DIFFERENTIAL GEOMETRY OF COMPLEX PROJECTIVE PLANE CONICS

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Abstract. In this paper we study properties of complex plane projective curves from a geometric point of view. We focus our attention on properties of conics. We find that Gauss curvature determines a conic up to a hermitian transformation preserving the Fubini-Study metric of the complex projective plane and we discuss some other geometric properties of the conics. Finally we study the deformation of smooth conics onto pair of lines and the classification of cubics up to hermitian transformations.

Introduction

The goal of this paper is to study projective plane curves as riemannian surfaces or hermitian curves. Let $\mathbb{P}^2$ be the complex projective plane and let $\mathcal{C}$ be a smooth plane curve (or, more generally, a curve with smooth branches). As a differentiable manifold $\mathcal{C}$ is a 2-dimensional submanifold of the four-dimensional submanifold $\mathbb{P}^2$; as analytic manifolds, the dimensions are divided by 2.

Let us consider the Fubiny-Study metric in $\mathbb{P}^2$; it is a hermitian metric and its real part is a riemannian metric. Both metrics restrict to $\mathcal{C}$ which becomes a riemannian surface (or a hermitian complex curve). Complex charts of $\mathcal{C}$ are isothermal for the riemannian metric. We want to study invariants of $\mathcal{C}$ as a geometric object and compare them with algebraic invariants.

This approach appears in the works of Linda Ness [5, 6] which studies the metric properties of deformations (Milnor fibers) for curves having ordinary singular points, i.e., formed by pairwise transversal smooth branches. In these papers, the author uses the Fubiny-Study metric on the projective plane to induce a hermitian metric on the smooth part of the curve. The geometric study of the singularities of a germ of plane curve singularity has been done by Evelia García Barroso and Bernard Teissier in [2, 3]; since their interest is mainly local, they prefer to use the euclidean hermitian metric.

We focus mainly on the study of reduced conics, determining the moduli space up to hermitian transformations and looking for geometric properties characterizing the hermitian conics. We show in concrete examples how smooth conics *degenerate* onto reducible conics. Finally, some aspects of cubics are studied.

1. Hermitian structure and Fubiny-Study metric

We state in this section the well-known properties of the Fubini-Study metric in order to fix normalizations and to help the reader. Let V be a $\mathbb{C}$-vector space of dimension $n+1$, its elements, considered both as points and vectors will be denoted with $\mathbf{u}, \mathbf{v}, \ldots$.

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Let $M := V \setminus \{0\}$; the projective space $\mathbb{P}(V)$ of V, i.e. the space of 1-dimensional complex vector subspaces of V, is naturally identified with the quotient $M/\mathbb{C}^*$ (action by scalar multiplication) and its points (orbits or vectorial lines) will be denoted with $\mathbf{p}, \mathbf{q}, \dots$, and the class of $\mathbf{u} \in M$ as $[\mathbf{u}]$.

Let us fix a hermitian scalar product h in V. Any such pair is isomorphic to $\mathbb{C}^{n+1}$ and the standard hermitian scalar product: if $\mathbf{u}, \mathbf{v} \in \mathbb{C}^{n+1}$ (as column vectors), then

$$h(\mathbf{u}, \mathbf{v}) := {}^t\mathbf{u} \cdot \overline{\mathbf{v}}.$$

Let $\mathrm{U}(V, h)$ be the group of h-unitary automorphisms. In the previous model it corresponds to

$$U(n; \mathbb{C}) := \left\{ P \in \mathrm{Mat}(n; \mathbb{C}) \mid {}^tP \cdot \overline{P} = \mathbf{1}_{n+1} \right\}.$$

Note that $T_\mathbf{u} M = T_\mathbf{u} V \equiv V$, $\forall \mathbf{u} \in M$, where a vector $\mathbf{v} \in V$ is identified with the tangent vector at $t = 0$ of the curve $t \mapsto \mathbf{u} + t\mathbf{v}$. The projective space $\mathbb{P}(V)$ acquires a natural structure of analytic manifold such that the quotient $\pi : M \to \mathbb{P}(V)$ is an analytic submersion.

Given $\mathbf{u} \in M$, let us denote by $\mathbf{p} = [\mathbf{u}]$ its class in $\mathbb{P}(V)$. If $\underline{\mathbf{u}} := (\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_n)$ is a basis of V, then the map

$$\varphi_{\underline{\mathbf{u}}} : \mathbb{C}^n \to \mathbb{P}(V), \quad \underline{z} := (z_1, \dots, z_n) \mapsto \left[\mathbf{u}_0 + \sum_{j=1}^n z_j \mathbf{u}_j \right] = \underline{\mathbf{u}} \begin{bmatrix} 1 \\ z_1 \\ \vdots \\ z_n \end{bmatrix}$$

is an analytic chart of $\mathbb{P}(V)$. Changing the basis, the projective space is covered and the change of charts are analytic. The following commutative diagram holds if λ_t is the multiplication by $t \in \mathbb{C}^*$:

$$\begin{tikzcd}
H_\mathbf{p} \arrow[r, hook] \arrow[dd, "\cdot t"'] \arrow[rrd, bend left, "d\pi_{\mathbf{u}|}"] & V \equiv T_\mathbf{u} M \arrow[rd, "d\pi_\mathbf{u}"] \arrow[dd, "d\lambda_{t|\mathbf{u}}"', "=\cdot t"] & \\
& & T_\mathbf{p}\mathbb{P}^n \\
H_\mathbf{p} \arrow[r, hook] \arrow[rru, bend right, "d\pi_{t\mathbf{u}|}"'] & V \equiv T_{t\mathbf{u}} M \arrow[ru, "d\pi_{t\mathbf{u}}"'] &
\end{tikzcd} \tag{1.1}$$

Note that $\mathbf{p} = \mathbb{C}\langle \mathbf{u} \rangle = \ker d\pi_\mathbf{p}$ and let us define $H_\mathbf{p} := \mathbf{p}^\perp$. The restrictions to $H_\mathbf{p}$ are isomorphisms in (1.1).

Let us consider the following hermitian metric $\mathbf{h}$ on M. For $\mathbf{u} \in M$ we define $\mathbf{h}_\mathbf{u} := \frac{1}{\|h(\mathbf{u})\|^2} h$. With this hermitian metric the maps λ_t are isometries and (1.1) allows us to define a hermitian metric $\mathbf{h}^\mathbb{P}$ on $\mathbb{P}^n$, called Fubiny-Study metric.

Example 1.1. Let us consider the chart $\varphi : \mathbb{C} \to \mathbb{P}^1$ given by $\varphi(t) := [1 : t]$. This chart factorizes through $\Phi : \mathbb{C} \to M$, $\Phi(t) := (1, t)$ (with coordinates $\mathbf{z} := (z_0, z_1)$ in M). Let

$\mathbf{u} := \frac{\partial}{\partial t}_{|[1:t]} = d\varphi_{|t}\left(\frac{\partial}{\partial t}_{|t}\right)$. Let us consider $d\Phi_{|t}\left(\frac{\partial}{\partial t}_{|t}\right) = \frac{\partial}{\partial z_1}_{|(1,t)}$. Note that

$$h\left(\frac{\partial}{\partial z_1}_{|(1,t)}, \underbrace{\frac{\partial}{\partial z_0}_{|(1,t)} + t\frac{\partial}{\partial z_1}_{|(1,t)}}_{\text{radial vector}}\right) = \bar{t}$$

and

$$\frac{\partial^\perp}{\partial z_1}_{|(1,t)} := \frac{\partial}{\partial z_1}_{|(1,t)} - \frac{\bar{t}}{1+t\bar{t}}\left(\frac{\partial}{\partial z_0}_{|(1,t)} + t\frac{\partial}{\partial z_1}_{|(1,t)}\right) \in H_{[1:t]}.$$

Since

$$h\left(\frac{\partial^\perp}{\partial z_1}_{|(1,t)}, \frac{\partial^\perp}{\partial z_1}_{|(1,t)}\right) = \frac{1}{1+t\bar{t}} \Longrightarrow \mathbf{h}_{(1,t)}\left(\frac{\partial^\perp}{\partial z_1}_{|(1,t)}, \frac{\partial^\perp}{\partial z_1}_{|(1,t)}\right) = \frac{1}{(1+t\bar{t})^2}.$$

Hence,

$$\mathbf{h}^{\mathbb{P}}_{[1:t]}(\mathbf{u},\mathbf{u}) = \frac{1}{(1+t\bar{t})^2}.$$

This number provides the matrix of the hermitian form in this chart. Its real part is the riemannian metric of the euclidean sphere $\mathbb{S}^2$ of radius $\frac{1}{2}$ as it can be checked from the stereographic projection.

Example 1.2. Let us consider the chart $\varphi : \mathbb{C}^2 \to \mathbb{P}^2$ given by $\varphi(\mathbf{t}) := [1 : t_1 : t_2]$, where $\mathbf{t} := (t_1, t_2)$ are the coordinates in $\mathbb{C}^2$. As in the previous example, it factorizes through $\Phi : \mathbb{C}^2 \to M$, $\Phi(\mathbf{t}) := (1, \mathbf{t})$ (with coordinates $\mathbf{z} := (z_0, z_1, z_2)$ in M). Let $\mathbf{u}_i := \frac{\partial}{\partial t_i}_{|[1:\mathbf{t}]} = d\varphi_{|\mathbf{t}}\left(\frac{\partial}{\partial t_i}_{|\mathbf{t}}\right)$. Let us consider $d\Phi_{|\mathbf{t}}\left(\frac{\partial}{\partial t_i}_{|\mathbf{t}}\right) = \frac{\partial}{\partial z_i}_{|(1,\mathbf{t})}$. We obtain

$$\frac{\partial^\perp}{\partial z_i}_{|(1,\mathbf{t})} := \frac{\partial}{\partial z_i}_{|(1,\mathbf{t})} - \frac{\bar{t}_i}{1+\|\mathbf{t}\|^2}\left(\underbrace{\frac{\partial}{\partial z_0}_{|(1,\mathbf{t})} + t_1\frac{\partial}{\partial z_1}_{|(1,\mathbf{t})} + t_2\frac{\partial}{\partial z_2}_{|(1,\mathbf{t})}}_{\text{radial vector}}\right) \in H_{[1:\mathbf{t}]}.$$

Hence,

$$\begin{pmatrix} \mathbf{h}^{\mathbb{P}}_{[1:\mathbf{t}]}(\mathbf{u}_1,\mathbf{u}_1) & \mathbf{h}^{\mathbb{P}}_{[1:\mathbf{t}]}(\mathbf{u}_1,\mathbf{u}_2) \\ \mathbf{h}^{\mathbb{P}}_{[1:\mathbf{t}]}(\mathbf{u}_2,\mathbf{u}_1) & \mathbf{h}^{\mathbb{P}}_{[1:\mathbf{t}]}(\mathbf{u}_2,\mathbf{u}_2) \end{pmatrix} = \frac{1}{(1+\|\mathbf{t}\|^2)^2}\begin{pmatrix} 1+|t_2|^2 & -\bar{t}_1 t_2 \\ -t_1\bar{t}_2 & 1+|t_1|^2 \end{pmatrix}.$$

Let $C \subset \mathbb{P}^2$ be a smooth projective plane curve. The restriction of this hermitian form to TC provides a hermitian form in C, which can be studied via analytic charts (local parametrizations).

Remark 1.3. If we replace $\mathbb{C}$ by $\mathbb{R}$, the above matrix is the matrix of a riemannian metric in a chart of the real projective plane; this metric coincides by its very construction with the metric obtained on $\mathbb{RP}^2$ as the quotient of $\mathbb{S}^2$ by the antipodal action.

The above examples can be extended easily to dimensions greater than 2. Both M and $\mathbb{P}^n$ are provided with a hermitian metric for which the map $\pi : M \to \mathbb{P}^n$ is riemannian submersion, see [1, Ch. 8]. Let us recall this result.

Proposition 1.4 ([4, Corollary 26.12])**.** *Let $F : M \to N$ be a riemannian submersion and let $\gamma : I \to M$ be a geodesic such that $\dot\gamma(t) \perp \ker dF_{\gamma(t)}$, $\forall t \in I$. Then, $\gamma \circ F$ is a geodesic.*

Corollary 1.5. *Let $\mathbf{u}, \mathbf{v} \in \mathbb{C}^{n+1}$ be two points which are unitary and orthogonal as vectors. Then, $t \mapsto [\cos t\, \mathbf{u} + \sin t\, \mathbf{v}]$ is a geodesic. In particular, the projective lines are totally geodesic submanifolds and the diameter of $\mathbb{P}^n$ equals $\frac{\pi}{2}$.*

Proof. It is easily seen that γ is a geodesic of M orthogonal to the fibers of π. $\square$

Corollary 1.6. *Let* $\mathbf{u} \in M$*,* $\mathbf{p} := [\mathbf{u}]$*. Then*

$$\left\{\mathbf{q} \in \mathbb{P}^n \middle| \, d(\mathbf{p}, \mathbf{q}) = \frac{\pi}{2}\right\} = \left\{\mathbf{q} \in \mathbb{P}^n \middle| \, d(\mathbf{p}, \mathbf{q}) \geq \frac{\pi}{2}\right\} = \{\mathbf{q} \in \mathbb{P}^n \mid \mathbf{p} \perp \mathbf{q}\}.$$

More precisely, for $\mathbf{p}, \mathbf{q} \in M$*:*

$$d([\mathbf{u}], [\mathbf{v}]) = \arccos \frac{|h(\mathbf{u}, \mathbf{v})|}{\|\mathbf{u}\| \, \|\mathbf{v}\|} \in \left[0, \frac{\pi}{2}\right].$$

2. Space of conics up to hermitian automorphisms

A polynomial $F \in \mathbb{C}[x_0, x_1, x_2]$ is *homogeneous* of degree d if $F(\lambda X) = \lambda^d F(X)$ for every $\lambda \in \mathbb{C}$. A *conic* in $\mathbb{P}^2$ is a hypersurface $C = \{[x] \in \mathbb{P}^2 | F(x) = 0\}$ where F is a non-zero homogeneous polynomial of degree 2. This equation can be also expressed as

$$\begin{pmatrix} x_0 & x_1 & x_2 \end{pmatrix} A \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = 0,$$

where A is a non-zero symmetric matrix. Two conics coincide if and only if their defining (polynomial or matrix) equations are proportional. In particular, the space of conics is a projective space of dimension 5. If $\det A \neq 0$ (resp. $\det A = 0$) we say that C is an irreducible or smooth (resp. reducible or singular) conic.

Definition 2.1. Let C be a projective curve defined as the zero locus of $F(X_0, X_1, X_2) = 0$, where F is square-free. We say that $\mathbf{p} := [x_0 : x_1 : x_2] \in C$ is *smooth* in C if $F_i(x_0, x_1, x_2)$ are not canceled simultaneously, where F_i denotes $\frac{\partial F}{\partial X_i}$. If not, $\mathbf{p}$ is a singular point. The curve C is smooth if all its points are smooth. Moreover, the *tangent line* to C at a smooth point P is the line

$$F_0(x_0, x_1, x_2) X_0 + F_1(x_0, x_1, x_2) X_1 + F_2(x_0, x_1, x_2) X_2 = 0.$$

In terms of matrices, a conic defined by a symmetric matrix A is smooth if and only if A is non-degenerate.

In order to work in a coordinate-free way, we replace the matrix by a symmetric bilinear form and work in arbitrary finite dimension. Let V be a $\mathbb{C}$-vector space, $\dim_{\mathbb{C}} V = n + 1$, with forms $Q, h : V \times V \to \mathbb{C}$, h is a hermitian scalar product and Q a non-zero symmetric bilinear form. Let us express both forms in terms of an h-unitary basis $\mathbb{v} = (v_0, v_1, \ldots, v_n)$ of V. For $\mathbf{v}, \mathbf{w} \in V$ such that $\mathbf{v} = \mathbb{v} X$ and $\mathbf{w} = \mathbb{v} Y$, we have the following expressions for h and Q:

$$h(\mathbf{v}, \mathbf{w}) = X^t \bar{Y} \qquad Q(\mathbf{v}, \mathbf{w}) = X^t A Y$$

where $A = (Q(v_i, v_j))$ is the symmetric matrix of Q with respect to $\mathbb{v}$. Also, we define

$$\mathbb{S}_h^{2n+1} := \{\mathbf{v} \in V \mid \|\mathbf{v}\| = \sqrt{h(\mathbf{v}, \mathbf{v})} = 1\} \subset V \setminus \{0\}$$

which is a compact manifold diffeomorphic to the sphere of dimension $2n + 1$.

Lemma 2.2. *In the above conditions, let* $F : \mathbb{S}_h^{2n+1} \to \mathbb{R}$*,* $F(\mathbf{v}) := Q(\mathbf{v}, \mathbf{v}) \overline{Q(\mathbf{v}, \mathbf{v})}$*. Let* $\mathbf{v} \in \mathbb{S}_h^{2n+1}$ *such that* F *reaches a maximum in* $\mathbf{v}$*. Then* $\mathbb{C}\langle \mathbf{v} \rangle^{\perp h} = \mathbb{C}\langle \mathbf{v} \rangle^{\perp Q}$*.*

Proof. Let us prove that $(\mathbb{C}\langle\mathbf{v}\rangle)^{\perp h} \subseteq (\mathbb{C}\langle\mathbf{v}\rangle)^{\perp Q}$. We define $g : \mathbb{R} \to \mathbb{R}$, $g(t) := F(\gamma(t))$ with $\gamma(t) = \cos t\,\mathbf{v} + \sin t\,\mathbf{w}$ where $\mathbf{w} \in \mathbb{C}\langle\mathbf{v}\rangle^{\perp h}$ and $\|\mathbf{w}\| = 1$. If $\mathbf{v} = \mathbb{v}X$, $\mathbf{w} = \mathbb{v}Y$, then

$$g(t) = \left((\cos t\,X + \sin t\,Y)^t A(\cos t\,X + \sin t\,Y)\right) \cdot \overline{((\cos t\,X + \sin t\,Y)^t A(\cos t\,X + \sin t\,Y))}$$

As $t = 0$ is a relative maximum for g it follows that

$$0 = \frac{1}{2}g'(0) = (Y^tAX)\overline{(X^tAX)} + (X^tAX)\overline{(Y^tAX)} = 2\Re\left((Y^tAX)(X^tAX)\right).$$

As $i\mathbf{w} \in (\mathbb{C}\langle\mathbf{v}\rangle)^{\perp h}$ and $\|i\mathbf{w}\| = 1$, we can repeat the above argument for $\mathbf{w}$, obtaining the equality $\Re\left(i(Y^tAX)(X^tAX)\right) = 0$, i.e., $(Y^tAX)(X^tAX) = 0$. Since $X^tAX \neq 0$ (F reaches its maximum at $\mathbf{v}$), then $Y^tAX = 0$ and $\mathbf{w} \in \mathbb{C}\langle\mathbf{v}\rangle^{\perp Q}$.

As both spaces are of the same dimension, the result follows. □

Let us state the complex Spectral Theorem with an elementary proof for completeness.

Proposition 2.3. *Let $Q : V \times V \to \mathbb{C}$ be a non-zero symmetric bilinear form. There exists an h-orthonormal basis $\{v_0, v_1, \dots, v_n\}$ of V such that the matrix A of Q on this basis is real diagonal with $r_0 \geq r_1 \geq \dots \geq r_n > 0$ such that*

$$A = \begin{pmatrix} r_0 & 0 & \cdots & 0 \\ 0 & r_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_n \end{pmatrix}.$$

Proof. Let us prove first that a diagonal matrix can be obtained. If $\dim V = 1$ then the result is trivial. If $\dim V > 1$ then, by Lemma 2.2, there is $0 \neq \mathbf{v} \in V$ such that $\|\mathbf{v}\| = 1$ and $W := (\mathbb{C}\langle\mathbf{v}\rangle)^{\perp h} = (\mathbb{C}\langle\mathbf{v}\rangle)^{\perp Q}$. The result follows by applying induction on W, $h_{|W}$ and $Q_{|W}$. Multiplying the elements of the basis by suitable complex numbers in $\mathbb{S}^1$ the result follows. □

Remark 2.4. As we can multiply by a non-zero scalar and the conic does not change, we can assume $r_0 = 1$.

Hence, it is possible to make a change of coordinates, preserving the hermitian product, such that any conic C can be transformed in

$$\begin{pmatrix} x_0 & x_1 & x_2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & r_1 & 0 \\ 0 & 0 & r_2 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = x_0^2 + r_1 x_1^2 + r_2 x_2^2.$$

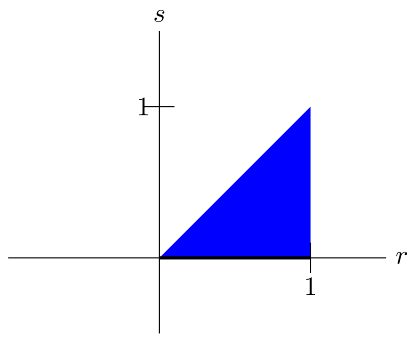
Then, if we choose $r, s \geq 0$ such that $r_1 = r^2$ and $r_2 = s^2$, we have that the moduli space of irreducible conics is given by

$$C_{r,s} = \{[x_0 : x_1 : x_2] \in \mathbb{P}^2 | x_0^2 + r^2 x_1^2 + s^2 x_2^2 = 0\}$$

with $0 < s \leq r \leq 1$. Hence, taking the limit cases, we have the moduli space of reducible conics as

$$C_{r,s} = \{[x_0 : x_1 : x_2] \in \mathbb{P}^2 | x_0^2 + r^2 x_1^2 = 0\}$$

with $0 \leq r \leq 1$ and $s = 0$.



- Moduli space of smooth conics
- Moduli space of reducible conics

Figure 1. Moduli space of conics

Remark 2.5. Let us consider the case of reducible conics, $r > 0$. The conic is formed by two projective lines L_1, L_2 intersecting at $\mathbf{p}_0 = [0 : 0 : 1]$; let $\mathbf{p}_j \in L_i$ such that $d(\mathbf{p}_0, \mathbf{p}_j) = \frac{\pi}{2}$, i.e.. maximal. Actually, these points are $\mathbf{p}_1 = [r : i : 0]$ and $\mathbf{p}_2 = [-r : i : 0]$. Moreover,

$$\cos d(\mathbf{p}_1, \mathbf{p}_2) = \frac{1 - r^2}{1 + r^2} \Longrightarrow r = \tan \frac{d(p_1, p_2)}{2}. \tag{2.1}$$

Hence, r can be geometrically recovered from $d(\mathbf{p}_1, \mathbf{p}_2)$. The limit case, when $r = 0$ correspond to the double line, when $\mathbf{p}_1 = \mathbf{p}_2$.

This is the kind of results we are looking for. Namely, we can compute the moduli space of a family of curves (products of two distinct lines) and we find geometric invariants which characterize to which element in the moduli space a particular curve belong. The first goal has been achieved for the smooth conics.

Theorem 2.6. *For any smooth conic C there is a hermitian change of coordinates which sends C to a conic $C_{r,s}$ where $0 < s \leq r \leq 1$.*

Before studying $C_{r,s}$ as a hermitian or riemannian manifold, we will focus on their behavior with respect to the distance of $\mathbb{P}^2$. Let C be a smooth conic defined by a regular symmetric matrix A. Let $\mathbf{p} = [x_0 : x_1 : x_2] \in C$, then $\mathbf{p}^\perp$ is the projective line

$$\mathbf{p}^\perp = \{\bar{x}_0 X_0 + \bar{x}_1 X_1 + \bar{x}_2 X_2 = 0\}.$$

This line is determined by its Plücker coordinates $\bar{\mathbf{p}} := [\bar{x}_0 : \bar{y}_0 : \bar{z}_0]$. It is easily checked that the Plücker coordinates of the tangent line to C at $\mathbf{p}$ in $\mathbb{P}^2$ are given by

$$\begin{pmatrix} a_0 & b_0 & c_0 \end{pmatrix} := \begin{pmatrix} x_0 & y_0 & z_0 \end{pmatrix} A.$$

As consequence

$$0 = \begin{pmatrix} x_0 & y_0 & z_0 \end{pmatrix} A \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} \Longleftrightarrow 0 = \begin{pmatrix} a_0 & b_0 & c_0 \end{pmatrix} A^{-1} A A^{-1} \begin{pmatrix} a_0 \\ b_0 \\ c_0 \end{pmatrix} = \begin{pmatrix} a_0 & b_0 & c_0 \end{pmatrix} A^{-1} \begin{pmatrix} a_0 \\ b_0 \\ c_0 \end{pmatrix}.$$

Hence, the Plücker coordinates of the tangent lines to C are the points of a conic determined by A^{-1}. We use it to check when orthogonal lines to points in a conic are tangent to the conic.

Theorem 2.7. *Let $C_{r,s}$ a smooth conic, $0 < s \leq r \leq 1$. The points $\mathbf{p} \in C_{r,s}$ such that $\mathbf{p}^\perp$ is tangent to $C_{r,s}$ are exactly:*

- *If $s = r = 1$, all points $\mathbf{p} \in C_{r,s}$.*
- *If $s = r < 1$, the two points $\mathbf{p} = [0 : \pm i : 1] \in C_{r,s}$.*
- *If $s < r = 1$, the two points $\mathbf{p} = [\pm i : 1 : 0] \in C_{r,s}$.*
- *If $s < r < 1$, the four points $\mathbf{p} = [\pm\sqrt{r^4 - s^4} : \pm ir\sqrt{1 - s^4} : s\sqrt{1 - r^4}] \in C_{r,s}$.*

The prove of this result is a straightforward computation and it provides a geometric characterization of the smooth conics. In the next section, we compute the curvature and we check that it also characterizes the geometry of the smooth conics.

3. Curvature of smooth conics

In this section, we will characterize the conics of $\mathbb{P}^2$ through Gauss curvature, which can be computed using [5, Theorem 1]. Since smooth conics are rational we are going to compute Gauss curvature using a parametrization Φ of $C_{r,s}$:

$$\begin{aligned} \mathbb{P}^1 &\xrightarrow{\Phi} C_{r,s} \subset \mathbb{P}^2 \\ [t_0 : t_1] &\longmapsto [f_2(t_0, t_1) : g_2(t_0, t_1) : h_2(t_0, t_1)] . \end{aligned}$$

where $f_2(t_0, t_1), g_2(t_0, t_1), h_2(t_0, t_1)$ are homogeneous polynomials of degree 2. A particular one is obtained using the pencil of lines through $[r : i : 0] \in C_{r,s}$. For the sake of simplicity we replace homogeneous coordinates by affine ones, and we obtain a chart of the conic for which we only miss one point:

$$\begin{aligned} \mathbb{P}^1 \supseteq \mathbb{C} &\xrightarrow{\Phi} C_{r,s} \setminus \{[-r : i : 0]\} \subset \mathbb{P}^2 \\ t &\longmapsto \left[r(s^2 - t^2) : i(s^2 + t^2) : 2rt\right] \end{aligned}$$

Although the image is not contained in a standard chart of the projective plane as in §1 (for which we have expressions of the hermitian product), we can compute the restricted hermitian product for $t \neq 0$; the computations can be extended by continuity.

As in Example 1.1, let $\mathbf{u} := \frac{\partial}{\partial t}_{|\Phi(t)}$. Using the expression given for the Fubini-Study metric in the Example 1.2, the restriction of $\mathbf{h}^{\mathbb{P}}$ on $C_{r,s}$, denoted as $\mathbf{h}^{r,s}$ is determined by the following formula:

$$\mathbf{h}^{r,s}_{\Phi(t)}(\mathbf{u}, \mathbf{u}) = \frac{4\left(r^2s^4 + r^2s^2t^2 + 4\,s^4t\bar{t} + r^2s^2\bar{t}^2 + r^2t^2\bar{t}^2 + s^4 - s^2t^2 - s^2\bar{t}^2 + t^2\bar{t}^2\right)r^2}{\left(r^2s^4 - r^2s^2t^2 - r^2s^2\bar{t}^2 + r^2t^2\bar{t}^2 + s^4 + s^2t^2 + 4\,r^2t\bar{t} + s^2\bar{t}^2 + t^2\bar{t}^2\right)^2}.$$

These computations (and the following ones) have been performed using `Sagemath` [8]; `Binder` [7] can be used in combination with `https://github.com/enriqueartal`. We are going to consider now real coordinates $t = x + iy$. The real part of this hermitian metric is a riemannian metric, which is isothermal in this chart, i.e., of the form $h_{r,s}(dx \otimes dx + dy \otimes dy)$, where $h_{r,s}$ is a real analytic function. If we define

$$K_1 = \left(r^2 + 1\right)s^4 + 4\left(x^2 + y^2\right)s^4 + 2\left(r^2 - 1\right)\left(x^2 - y^2\right)s^2 + \left(r^2 + 1\right)\left(x^2 + y^2\right)^2,$$

$$K_2 = \left(r^2 + 1\right)s^4 - 2\left(r^2 - 1\right)\left(x^2 - y^2\right)s^2 + \left(r^2 + 1\right)\left(x^2 + y^2\right)^2 + 4\left(x^2 + y^2\right)r^2,$$

the real coordinate expression can be written as

$$h_{r,s} = 4r^2 \frac{K_1}{K_2^2}.$$

As our metric is a conformal application, and the chart chosen to parameterize the conics is isothermal, we will be able to obtain the expression of the Gauss curvature without computing the symbols of Christoffel since we can do it using the following expression

$$K_{r,s} = -e^{-2\tilde{h}_{r,s}}((\tilde{h}_{r,s})_{xx} + (\tilde{h}_{r,s})_{yy})$$

where $h_{r,s} = e^{2\tilde{h}_{r,s}}$ and $K_{r,s}$ denotes the Gauss curvature of the conics $C_{r,s}$. With `Sagemath`, one can check:

$$K_{r,s} = 4 - \frac{2s^4 K_2^3}{r^2 K_1^3}.$$

Finally, we can give the desired characterization.

Theorem 3.1. *Let $C_{r,s}$ a smooth conic as above. Then, its Gauss curvature has the following properties.*

- *If $s = r = 1$, we have constant curvature $K = 2$.*
- *If $s = r < 1$, we have $K_{\max} = 4 - 2s^2$ in $[-i(y_0^2 s^2 + y_1^2) : y_0^2 s^2 - y_1^2 : 2sy_0y_1]$, $y_0, y_1 \in \mathbb{R}^2 \setminus \{0\}$, and $K_{\min} = \frac{2(2s^4-1)}{s^4}$ at the points $[0 : \pm i : 1]$.*
- *If $s < r = 1$, we have $K_{\max} = 4 - 2s^4$ at the points $[\pm i : 0 : 1]$ and $K_{\min} = \frac{2(2s^2-1)}{s^2}$ on the rational curve $[s^2x_0^2 - x_1^2 : i(x_0^2 s^2 + x_1^2) : 2x_0x_1]$, $x_0, x_1 \in \mathbb{R}^2 \setminus \{0\}$.*
- *If $s < r < 1$, we have $K_{\max} = \frac{2\left(2\,r^2-s^4\right)}{r^2}$ at the points $[r : \pm i : 0]$ and $K_{\min} = \frac{2\left(2\,r^2 s^2-1\right)}{r^2 s^2}$ at the points $[0 : \pm is : r]$. Also in this case, we have two saddle points in $[\pm is : 0 : 1]$ where $K = \frac{2\left(2\,s^2-r^4\right)}{s^2}$.*

The computations for the proof are detailed in the worksheet of `Sagemath`. Note also that the case $r = s = 1$ is quite special as they have the same riemannian metric as a sphere of radius $\frac{1}{\sqrt{2}}$ in $\mathbb{R}^3$, in particular the conic with equation $x_0^2 + x_1^2 + x_2^2 = 0$ is homothetic to the projective line.

Corollary 3.2. *Two smooth conics are isometric if and only if their parameters r, s coincide. Moreover, they are homothetic if and only if they are isometric.*

Proof. The case $r = s = 1$ is characterized by constant curvature. The case $s = r < 1$ is characterized by the fact that the curvature is not constant and there are infinite maxima and s can be recovered from the maximum value. A similar property holds for $s < r = 1$, where maxima are replaced by minima. They are distinguised from the case $s < r < 1$ with two maxima and minima. From the maximum and minimum values we can recover $\frac{s^2}{r}$ and $\frac{r^2}{s}$, hence also s, r.

If the riemannian metrics are homothetic by a factor $a > 0$, their Gauss curvatures are homothetic by $\frac{1}{a^2}$. Hence the ratios of the critical values must be preserved. Clearly, the cases $0 < s = r = 1$, $0 < s = r < 1$, $0 < s < r = 1$ and $0 < s < r < 1$ can be treated apart.

The case $0 < s = r = 1$ is trivial. For the case $0 < s = r < 1$ the critical values can be seen as homogenous coordinates $\left[4 - 2s^2 : \frac{2(2s^4-1)}{s^4}\right] = \left[1 : \frac{2s^4-1}{s^4(2-s^2)}\right]$. Since the second coordinate is an increasing function of s in $(0, 1)$, it determines the value s.

For the case $0 < s < r = 1$ the critical values can be seen as homogenous coordinates $\left[4 - 2s^2 : \frac{2(2s^2-1)}{s^2}\right] = \left[1 : \frac{(2s^2-1)}{s^2(2-s^2)}\right]$. The second coordinate is again an increasing function of s in $(0, 1)$ and it determines the value s.

For the case $0 < s < r < 1$, the homogeneous coordinates of the critical values are given by $\left[s^2(2r^2 - s^4) : r^2(2s^2 - r^4) : 2r^2s^2 - 1\right] = [a : b : c]$. Note that $a \neq 0$ and $a > b > c$. We get

$$r^2 = \frac{s^4 c}{2(c-a)}, \qquad s^6 = 8\frac{(c-a)^2(bc-a^2)}{c^3 b}$$

The result follows if $bc \neq 0$. If $b = 0$, we have $s^2 = \frac{r^4}{2}$, and the coordinates are given by $\left[r^6(8 - r^6) : 0 : 8(r^6 - 1)\right]$ which determine r^6. If $c = 0$, we have $s^2 = r^{-2}$ and the homogeneous coordinates $\left[2 - r^{-6} : 2 - r^6 : 0\right]$ also determine r^6. □

4. Deformation properties

Let us consider the family of conics $C_s : x_1^2 + x_2^2 + s^2x_3^2 = 0$, for $0 \le s \le 1$. For the limit cases, C_0 is a union of two lines and C_1 is a smooth conic with constant curvature. Recall that the critical values of the Gauss curvature are

$$K_{\min} = 2\frac{2s^2-1}{s^2}, \qquad K_{\max} = 2(2 - s^4).$$

The maximum value goes from $K_{\max} = 2$, when $s = 1$ and $K_{\max} = 4$, for $s = 0$. In the latter case, the curvature is constant equal to 4, outside the singular point, where the function is not defined. If $s > \frac{\sqrt{2}}{2}$, the curvature is strictly positive.

For the case $s = s_0 := \frac{\sqrt{2}}{2}$, the curvature of C_{s_0} is non negative. As it was stated in Theorem 3.1, the minimum appears for the real curve $[s^2x_0^2 - x_1^2 : i(x_0^2s^2 + x_1^2) : 2x_0x_1]$, when $[x_0 : x_1] \in \mathbb{RP}^1$. Topologically, we have that the curvature vanishes at the *equator* of a sphere and it positive in the open hemispheres. In the chart this curve correspond to $x^2 + y^2 = s^2$. In order to have point with negative curvature, we need $s < s_0$.

If $K = 0$, then

$$0 = \left(x^2 + y^2\right)^2 - 2s^{\frac{4}{3}}\frac{2^{\frac{1}{3}}s^{\frac{8}{3}} - 1}{s^{\frac{4}{3}} - 2^{\frac{1}{3}}}\left(x^2 + y^2\right) + s^4$$

which is equivalent to

$$0 = \left(x^2 + y^2 - \frac{\left(2^{\frac{1}{3}}s^{\frac{8}{3}} - 1\right)s^{\frac{4}{3}}}{s^{\frac{4}{3}} - 2^{\frac{1}{3}}}\right)^2 - \frac{\left(s^4 - 1\right)\left(2^{\frac{2}{3}}s^{\frac{4}{3}} - 1\right)s^{\frac{8}{3}}}{\left(s^{\frac{4}{3}} - 2^{\frac{1}{3}}\right)^2}.$$

When $s < \frac{\sqrt{2}}{2}$, the conic C_s is decomposed into a annulus where the curvature is non-positive (and negative in its interior) and two disks where the curvature is non-negative and positive in its interior. There is a deformation in this family were the negative annulus collapses to the singular point $P = [1 : -i : 0]$ and the the two disks end in the complement of the lines to P.

5. Metric properties of cubics

The geometric study of cubics may need another work. In this work we will provide a small introduction. Let us consider the space of all cubics composed by three non-concurrent lines, or equivalently (by duality or by taking the pairwise intersections) the space of three non-aligned points.

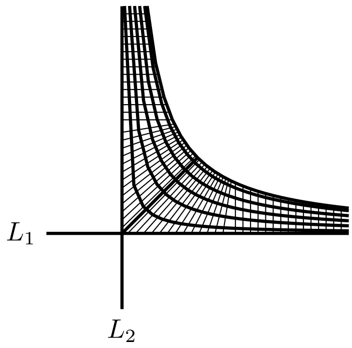


FIGURE 2. Deformation of a family of smooth conics onto a pair of lines

The moduli space of these curves, up to projective transformations, is reduced to one point. In this section we are going to describe the moduli space up to unitary transformations. For the sake of simplicity, we will consider the space of three ordered lines; the original one can be obtained by applying the action of the symmetric group of three figures.

Let L_a, L_b, L_c be three non-concurrent lines and let us denote $A := L_b \cap L_c$, $B := L_a \cap L_c$, and $C := L_a \cap L_b$; the points A, B, C are not aligned. After a unitary transformation, we may assume that $C = [0 : 0 : 1] =: P_2$ and $L_b = \{X_1 = 0\} =: L_1$. A unitary transformation preserving (C, L_b) must preserve also $C^\perp = \{X_2 = 0\}$ and $L_b^\perp = [0 : 1 : 0]$. Hence, these transformations are diagonal. Let us summmarize these first ideas.

Lemma 5.1. *Let $\mathcal{M}$ be the moduli space of the space of three ordered non-concurrent lines by the action of the projective unitary group. Then $\mathcal{M}$ is isomorphic to $V := (L_1 \setminus \{P_2\}) \times \left(\mathbb{P}^2 \setminus L_1\right)$ by the action of $\mathbb{S}^1 \times \mathbb{S}^1$ as a diagonal matrix group.*

The space V is isomorphic to $\mathbb{C}^3$ via

$$(a, b, c) \overset{\Phi}{\longmapsto} ([1 : 0 : a], [b : 1 : c]) = (A, B)$$

and the action of $(\lambda, \mu) \in \mathbb{S}^1 \times \mathbb{S}^1$ is given by

$$(\lambda, \mu) \cdot (a, b, c) = (\overline{\lambda}\mu a, \lambda b, \mu c).$$

There is a natural stratification of V by the isotropy groups of its points.

Lemma 5.2. *The stratification by the action of $\mathbb{S}^1 \times \mathbb{S}^1$ is given by:*

(1) $\mathcal{S}^0 := \{\Phi(0, 0, 0)\}$. *The isotropy group is the whole group $\mathbb{S}^1 \times \mathbb{S}^1$ and the pair $\Phi(0, 0, 0) = ([1 : 0 : 0], [0 : 1 : 0])$ corresponds to the case where the three points are at distance $\frac{\pi}{2}$.*
(2) $\mathcal{S}^1_0 = \Phi(\mathbb{C}^* \times \{(0, 0)\})$. *The isotropy group is the diagonal subgroup. The pairs correspond to the case $d(A, B) = d(B, C) = \frac{\pi}{2}$ and $d(A, C) < \frac{\pi}{2}$.*
(3) $\mathcal{S}^1_1 = \Phi(\{0\} \times \mathbb{C}^* \times \{0\})$. *The isotropy group is $\{1\} \times \mathbb{S}^1$. The pairs correspond to the case $d(A, C) = d(B, C) = \frac{\pi}{2}$, $d(A, B) < \frac{\pi}{2}$.*
(4) $\mathcal{S}^1_2 = \Phi(\{(0, 0)\} \times \mathbb{C}^*)$. *The isotropy group is $\mathbb{S}^1 \times \{1\}$. The pairs correspond to the case $d(A, B) = d(A, C) = \frac{\pi}{2}$ and $d(B, C) < \frac{\pi}{2}$.*
(5) $\mathcal{S}^3$ *is the complement in V of the union of the other strata. The isotropy group is trivial. At least two of the distances are $< \frac{\pi}{2}$.*

The action of the group on $V \equiv \mathbb{C}^3$ *kills* the arguments of complex numbers outside the stratum $\mathcal{S}^3$ (or even in $\mathbb{C}^3 \setminus \{abc = 0\}$). A simple computation gives the following result.

Proposition 5.3. *The moduli space $\mathcal{M}$ is homeomorphic to the quotient of $\mathbb{R}^3_{\geq 0} \times \mathbb{S}^1$ by the identification generated by $((a,b,c),\lambda) \sim ((a,b,c),1)$ when $abc = 0$.*

This proposition justifies to study separately an open dense subset of $\mathcal{S}^3$. Let us denote $\check{\mathcal{S}}^3 := \Phi((\mathbb{C}^*)^3)$; its image in $\mathcal{M}$ is denoted as $\check{\mathcal{M}}$ and by the proposition it is homeomorphic to $\mathbb{R}^3_{>0} \times \mathbb{S}^1$.

For these cases, we can use the group action to assume that $B = [\cos u \sin v : \sin u \sin v : \cos v]$, where $v = d(B, C) \in (0, \frac{\pi}{2})$, and $u \in (0, \frac{\pi}{2})$. For this assumption, the whole group action has been used and hence we have $A = [\lambda \sin w : 0 : \cos w]$, where $w = d(A, C) \in (0, \frac{\pi}{2})$ and $\lambda \in \mathbb{S}^1$. We have a homeomorphism of $\check{\mathcal{M}}$ with $\left(0, \frac{\pi}{2}\right)^3 \times \mathbb{S}^1$, $(A, B) \equiv (u, v, w, \lambda)$. Let us find a geometric meaning for u and λ. For u we proceed as follows. The point in $C^\perp$ at distance $\frac{\pi}{2}$ of B is the point $\tilde{B} = [-\sin u : \cos u : 0]$, and the distance of $\tilde{B}$ with $L_b^\perp$ equals u.

To understand the meaning of λ we consider the remaning distance $q := d(A, B)$, using Corollary 1.6:

$$\cos q = |\lambda \cos u \sin v \sin w + \cos v \cos w| \,. \tag{5.1}$$

Let us use also the geodesic triangle with vertices A, B, C. The edge BC is $\gamma_a : [0, v] \to \mathbb{P}^2$ given by $\gamma_a(t) := [\cos u \sin t : \sin u \sin t : \cos t]$. The edge AC is $\gamma_b : [0, w] \to \mathbb{P}^2$ given by $\gamma_b(t) := [\lambda \sin t : 0 : \cos t]$. Let us consider the lifts $\tilde{\gamma}_a, \tilde{\gamma}_b$ to $\mathbb{C}^3 \setminus \{\mathbf{0}\}$ of these paths. The tangent vectors at $t = 0$ are

$$\dot{\tilde{\gamma}}_a(0) = (\cos u, \sin u, 0), \quad \dot{\tilde{\gamma}}_b(0) = (\lambda, 0, 0).$$

As these vectors are unitary and orthogonal to the radius vector $(0, 0, 1)$, we obtain that

$$\mathbf{h}^{\mathbb{P}}_C(\dot{\gamma}_a(0), \dot{\gamma}_b(0)) = \lambda \cos u.$$

As a consequence, if γ is the angle between the two edges at C, then

$$\cos \gamma = \Re \lambda \cos u. \tag{5.2}$$

Following (5.1) and (5.3), we obtain

$$\cos q = |\cos v \cos w + \cos \gamma \cos \gamma \sin v \sin w + i \Im \lambda \cos u \sin v \sin w| \,. \tag{5.3}$$

This formula coincides with the spherical cosinus law when $\lambda = \pm 1$, so λ *measures* how far this triangle is to be a spherical one.

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