DICRITICAL DIVISORS FROM TOPOLOGY

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Introduction

As it stated in [10], Professor Abhyankar became interested in the theory of dicritical divisors around 2008. This terminology came from the theory of dynamical systems and had been applied by several authors to the study of meromorphic functions (or, equivalently, pencils of curves), having problems in affine algebraic geometry in mind. At our knowledge, this term was use at least since Dulac [8, 9].

Most probably, Ram Abhyankar immediately understood that he had already encountered these divisors (under other names) starting from the fifties, but he decided to come back to the subject. He was not very happy with the topological arguments but his goal was to understand every detail in order to translate the topological intuition to algebraic statements. The first part of the process was to understant statements that were valid for the ring of convergent power series in two variables with complex coefficients. Most of them could be translated to the ring of formal power series but his main goal was to figure out which results could be stated and proved in more general settings, namely to drop completness, to allow positive characteristic. The main test for such a statement was to check it in the mixed characteristic case. Besides his interest for dicritical components, he understood the importance of the concept of curvettes (see [14] for some history on this concept)

In this note we will try to explain the topological ideas Abhyankar was interested in, their algebraic translation, namely explaining the topological meanings of dicritical divsors and curvettes. We finish the paper with some comments on how does it work when we replace complex coefficients by arbitrary coefficients, including finite fields. We illustrate these ideas through some explicit examples. We will restrict ourselves to the 2-dimensional case, which, by the way, was the place of our common works with Abhyankar. It was planned to pursue the study in higher dimension.

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1. Resolution of a meromorphic germ and special pencils

Unless explicitely stating the opposite, we will work over the complex numbers, but most results should be true in a more general setting after some tuning.

Let F be a germ of meromorphic function in the neighbourhood of the origin in $\mathbb{C}^2$. More precisely, $F := \frac{f}{g}$, where $f, g \in \mathbb{C}\{X, Y\}$, $g \neq 0$. The algebraic counterpart of this setting is that $R := \mathbb{C}\{X, Y\}$ is a 2-dimensional local ring and, hence, F is an element of its fraction field $Q(R)$. Of course, it is not complete, but it shares many properties of complete local rings, see [3, §4]. It is well-known that R is a factorial domain and f, g can be chosen pairwise coprime. From the algebraic point view, the study of F is related to the study of the ideal $I = \langle f, g\rangle$ and its integral closure (which are somewhat *fat points*).

In order to apply a more geometric language, we may choose a small neighbourhood Δ of the origin in $\mathbb{C}^2$. This neighbourhood will be theoretically fixed but we may require to choose a smaller one if needed. We require that representatives of the germs f, g are defined in Δ and also that the intersection of their zero loci contains at most the origin. All the arguments below do not depend on the particular choice of Δ as far as the above properties are fullfilled. For the sake of simplicity, we still denote by f, g, F the corresponding representatives on Δ.

Whereas f, g are holomorphic functions $\Delta \to \mathbb{C}$, the meromorphic function F can be seen as a rational map $\Delta \dashrightarrow \mathbb{P}^1$, where $\mathbb{P}^1$ is naturally identified with $\mathbb{C} \cup \{\infty\}$. In general, F is a well-defined morphism only on $\Delta \setminus \{0\}$. More precisely, F defines a morphism on Δ if and only if either f, g are units, i.e., $I = R$. From now on we are interested in the case where F is not well defined at the origin, i.e., $f, g \in M(R) = \langle X, Y\rangle$, the maximal ideal of R. The main classical result is the following one, see e.g. [12].

Theorem 1.1. *There exists a proper birational morphism $\pi : Z \to \Delta$ (the composition of a sequence of blow-ups over the origin of Δ), which is an isomorphism outside the origin, and such that the pull-back of F extends to a well-defined morphism $F : Z \to \mathbb{P}^1$ called a* resolution *of F.*

An embedded resolution of a pair of (the closure of) generic fibers of F yields such a resolution, by the way the minimal resolution, since the main goal is to *separate* fibers. Let $\pi^{-1}(0) =: E = \bigcup_{j=1}^{r} E_j$ be the exceptional divisor of π, E_j being its irreducible components. Note that each E_j is isomorphic to $\mathbb{P}^1$ and hence the morphism $\pi_j := \pi_{|E_j} : E_j \to \mathbb{P}^1$ gives a well-defined rational function (up to a choice of coordinates in E_j).

Definition 1.2. An irreducible component E_j of E is a *dicritical divisor* of F if π_j is non-constant (i.e., surjective) and it is a *constant divisor* if π_j is constant. The *multiplicity* of a divisor E_j is the degree of π_j (a divisor is constant if and only if its multiplicity vanishes).

The dicritical divisors are essentially *canonical*. Two resolutions are related by a birational map which defines a natural bijection between the sets of dicritical divisors of both extensions of F.

Example 1.3. Let $f = y^2$, $g = x^3$. We start by blowing-up the origin, i.e.,

$$\Delta_1 := \{((x, y), [u : v]) \in \Delta \times \mathbb{P}^1 \mid xv = yu\}.$$

This space needs two charts to be covered. Let us study one of them:

$$(x_1, y_1) \mapsto ((x_1, x_1 y_1), [1 : y_1]) .$$

The extension of F is given by:

$$(x_1, y_1) \mapsto ((x_1, x_1 y_1), [1 : y_1]) \mapsto (x_1, x_1 y_1) \mapsto \frac{x_1^2 y_1^2}{x_1^3} = \frac{y_1^2}{x_1}$$

Note the extension fails to be defined at the origin of this chart. It is not difficult to see that it is well-defined at the other chart. This first blowing-up has a exceptional divisor E_1 (with self-intersection -1); the equation of E_1 in the chosen chart is $x_1 = 0$. Outside the origin, the restriction of the extension to E_1 is ∞-constant.

We proceed by blowing-up the origin of the chart. As before, there are two charts and we focus our attention in one of them since the extension will be well-defined on the other one:

$$(x_2, y_2) \to (x_2 y_2, y_2) \mapsto \frac{y_2^2}{x_2 y_2} = \frac{y_2}{x_2}.$$

Again, the extension of F is not defined at the origin of this chart. The new exceptional divisor E_2, with self-intersection -1, is defined by $y_2 = 0$, while the strict transform of E_1 is defined by $x_2 = 0$ and has self-intersection -2. For the sake of simplicity, strict transforms will keep the notation of the original divisor. Note that E_2 becomes 0-constant.

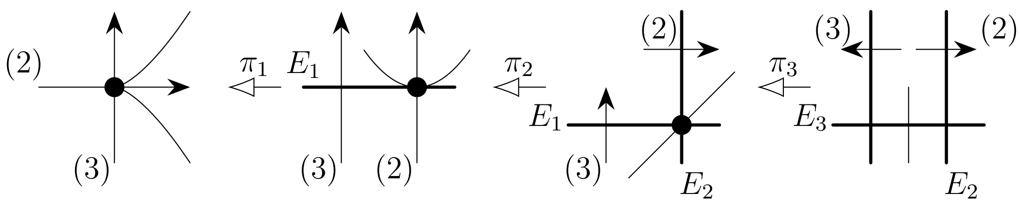

Figure 1. Sequence of blowing-ups.

An easy computation shows that the blowing-up at the origin of the chart solves F, i.e., we have a resolution $\pi : \Delta_3 \to \Delta$ such that $E = E_1 \cup E_2 \cup E_3$ where E_3 is the last exceptional component and it is dicritical of of multiplicity 1. This normal crossing divisor is represented as usual by its dual graph weighted with the self-intersections, see Figure 2. Note that in this case the resolution of F involves more blowing-ups than the resolution of $fg = y^2x^3 = 0$.

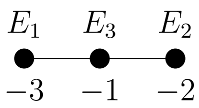

Figure 2. Dual graph of the exceptional divisor for the resolution of $\frac{y^2}{x^3}$.

Example 1.4. Next example is a little more involved. Let

$$f_0 := \left(\left(y^2 - x^3\right)^2 - x^5 y\right)^3 + \left(y^2 - x^3\right)^3 x^4 y^4, \quad f_1 = y^2 + x^3,$$

and

$$g_0 = \left(y^2 - x^3\right)^3 - x^8 y, \quad g_1 = \left(\left(y^2 - x^3\right)^2 - x^5 y\right)^2 + \left(y^2 - x^3\right) x^4 y^4.$$

We set $f = f_0 f_1$ and $g = g_0 g_1$.

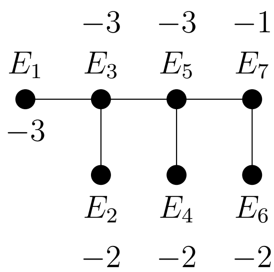

Figure 3. Dual graph of the exceptional divisor for the resolution of $\frac{f}{g}$.

We proceed as in Example 1.3 to compute a resolution of F, using a composition of seven blowing-ups. Let $\tilde{F} : \hat{\Delta} \to \mathbb{P}^1$ be the extension and $E := \bigcup_{j=1}^7 E_j$ be the exceptional divisor, indexed by the order of the blowing-ups. The dual graph with self-intersections is in Figure 3. The components behave as follows:

- E_3, E_5, E_7 are dicritical divisors, all of them of multiplicity 1.
- E_1 is 1-constant.
- E_2 is -1-constant.
- E_4 is 2-constant.
- E_6 is -2-constant.

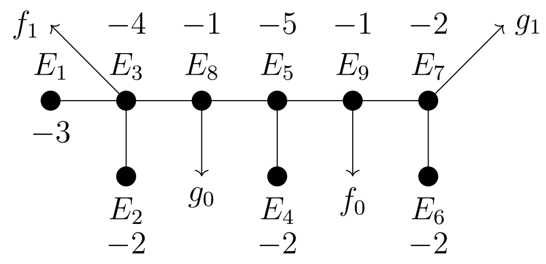

Figure 4. Dual graph of the exceptional divisor for the resolution of $fg = 0$.

The resolution of $fg = 0$ is a non-minimal resolution of F. Note that in the minimal resolution of $fg = 0$, see Figure 4, E_8 is ∞-constant and E_9 is 0-constant.

We end this section with an interesting particular example of meromorphic functions (or pencils of germs), the so-called special pencils.

Definition 1.5. A meromorphic function $F \in Q(R)$ is an *special pencil* if g is the power of a smooth function of R, i.e., a regular generator of the maximal ideal. For the sake of simplicity, we may assume $g = x^m$.

Example 1.6. Special pencils were introduced in [12], see also [10], since they appear in a natural way from polynomial maps. Let $f \in \mathbb{C}[X, Y]$ be non-constant polynomial. The map $f : \mathbb{C}^2 \to \mathbb{C}$ is non proper; if $\deg f = d$ and $\tilde{f}(x, y, z)$ is the homogeneization of f, then $\frac{\tilde{f}}{z^d} : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ is a rational map extending f (using the canonical inclusion $\mathbb{C}^2 \hookrightarrow \mathbb{P}^2$ via $(x, y) \mapsto [x : y : 1]$). This map is well-defined outside a finite numbers of points in the line at infinity $L_\infty := \mathbb{P}^2 \setminus \mathbb{C}^2$; around these points the extension behaves as a special pencil.

These pencils were extensively studied in [12, 6, 2, 3]. By this definition, the ∞-fiber plays a special role. In [12], the authors proved that, for special pencils (arising from polynomial maps), the maps π_j over the dicritical components was a polynomial map (the preimage of ∞ was reduced to ∞). This result has been extended for any special pencil (and for more general regular local rings) in [6] using Newton techniques, see [7] for a down-to-earth explanation.

2. Curvettes

The following problem was addressed in [5] (*translated* from algebraic to topological language).

Problem 2.1. *Let $\pi : Z \to \Delta$ be the composition of a sequence of blowing-ups. Let us pick up a subset $\mathcal{W}$ of the exceptional divisors and a family of multiplicities $M_{\mathcal{W}} :=$*

$\{m_W\}_{W\in\mathcal{W}}$. *Does there exist a meromorphic function F having exactly the divisors in $\mathcal{W}$ as dicritical divisors with multiplicities $M_{\mathcal{W}}$?*

The answer given by Abhyankar and Heinzer is positive with no restriction. A similar question arises for special pencils. In that case, a necessary condition must be imposed due to [1, Proposition 3.5], see (2.2); in [4, Theorem 8.2], Abhyankar and Heinzer proved that the condition is also sufficient.

Later on, in [3] Abhyankar and the first author reproved both results with a more geometrical flavour. While the statements are still algebraic, the intuition come from the *topological* (analytic) case. In particular, the cited condition (2.2) for special pencils rely on an algebraic invariant $c(R,V,W)$, $V,W\in\mathcal{W}$, (which is not *obviously* symmetric from the definition in [1]), and which is translated into (2.3) in terms of intersection numbers; in particular, the invariant is symmetric in V,W.

These proofs strongly use the idea of curvette, algebraicly developed in [2, 3], see also the interesting work of Moyano [14] for some history of the term, which comes from the term *curvetta* of the Italian school of algebraic geometry. The topological notion of curvette is defined as follows. Consider $\pi: W\to\Delta\subset\mathbb{C}^2$ a composition of blowing-ups (over 0), and let $\pi^{-1}(0)=:E=\bigcup_{j=1}^r E_j$ be the exceptional divisor. Let $p\in E_j$ be a regular point of E (i.e. it is not a crossing point) and let Γ_p be the germ at p of a smooth germ transversal to E_j. The image $\pi(\Gamma_p)$ defines a Weil divisor (restricting Δ if necessary) and hence a Cartier divisor; in fact $\pi(\Gamma_p)$ is the zero locus of an irreducible $\delta_p\in R$. If $u_p=0$, $u_p\in Q(R)$, is a local equation of E_j around p, then $\gamma_p:=\frac{\pi^*(\delta_p)}{u_p^{N_j}}$ (for a suitable N_j independent of p) defines an equation of Γ_p.

Definition 2.2. The germ of curve Γ_p is called a *curvette* of E_j at p. Following [2], we will call δ_p a *root curvette* of E_j (based at p). By abuse of notation $\gamma_p\in Q(R)$ is also called a *curvette*. The number N_j is independent of p and it is denoted by $E_j(\gamma_p)$; in fact, the divisor E_j corresponds to a valuation of $Q(R)$ positive on $M(R)$.

Example 2.3. Let us consider Example 1.3. Note that x is a root curvette of E_1, y is a root curvette of E_2, and y^2-tx^3, $t\in\mathbb{C}^*$, is a curvette of E_3. Let us consider now Example 1.4. The germs x,y,f_1 are root curvettes for E_1,E_2,E_3, respectively, as above. As we can check in Figure 4 g_1,g_0,f_0 are curvettes of E_7,E_8,E_9, respectively.

Remark 2.4. Note that an irreducible curve is the root curvette of infinitely many divisors.

Let us explain the geometric idea in order to prove the existence of a meromorphic germ with a prescribed set of dicritical divisors (counted with multiplicity). For each component $W=E_{j(W)}\in\mathcal{W}$ we choose two packets of m_W root curvettes (eventually counted with multiplicity), say $\delta_{W,j}, \tilde{\delta}_{W,j}$, $1 \leq j \leq m_W$, with the unique condition that the set of base points of the two packets are disjoint. intersection points of the two sets of the root curvettes with E_w are disjoint. With this settings, we define:

$$f = \prod_{W \in \mathcal{W}} \prod_{j=1}^{m_W} \delta_{W,j}, \quad g = \prod_{W \in \mathcal{W}} \prod_{j=1}^{m_W} \tilde{\delta}_{W,j}. \tag{2.1}$$

Then, $F = \frac{f}{g}$ is a function satisfying the desired property.

In the case of a special pencil, $g = \delta^m$, for some smooth curve $\delta = 0$ and f is constructed as in (2.1) (with the only restriction that the curvettes of f are disjoint to the strict transform of $\delta = 0$). Abhyankar condition is:

$$mW(\delta) = \sum_{V \in \mathcal{W}} m_W c(R, V, W). \tag{2.2}$$

The definition of the invariant $c(R, V, W)$ can be found in [1] and is quite algebraic. All the terms in (2.2) can be interpreted in terms of intersection theory which is by the way a key ingredient for the topological proof.

Definition 2.5. Let $f, g \in R$ be coprime. Then, the *intersection number* of f, g (or the divisors defined by the functions) is given by

$$\{f = 0\} \cdot \{g = 0\} = f \cdot g = \dim_{\mathbb{C}} R/\langle f, g\rangle.$$

General definition of intersection theory of surfaces can be found in [13]. Note that these definitions apply to complex surfaces and also to more algebraic settings (with arbitrary base fields). Let us state two properties which are essential for the proof.

Proposition 2.6. *Let X be a surface and let A, B be two divisors in X where the intersection $(A \cdot B)_X$ is defined. Then*

(1) *If $\pi : Y \to X$ is a proper birational morphism, then*

$$(A \cdot B)_X = (\pi^*(A) \cdot \pi^*(B))_Y.$$

(2) [11, Proposition 3.8] *If A is the divisor of a meromorphic function and B is a smooth compact divisor then $(A \cdot B)_X = 0$.*

With these results it is easy to prove that for generic choices of root curvettes $\delta_W, \tilde{\delta}_W$:

$$(\{\delta = 0\} \cdot \pi(\Gamma_W)) = (\delta \cdot \delta_W) = W(\delta), \quad (\pi(\Gamma_V) \cdot \pi(\Gamma_W)) = (\delta_V \cdot \delta_W) = c(R, V, W). \tag{2.3}$$

The idea is simple. Let us compute the second intersection number. Note that

$$\pi^*(\pi(\Gamma_V)) = \Gamma_V + \sum_{j=1}^{r} m_{E_j}(V) E_j$$

where $m_{E_j}(V)=E_j(\pi(\Gamma_V))$ and it turns out to be equal to $c(R,V,E_j)$. Then:

$$\begin{aligned}\delta_V\cdot\delta_W=&\left(\Gamma_V+\sum_{j=1}^r m_j(V)E_j\right)\cdot\left(\Gamma_W+\sum_{j=1}^r m_j(W)E_j\right)\\=&\Gamma_V\cdot\left(\Gamma_W+\sum_{j=1}^r m_j(W)E_j\right)=\sum_{j=1}^r m_j(W)(\Gamma_V\cdot E_j)=m_W(V).\end{aligned}$$

In particular, $m_W(V)=m_V(W)$ and they coincide with the intersection number of the corresponding root curvettes. With the same ideas we have the following Lemma relating intersection form and valuation theory.

Lemma 2.7. *For any germ $f\in R$, $E_j(f)=\{f=0\}\cdot\delta_{E_j}$, where δ_{E_j} is a generic curvette of E_j.*

Remark 2.8. The intersection matrix $(E_i\cdot E_j)$ is negative definite and unimodular.

To prove the statement we follow several steps.

Step 1. A component E_j is 0-constant if and only if $E_j(f)>E_j(g)$ and ∞-constant if and only if $E_j(f)<E_j(g)$. In particular for our candidate function F all the components E_j are either dicritical or t-components for some $t\in\mathbb{C}^*$

Step 2. The function F is well-defined on Z. The main point is that any connected component of a fiber of a resolution of F must contain the strict transform of a curve. It is a direct consequence of Remark 2.8 and Proposition 2.6(1).

Step 3. As a consequence, the dicritical divisors are the expected ones.

Remark 2.9. In fact, for Abhyankar-Heinzer Problem 2.1, we could proceed separately for each component. Namely, we could find functions F_W such that W is the unique dicritical component (with multiplicity 1). For a suitable choice of these functions, the product $\prod_{W\in\mathcal{W}}F_W^{n_W}$ answers positively the question.

Example 2.10. Note that neither Example 1.3 nor Example 1.4 follow this strategy, since the fibers at $0,\infty$ are not composed by root curvettes. Nevertheless, $\frac{af+bg}{cf+dg}$ for a generic invertible complex matrix $A:=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$ is obtained as above.

3. Remarks on positive and mixed characteristic

Most techniques and definitions used in §2 can be refined to be applied (with some minor technical modifications) to any local regular ring of dimension 2, say for example $R=K[[X,Y]]$. The actual computations and answers to Problem 2.1 work when the residual field K is infinite, or at least when there are *enough* curvettes with distinct base points, more precisely enough points in $\mathbb{P}^1(K)$. In principle, it might work also in the finite field case, because, actually there are infinitely many *points* but we have to be careful with the multiplicities of the points, see [2], if K is not algebraic closed. The intersection matrices for the strict transforms of exceptional divisors of a sequence of blowing-ups have been studied in [14]. In this case, these matrices are still negative definite but it may fail to be unimodular if we blow up on points whose residual field is an algebraic extension of K.

As stated in [10] a similar question to Problem 2.1 remains open in the case of finite residual fields. There is an obvious obstruction to extend the idea of the proof in §2 even in the case of $\mathbb{F}_q[[X,Y]]$ (q a prime power): we may have not enough curvettes based at distinct points, since in that case the projective plane $\mathbb{P}^2(\mathbb{F}_q)$ has only $q+1$ points with residual field $\mathbb{F}_q$ and any other point will have multiplicities due to the index of its residual field over $\mathbb{F}_q$.

Example 3.1. Does there exist a pencil in $\mathbb{F}_2[[X,Y]]$ such that its unique dicritical divisor is E_3 like in Figure 2? If the multiplicity m of the dicritical divisor is greater than 1, we can apply the method of §2; consider a curvette over the unique rational free point in E_3 (with multiplicity m) for one function and a curvette with multiplicity over a point whose residual field is $\mathbb{F}_{2^m}$ (which contributes with multiplicity m). This argument does not work for $m=1$. Nevertheless, the answer is also yes in this case, Example 1.3 over $\mathbb{F}_2$ is an example. In this case the triple curvette of E_1 and the double curvette of E_2 have also the behavior of curvettes over E_3.

The study of these generalized curvettes was one of the dramatically truncated goals of the joint work of Abhyankar and the first author. We end this work with a partial result which uses these generalized curvettes. If we consider the sequence of blow-ups of an irreducible germ, we can prove that there is an $F \in Q(R)$ such that the last exceptional component is the only dicritical with multiplicity one, even if it may be impossible to construct it using curvettes. Intersection form allows to find some generalized curvettes. The general case remains open since now the comment in Example 2.10 does not apply if the field is small.

Proposition 3.2. *The answer to* Problem 2.1 *is positive for* $K[[X,Y]]$ *for any* K *if* $\#\mathcal{W}=1$.

Sketch of the proof. The sequence of blowing-ups to reach a divisor W corresponds to the minimal resolution of an irreducible element of R. If K is infinite or if $K=\mathbb{F}_q$, $q>2$, it is not hard to adapt the techniques of §2; the dual graph looks like in Figure 3 and, in the finite case, for the last component W we have at least $q-1\geq 2$ free points to attach curvettes. The problem is that for $\mathbb{F}_2$, only one point is free. The valency of W in the dual graph is two, at least one of the connected subgraphs obtained erasing W is linear.

Let us consider a curvette Γ of the last component of this subgraph. Using intersection theory it is not hard to prove that if d is the determinant of the interesection matrix of the subgraph, then if δ is the corresponding root curvette, then δ^d is a generalized curvette. $\square$

Example 3.3. As we have pointed out if the base field is *small*, the general case cannot be obtained by multiplying suitable solutions for each particular dicritical. The main difficulty comes from the word *suitable*, since it is not possible in general to find functions such that almost all non-dicritical components are t-constant for $t \in K^*$. For example, consider divisors like in Figure 3 and look for $F \in Q(R)$ such that E_3, E_5, E_7 are dicritical divisor of multiplicity one. It is clear that we cannot find F using curvettes when $K = \mathbb{F}_2$. Moreover, the reduction of $F \mod 2$ does not work. We can proceed using Figure 4. We keep for g curvettes of E_7, E_8; for f we pick up a curvette for E_9 but we canoot keep a curvette for E_3 (no free point). As in Proposition 3.2, we can replace this unexisting curvette by a double curvette of E_2. It is a straightforward to check that it works.

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