal{O}_{\mathbb{P}^2}(3)) \to \mathbb{C}^9\right) = 1.$$

This example was studied in [2]. In this paper another arrangement of nine lines is studied, with the same number of double and triple points and where σ_ξ is surjective.

Example 4.10. Let us consider the Hesse arrangement, i.e. the lines joining the nine inflection points of a smooth cubic, which admits the following equation

$$xyz(x^3y^3z^3 - 27(x^3 + y^3 + z^3)^3) = 0.$$

It has 9 quadruple points and 12 double points. Let us consider the character ξ defined by $\xi(\mu_i) := -1$. Then $\ell = 6$, $s_P = 1$ and

$$\dim \operatorname{coker}\left(\sigma_\xi : H^0(\mathbb{P}^2; \mathcal{O}_{\mathbb{P}^2}(3)) \to \mathbb{C}^9\right) = 1.$$

4.1. **Quasi-adjunction polytopes.** This concept is introduced in [19], in order to study all fully-ramified character in finite time.

Let us consider the semi-open cube $K := [0, 1)^{n+1}$. In this cube we will consider the *level cubes* $K_\ell := K \cap \{r_0 + \cdots + r_n = \ell\}$, $\ell \in \{1, \ldots, n\}$.

Let us fix a point $P \in \mathcal{P}_{>2}$ and associate the following subsets

$$K_{P,k} := \begin{cases} \{\sum_{P \in L_j} r_j < 2\} & \text{if } k = 0 \\ \{k + 1 \leq \sum_{P \in L_j} r_j < k + 2\} & \text{if } 1 \leq k < m_P - 1 \end{cases}$$

These subsets have the following property. Let ξ be a character and let $(r_0,\ldots,r_n)$ be its real representative. Then:

$$(r_0,\ldots,r_n)\in K_{P,k}\Longleftrightarrow \mathcal{J}_{P,\xi}=\mathcal{M}_P^k,$$

For a fixed ℓ, the sets $\{K_{P,k}\mid P\in\mathcal{P}_{\mathcal{A}}\}$ induce a finite partition of K_ℓ. The partition subsets are called the *quasi-adjunction polytopes.*

Proposition 4.11. *Two characters in the same quasi-adjunction polytope share the map σ_ξ.*

Remark 4.12. The structure of characteristic varieties impose conditions on the polytopes containing a character ξ and its conjugate $\bar{\xi}$, which may impose conditions on the position of the points in $\mathcal{P}_{>2}$.

What is behind this section is the following. The information about fully-ramified characters can be obtained studying cyclic ramified coverings of $\mathbb{P}^2$, following the ideas of Zariski [26], Esnault-Viehweg [11, 12, 13], Libgober [16], Loeser-Vaquié [21] and the author [3].

5. Coordinate components and quasi-projective term

Let $\mathcal{A}:=\{\bar{L}_0,\bar{L}_1,\ldots,\bar{L}_n\}$ be a line arrangement in $\mathbb{P}^2$. Let us consider the torus $\mathbb{T}(\mathcal{A})$; using the coordinates $(t_0,t_1,\ldots,t_n)$ we consider $\mathbb{T}(\mathcal{A})$ as the subtorus $\prod_{i=0}^n t_i=1$ in $(\mathbb{C})^{n+1}$. Let $\emptyset\neq J\subset\{0,1,\ldots,n\}$; let us consider the subarrangement $\mathcal{A}_J:=\{\bar{L}_j\mid j\in J\}$. Note that the torus $\mathbb{T}(\mathcal{A}_J)$ is in a natural way a subtorus of $\mathbb{T}(\mathcal{A})$, namely

$$\mathbb{T}(\mathcal{A}_J)=\mathbb{T}(\mathcal{A})\cap\mathbb{T}_J,\quad \mathbb{T}_J:=\{(t_0,t_1,\ldots,t_n)\in\mathbb{C}^{n+1}\mid t_i=1\text{ if }i\notin J\}.$$

Moreover, it is clear that $\mathcal{V}_k(\mathcal{A}_J)\subset\mathcal{V}_k(\mathcal{A})$ but it as we will see in general $\mathcal{V}_k(\mathcal{A}_J)\subsetneqq\mathcal{V}_k(\mathcal{A})\cap\mathbb{T}_J$.

Definition 5.1. Let Σ be an irreducible component of $\mathcal{V}_k(\mathcal{A})$. The component is said to be:

(1) *Coordinate* if Σ is contained in $\mathbb{T}_J$ for some $\emptyset\neq J\subsetneqq\{0,1,\ldots,n\}$.
(2) *Non-coordinate* if it is not coordinate.
(3) *Non-essential coordinate* if Σ is an irreducible component of $\mathcal{V}_k(\mathcal{A}_J)$ for some $\emptyset\neq J\subsetneqq\{0,1,\ldots,n\}$.
(4) *Essential coordinate* if it is coordinate but not non-essential.

In order to compute the characteristic varieties of $\mathcal{A}$, we may apply induction on the number of lines. For $n=0$ it is clear that the characteristic varieties are empty. Let us assume that $n>0$. Non-coordinate components are computed using

Section 4 and non-essential coordinate components are assumed to be computed using induction hypothesis. We need only to compute the essential coordinate components.

Remark 5.2. Let us consider the irreducible components of $\mathcal{V}_k(\mathcal{A})\cap\mathbb{T}_J$. There are three types of components: essential and non-essential coordinate components in $\mathbb{T}_J$ and intersection with $\mathbb{T}_J$ of irreducible components not contained in $\mathbb{T}_J$.

From now on we fix a torsion character $\xi\in\mathbb{T}(\mathcal{A})$ (of order h) and we assume that it is not fully-ramified, i.e., $\mathcal{A}_0^\xi$ is non-empty. For simplicity, we will assume that $\mathcal{A}_0^\xi=\{\bar{L}_m,\dots,\bar{L}_n\}$, for some $m<n$.

Recall that $M(\mathcal{A})=X_\xi\setminus D_\xi$ and $\mathcal{D}_\xi$ is the set of irreducible components of D_ξ. Let us show how X_ξ is constructed. Let $\pi:Y\to\mathbb{P}^2$ the blowing-up of $\mathcal{P}_{>2,\mathcal{A}}$. The divisor $\bigcup\tilde{\mathcal{A}}:=\pi^{-1}(\bigcup\mathcal{A})$ is a normal crossing divisor and $M(\mathcal{A})=Y\setminus\bigcup\tilde{\mathcal{A}}$. Its irreducible components are:

- $\tilde{L}_j$, $0\leq j\leq n$, strict transforms of the lines in $\mathcal{A}$;
- E_P, $P\in\mathcal{P}_{>2,\mathcal{A}}$, the exceptional components of π.

Lemma 5.3. *The intersection of the components of $\tilde{\mathcal{A}}$ is given by:*

(1) $E_P^2=-1$;
(2) $(\tilde{L}_j)^2=1-a_j$, $a_j:=\#\{P\in\mathcal{P}_{>2}\mid P\in\bar{L}_j\}$;
(3) *for* $i\neq j$, $\tilde{L}_i\cap\tilde{L}_j\neq\emptyset\Longleftrightarrow\bar{L}_i\cap\bar{L}_j\notin\mathcal{P}_{>2}$;
(4) *for* $P\neq Q$, $E_P\cap E_Q=\emptyset$.
(5) $E_P\cap\tilde{L}_i\neq\emptyset\Longleftrightarrow P\in\bar{L}_i$.

Lemma 5.4. *For $P\in\mathcal{P}_{>2,\mathcal{A}}$, the homology class of a meridian of E_P is given by* $t_P:=\prod_{P\in\bar{L}_j}t_j\in H_1(M(\mathcal{A});\mathbb{Z})$.

Definition 5.5. Let $B\in\tilde{\mathcal{A}}$. The *neighboring subgroup* G_B of B is the subgroup of $\mathbb{C}^*$ generated by $\xi(t_B)$ and $\xi(t_C)$ for the components $C\in\tilde{\mathcal{A}}$ such that $C\cap B\neq\emptyset$. The *neighboring index* is $n_B:=\frac{\#\operatorname{im}\xi}{\#G_B}=\frac{h}{\#G_B}$.

We defined ρ_ξ as a branched covering $X_\xi\to\mathbb{P}^2$. It is more useful now to consider ρ_ξ as a branched covering $X_\xi\to Y$. This map is constructed in two steps. First we consider the normal model of the branched covering of Y with respect to ξ; this normal model has some isolated singular points which are quotient singularities. The space X_ξ is the minimal resolution of the normal model. The divisor $D_\xi=\rho_\xi^{-1}(\bigcup\tilde{\mathcal{A}})$ is a normal-crossing divisor.

The set $\mathcal{D}_\xi$ has three types of elements:

- $A_1,\dots,A_s$, satisfying that $\rho_\xi(A_j)$ is a point, $\forall j=1,\dots,s$.

- For $B \in \tilde{\mathcal{A}}$, $\hat{\rho}_\xi^{-1}(B) \subset \mathcal{D}_\xi$, where $\hat{\rho}_\xi^{-1}$ *means* strict transform.

Definition 5.6. We say that $B \in \tilde{\mathcal{A}}$ is said *unramified* if $\xi(t_B) = 1$, and *inner unramified* if it is unramified and it is also the case for all its neighbors. We denote by $\mathcal{U}_\xi \subset \tilde{\mathcal{A}}$ the set of the inner unramified components.

Remark 5.7. A component $B \in \tilde{\mathcal{A}}$ is unramified if it is the strict transform of $\bar{L}_i \in \mathcal{A}_0^\xi$ or it is the exceptional component of some $P \in \mathcal{P}_{\geq 2,\mathcal{A}}$ such that $\prod_{P \in \bar{L}_i} t_i = 1$. In order to be inner unramified the following conditions must be fulfilled:

- $B = E_P$ and all the lines through P are in $\mathcal{A}_0^\xi$.
- $B = \tilde{L}_i$, all the lines intersecting $\bar{L}_i$ at double points are in $\mathcal{A}_0^\xi$ and all the points $P \in \mathcal{P}_{\geq 2,\mathcal{A}} \cap \bar{L}_i$ satisfies $t_P = 1$.

The intersection form on $H^2(X_\xi; \mathbb{Z})$ induces a non-degenerate hermitian form in $H^2(X_\xi; \mathbb{C})$. For this hermitian form, the decomposition in eigenspaces for $\tilde{\xi}$ is orthogonal.

Lemma 5.8. *There is an isomorphism* $U_\xi := \mathbb{C}\langle \mathcal{U}_\xi \rangle \equiv \left(\bigoplus_{D \in \mathcal{D}^\xi} \mathbb{C}\langle D \rangle \right)^\xi$*, given by*

$$B \leftrightarrow \frac{1}{\sqrt{h}} \sum_{j=1}^{h} \exp\left(-2i\pi \frac{j}{h}\right) B_j.$$

The induced hermitian form on $\mathbb{C}\langle \mathcal{U}_\xi \rangle$*, denoted by* $\cdot_\xi$*, is non-degenerate.*

Notation 5.9. $\Gamma_{\mathcal{U}_\xi}$ denotes the dual graph of $\mathcal{U}_\xi$.

Proposition 5.10. *If* $\Gamma_{\mathcal{U}_\xi}$ *is a tree, then* $\cdot_\xi$ *coincides with the restriction of the usual intersection form in* U_ξ*.*

Example 5.11. For which characters ξ, is $\mathcal{U}_\xi$ formed by the green lines? $\xi(t_1) = x \in \mathbb{C}^* \Longrightarrow \xi(t_2) = x^{-1} \Longrightarrow \xi(t_3) = x \Longrightarrow \xi(t_4) = x^{-1}$

$$\cdot_\xi : \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

Let us study the twisted intersection form when $\Gamma_{\mathcal{U}_\xi}$ is not a tree. Fix $\mathcal{T}_{\mathcal{U}_\xi}$ a maximal tree. The edges of $\Gamma_{\mathcal{U}_\xi}$ correspond to pairs B, C such that $B \cdot C = 1$; the oriented edge from B to C is denoted by $\overrightarrow{BC}$.

If $\overrightarrow{BC} \not\subseteq \mathcal{T}_{\mathcal{U}_\xi}$ then it defines a cycle $\gamma_{B,C} \in H_1(\Gamma_{\mathcal{U}_\xi}; \mathbb{Z})$ which is well-defined $\mod \ker \xi$ $(\gamma_{C,B} = \gamma_{B,C}^{-1})$.

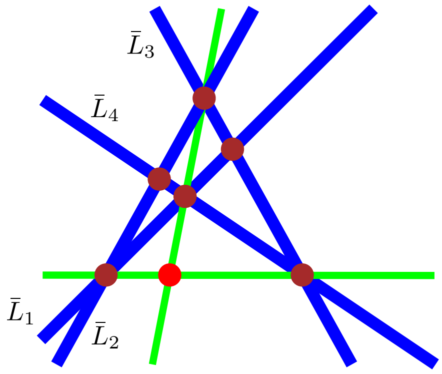

FIGURE 14. Ceva arrangement

FIGURE 15. $\Gamma_{\mathcal{U}_\xi}$

$$B \cdot_\xi C = \begin{cases} 0 & \text{if } B \cdot C = 0 \\ B \cdot C & \text{if either } B = C \text{ or } \overrightarrow{BC} \subset \mathcal{T}_{\mathcal{U}_\xi} \\ \xi(\gamma_{B,C})(B \cdot C) & \text{if } \overrightarrow{BC} \not\subset \mathcal{T}_{\mathcal{U}_\xi} \end{cases}$$

For an order in $\mathcal{U}_\xi$ consider the matrix $A(\mathcal{U}_\xi)$ of this twisted hermitian product on $\mathcal{U}_\xi$.

Example 5.12. Let us study now the extended Ceva arrangement.

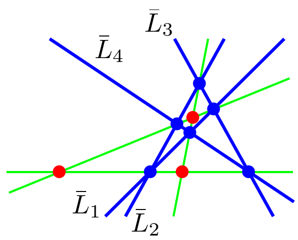

FIGURE 16. Extended Ceva arrangement

For which characters ξ, is $\mathcal{U}_\xi$ formed by the green lines? $\xi(t_1) = x \in \mathbb{C}^*$ $\Longrightarrow \xi(t_2) = x^{-1} \Longrightarrow \xi(t_3) = x \Longrightarrow \xi(t_4) = x^{-1}$ $x = x^{-1} = -1$, ρ 2-fold cover.

Theorem 5.13. $\operatorname{corank} A(\mathcal{U}_\xi) = \dim\ker \left(\bigoplus_{D \subset D_\xi} \mathbb{C}\langle D \rangle \to H^2(X_\xi, \mathbb{C}) \right)^\xi.$

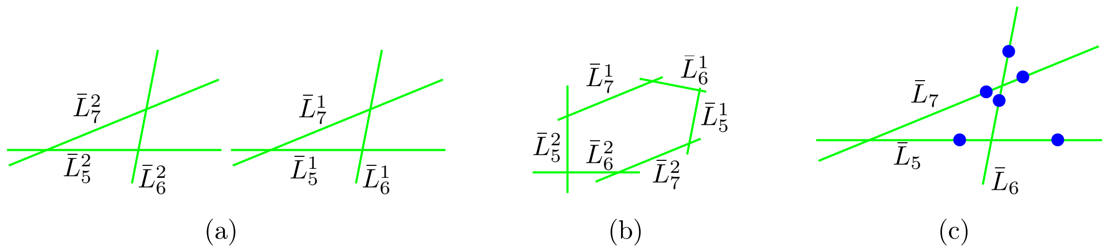

FIGURE 17. Steps

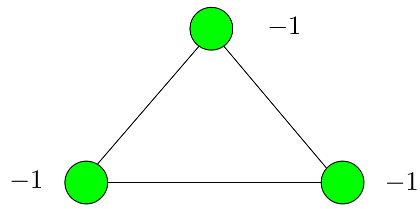

FIGURE 18. $\Gamma_{\mathcal{U}_\xi}$

- The key point in the proof is Hodge Index Theorem, and the fact that the positive part is included in the eigenspace for 1.
- The matrices $A(\mathcal{U}_\xi)$ are not *a priori* combinatorial invariants.
- One can proceed conversely. Fix a subset $\mathcal{U} \subset \tilde{\mathcal{A}}$ and consider the characters ξ for which $\mathcal{U} = \mathcal{U}_\xi$ and look for ξ such that corank $A(\mathcal{U}_\xi) > 0$.
- Look for isolated characters!
- Not isolated for Ceva arrangement.
- Corank 2 for extended Ceva Arrangement [9].

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