COURSE 4. TOPOLOGY OF PLANE CURVES

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Contents

CHAPTER 1

Day #1ay 1

1. Motivation

The study of the topology of complex algebraic varieties is a main subject in the intersection of geometric topology and algebraic geometry. In this course we will focus our attention on the study of the topology of (either projective or affine) plane curves as subspaces of the (either projective or affine) plane.

One of the reason is that it is the first non-trivial case when studying the embedded topology of such varieties. There is another one which justifies that we are mainly interested in the study of fundamental group as a main topological invariant.

Any projective variety can be seen as a branched covering of the projective space (of the same dimension) ramified along a hypersurface. These branched covers are measured by the fundamental group of the complement of the hypersurface in the projective space. Moreover, using Zariski-Lefschetz theory this group does not change if we consider a *generic* plane section. Hence, we see the value of the study of the fundamental group of the complement of an algebraic curve.

In this course we will study these fundamental groups using various techniques. We will illustrate these techniques with examples. In particular, after this course, the students should be able to find the fundamental group $\pi_1(\mathbb{P}^2 \setminus \mathcal{C})$ for any projective curve $\mathcal{C}$ of degree at most 4.

For general reference see [**4**].

2. First examples

The ambient space will be the complex projective plane $\mathbb{P}^2$. If we work with curves having a line L as an irreducible component, since $\mathbb{P}^2 \setminus L$ is isomorphic to $\mathbb{C}^2$, for some cases the affine plane will be considered as the ambient space; it is not compact, but some projection techniques are easier to apply in this context.

Having in mind the goal of computing $\pi_1(\mathbb{P}^2 \setminus \mathcal{C})$ for any projective curve $\mathcal{C}$ of degree at most 4, we start with the simplest case, a curve of degree 0, i.e., the empty set

Proposition 2.1. *The projective plane $\mathbb{P}^2$ is simply connected.*

Proof. We will give three different skecthes of the proof. In fact, we are going to prove that $\pi_1(\mathbb{P}^n)$ is trivial.

The first approach uses Seifert-van Kampen Theorem. Note that $\mathbb{P}^n$ is the union of $n+1$ open sets all of them isomorphic to $\mathbb{C}^n$ which is contractible and, hence, simply connected. Moreover, the intersection of any subfamily of these $n+1$ open sets is connected. By an easy induction process, the classical Seifert-van Kampen Theorem implies that $\pi_1(\mathbb{P}^n)$ is trivial.

The second approach uses the Hopf fibration, a locally trivial fibration $\rho_n : \mathbb{S}^{2n+1} \to \mathbb{P}^n$, which fiber isomorphic to $\mathbb{S}^1$. The long exact sequence of homotopy provides the desired information, not only for $\pi_1(\mathbb{P}^n)$ but also for some higher homotopy groups.

The third approach comes from the CW-complex decomposition of $\mathbb{P}^n$ which uses the above Hopf filtration. The natural decomposition has exactly one cell of dimension $2j$, $0 \leq j \leq n$. Then, the fundamental group is clearly trivial. $\square$

Following the program, we study the fundamental group of $\mathbb{P}^2 \setminus \mathcal{C}$, where $\mathcal{C}$ is a curve of degree 1, i.e., a line. As we have already used, in this case $\mathbb{P}^2 \setminus \mathcal{C} \cong \mathbb{C}^2$, a contractible space, and then, the fundamental group is clearly trivial.

Remark 2.2. Since we are interested only in the topology, we assume that the curves we are dealing with are reduced, i.e., their equations are square-free.

Exercise 2.1. Prove that any conic is either the union of two (distinct) lines or an irreducible conic. Any two irreducible conics are isomorphic.

Example 2.3. If $\mathcal{C}$ is the union of two lines, we may assume that its equation is $yz = 0$. In particular $\mathbb{P}^2 \setminus \mathcal{C} \cong \mathbb{C} \times \mathbb{C}^*$. Since $\mathbb{C}^*$ has the same homotopy type as $\mathbb{S}^1$, we have that $\pi_1(\mathbb{P}^2 \setminus \mathcal{C}) \cong \mathbb{Z}$. Recall that $\pi_1(\mathbb{C}^*; 1)$ is generated by the homotopy class of the loop $t \mapsto \exp(2i\pi t)$.

In the above examples, all the fundamental groups are abelian. Since the quotient $\pi_1(\mathbb{P}^2 \setminus \mathcal{C})/\pi_1(\mathbb{P}^2 \setminus \mathcal{C})'$ is isomorphic to $H_1(\mathbb{P}^2 \setminus \mathcal{C}; \mathbb{Z})$, let us recall how to compute it.

Proposition 2.4 ([**11**]). *Let $\mathcal{C} = \bigcup_{i=1}^r \mathcal{C}_i$ be the decomposition in irreducible components of an algebraic plane curve $\mathcal{C}$, $d_i := \deg \mathcal{C}_i$, $d := \gcd\{d_1, \ldots, d_r\}$. Then, $H_1(\mathbb{P}^2 \setminus \mathcal{C}; \mathbb{Z}) \cong \mathbb{Z}^{r-1} \oplus \mathbb{Z}/d\mathbb{Z}$. In particular, if one of the components is a line, then $H_1(\mathbb{P}^2 \setminus \mathcal{C}; \mathbb{Z}) \cong \mathbb{Z}^{r-1}$.*

Proof. We will make use of Lefschetz duality which implies that $H_1(\mathbb{P}^2 \setminus \mathcal{C}; \mathbb{Z}) \cong H^3(\mathbb{P}^2, \mathcal{C}; \mathbb{Z})$. Let us consider now the long exact sequence of cohomology of the pair $(\mathbb{P}^2, \mathcal{C})$ and recall that $H^3(\mathbb{P}^2; \mathbb{Z}) = 0$:

$$\mathbb{Z} \cong H^2(\mathbb{P}^2; \mathbb{Z}) \stackrel{\alpha}{\to} H^2(\mathcal{C}; \mathbb{Z}) \to H^3(\mathbb{P}^2, \mathcal{C}; \mathbb{Z}) \to 0 \tag{1}$$

Using Mayer-Viétoris exact sequence and the fact that the irreducible components of $\mathcal{C}$ (pairwise) intersect at points, we obtain that $H^2(\mathcal{C}; \mathbb{Z}) \cong \bigoplus_{i=1}^r H^2(\mathcal{C}_i; \mathbb{Z}) \cong \mathbb{Z}^r$. Using Poincaré duality and intersection theory, the map $\alpha : \mathbb{Z} \to \mathbb{Z}^r$ is given by $\alpha(1) = (d_1, \ldots, d_r)$ and the result follows. $\square$

Remark 2.5. There is a nice topological interpretation of this result. Let L be a generic line in $\mathbb{P}^2$ with respect to $\mathcal{C}$, i.e. $\#(L \cap \mathcal{C}) = d_1 + \cdots + d_r$. Note that $H_1(L \setminus \mathcal{C} : \mathbb{Z})$ is generated by small circles centered at the punctures (counterclockwise oriented) with the relation that the sum of all of them is trivial. The punctures associated to each irreducible components are equal in $H_1(\mathbb{P}^2 \setminus \mathcal{C}; \mathbb{Z})$.

One of the applications of this result is the following one. If it is possible to prove that such a fundamental group is abelian, then it is easily described. In fact Proposition 2.4 works also word by word if we consider hypersurfaces in higher dimensional projective spaces.

3. Projections and meridians

Let us fix now a projective curve $\mathcal{C} \subset \mathbb{P}^2$ and a point $P \in \mathbb{P}^2$. Note first that $\pi_1(\mathbb{P}^2 \setminus \mathcal{C}) \cong \pi_1(\mathbb{P}^2 \setminus (\mathcal{C} \cup \{P\}))$; it is trivial if $P \in \mathcal{C}$ and follows from Seifert-van Kampen Theorem if $P \notin \mathcal{C}$. The main goal of the well-known Zariski-van Kampen Theorem is to give an *algorithm* to compute $\pi_1(\mathbb{P}^2 \setminus \mathcal{C})$ using the projection $\pi_P : \mathbb{P}^2 \setminus \{P\} \to \mathbb{P}^1$. This method will be discussed in the next lectures but we are going to give the first ideas of the method.

REMARK 3.1. It is useful to choose suitable coordinates, e.g., such that $P = [0:1:0]$. The classical method of Zariski and van Kampen assumes that $P \notin \mathcal{C}$ but it is not necessary.

The projective line $\mathbb{P}^1$ is identified with the pencil of lines through P; for a point $p \in \mathbb{P}^1$, the associated line is denoted as $\bar{L}_p$, while $L_p := \bar{L}_p \setminus \{P\}$.

REMARK 3.2. In the classical method, we may identify $\mathbb{P}^1 \equiv \mathbb{C} \cup \{\infty\}$ where $\bar{L}_t = \{x = tz\}$, $t \in \mathbb{C}$, and $\bar{L}_\infty = \{z = 0\}$.

Let us denote $\mathbb{P}^1_* := \{p \in \mathbb{P}^1 \mid \bar{L}_p \not\subseteq \mathcal{C}\}$. Let $k := \sup_{p \in \mathbb{P}^1_*} \#(L_p \cap \mathcal{C})$.

EXERCISE 3.1. Prove that $k < \infty$ and

$$\mathcal{B} := \{p \in \mathbb{P}^1_* \mid \#(L_p) \cap \mathcal{C} < k\} \cup \{p \in \mathbb{P}_1 \mid \bar{L}_p \subset \mathcal{C}\}$$

is a finite set.

PROPOSITION 3.3. *The restriction* $\pi_{P|} : \mathbb{P}^2 \setminus (\mathcal{C} \cup \{P\} \cup \bigcup_{p \in \mathbb{P}^1_*} L_p) \to \mathbb{P}^1 \setminus \mathcal{B}$ *is a locally trivial fibration with fiber homeomorphic to* $\mathbb{P}^1 \setminus (k+1)$ points $\cong \mathbb{C} \setminus k$ points.

The long exact sequence of homotopy of this fibration induces a short exact sequence involving the fundamental group of the complement of a curve bigger than $\mathcal{C}$. We study this exact sequence and the relationship of this group with $\pi_1(\mathbb{P}^2 \setminus \mathcal{C})$ in the following section

PROPOSITION 3.4. *With the above notations, there is a short exact sequence:*

$$1 \to \pi_1(\mathbb{C} \setminus k \text{ points}) \to \pi_1(\mathbb{P}^2 \setminus (\mathcal{C} \cup \bigcup_{p \in \mathbb{P}^1_*} L_p)) \to \pi_1(\mathbb{P}^1 \setminus \mathcal{B}) \to 1. \tag{2}$$

The first and third terms of (2) are free groups coming from fundamental groups of punctured planes or 2-spheres. These groups have bases with geometric meaning. Let us consider $X := \mathbb{P}^1 \setminus \{p_0, p_1, \ldots, p_m\}$ (pairwise distinct points). Fix $q \in X$ a base point.

DEFINITION 3.5. A meridian of p_j in $\pi_1(X; q)$ is the homotopy class of a loop γ obtained as follows. Fix a closed disk $\mathbb{D}_j$ containing p_j in its interior and disjoint from the other points. Let $p'_j \in \partial\mathbb{D}_j$ and let β be the loop based at p'_j running counterclockwise $\partial\mathbb{D}_j$. Let α be a path with endpoints q and p'_j. Then $\gamma := \alpha \cdot \beta \cdot \alpha^-$.

EXERCISE 3.2. The following properties hold for meridians:

(1) Two meridians of p_j in $\pi_1(X; q)$ are conjugated and any element in the conjugacy class of a meridian is a meridian.
(2) The group $\pi_1(X; q)$ is free and there is a basis formed by meridians γ_j of p_j, $j = 1, \ldots, r$ such that $\gamma_0 := (\gamma_r \cdot \ldots \cdot \gamma_1)^{-1}$ is a meridian of p_0 (such a basis will be called *geometric*).
(3) Two meridians in $\pi_1(X; q)$ are conjugated if and only if they belong to the same point.

EXERCISE 3.3. Define meridians for irreducible components of plane curves in $\mathbb{P}^2$ and prove the statements in Exercise 3.2 (except the one claiming the freeness of the group).

The following two results are part of Zariski-van Kampen method.

PROPOSITION 3.6. *The group* $\pi_1(\mathbb{P}^2 \setminus (\mathcal{C} \cup \bigcup_{p \in \mathbb{P}^1_*} L_p))$ *is generated by meridians, more precisely, by a geometric basis of the fundamental group of a punctured fiber and by lifts of meridians of* $\pi_1(\mathbb{P}^1 \setminus \mathcal{B})$.

Proposition 3.7. *Let $\mathcal{C},\mathcal{D}\subset\mathbb{P}^2$ be two curves with no irreducible component in common. Let $\rho:\pi_1(\mathbb{P}^2\setminus(\mathcal{C}\cup\mathcal{D}))\to\pi_1(\mathbb{P}^2\setminus\mathcal{C})$ the morphism induced by the inclusion. Then,*

1. *The morphism ρ is surjective.*
2. $\ker\rho$ *is normally generated by the meridians of the irreducible components of $\mathcal{D}$.*

Proof. The proof uses that for any differentiable manifold M, $\pi_1(M)\cong\pi_1^{\mathcal{C}^\infty}(M)$ (i.e. both loops and homotopies are assumed to be differentiable). Both results come from transversality theory. $\square$

Example 3.8. The fundamental group G of the complement of a smooth conic is cyclic of order 2. In order to prove this result, we can consider the fundamental group of the affine complement of a parabole (which is isomorphic to $\mathbb{C}\times\mathbb{C}^*$), which is abelian. Its quotient G is also abelian and hence isomorphic to its abelianization.

Corollary 3.9. *Let us assume that $\mathcal{C}$ contains no line through P (in fact, no more than one line). Then, if $\bar{L}_p$ is a generic line through P, then the natural map $\pi_1(\bar{L}_p\setminus\mathcal{C})\to\pi_1(\mathbb{P}^2\setminus\mathcal{C})$ is surjective.*

Exercise 3.4. Let us assume that $\mathcal{C}$ is a reduced curve of degree d with equation $F(x,y,z):=f_{d-1}(x,y)z+f_d(x,y)$, where f_j is a homogeneous polynomial of degree j.

1. If $\gcd(f_{d-1}(x,y),f_d(x,y))=1$, then $\pi_1(\mathbb{P}^2\setminus\mathcal{C})$ is abelian.
2. If $f_{d-1}(x,y)\equiv 0$, then $\pi_1(\mathbb{P}^2\setminus\mathcal{C})$ is free of rank $d-1$.

Exercise 3.5. Compute the fundamental group of the complement of $z\prod_{i=1}^r(x-t_iz)\prod_{j=1}^m(y-s_jz)=0$.

Exercise 3.6. Prove that the fundamental group of the complement of $xz(xz-y^2)=0$ is not abelian.

4. Degenerations

We finish this section with two results concerning families of curves. For a 1-*dimensional family* of curves of degree d, we mean $\{\mathcal{C}_t\}_{t\in\mathbb{D}_\varepsilon}$, where $0<\varepsilon$ and the coefficients of the equations of $\mathcal{C}_t$ depend holomorphically on t. We say that the family is *equisingular* if there exist holomorphic maps $\varphi_j:\mathbb{D}_\varepsilon\to\mathbb{P}^2$, $P_j^t:=\varphi_j(t)$, $j=1,\dots,m$, such that

1. $\operatorname{Sing}(\mathcal{C}_t)=\{P_1^t,\dots,P_m^t\}$, $\#\operatorname{Sing}(\mathcal{C}_t)=m$.
2. The topological type of $(\mathcal{C}_t,P_j^t)$ does not depend on t.

The family is said to be *equisingular away from the origin* if the above holds for $t\in\mathbb{D}_\varepsilon^*$.

Theorem 4.1. *If $\{\mathcal{C}_t\}_{t\in\mathbb{D}_\varepsilon}$ is an equisingular family then the pairs $(\mathbb{P}^2,\mathcal{C}_0)$ and $(\mathbb{P}^2,\mathcal{C}_t)$ are homeomorphic, $\forall t\in\mathbb{D}_\varepsilon$.*

Sketch of the proof. Assume that $\mathcal{C}_t$ is smooth. We can use the incidence variety and submersion properties. $\square$

Theorem 4.2. *If $\{\mathcal{C}_t\}_{t\in\mathbb{D}_\varepsilon}$ is an equisingular family away from the origin and $\mathcal{C}_0$ is a reduced curve, then there is an epimorphism $\pi_1(\mathbb{P}^2\setminus\mathcal{C}_0)\to\pi_1(\mathbb{P}^2\setminus\mathcal{C}_t)$, $\forall t\in\mathbb{D}_\varepsilon^*$.*

Proof. From Theorem 4.1, it is enough to prove the statement for a particular t. Let U_0 be a regular compact neighborhood of $\mathcal{C}_0$ in $\mathbb{P}^2$ and let $\mathring{U}_0$ be its interior. We may assume that $\mathcal{C}_t\subset\mathring{U}_0$. The map we must consider is the induced by $\mathbb{P}^2 \setminus \mathring{U}_0 \hookrightarrow \mathbb{P}^2 \setminus \mathcal{C}_t$, taking into account that $\mathbb{P}^2 \setminus \mathring{U}_0$ is homotopy equivalent to $\mathbb{P}^2 \setminus \mathcal{C}_0$.

They key argument is provided by Corollary 3.9, where the fact the $\mathcal{C}_0$ is reduced is used. □

EXERCISE 4.1. Compute the fundamental groups of the complements of cubic curves.

5. Zariski-Lefschetz theory

Corollary 3.9 admits the following generalization due to Zariski and Lefschetz.

THEOREM 5.1. *Let $\bar{X}^n \subset \mathbb{P}^N$ be a projective irreducible variety of dimension n. Let $\bar{Y} \subset \bar{X}$ be an algebraic subvariety such that $X := \bar{X} \setminus \bar{Y}$ is smooth. Let $H \subset \mathbb{P}^N$ be a hyperplane and assume it is* generic *with respect to the pair $(\bar{X}, \bar{Y})$. Then, the inclusion $H \cap X \hookrightarrow X$ induces on the homotopy groups π_k isomorphisms for $0 \leq k < n-1$ and epimorphism for $k = n-1$.*

One mays suspect that this Theorem may be used as follows: take a high-dimensional variety, look for a generic plane section and compute the fundamental group in the surface case. This is not usually the case.

EXAMPLE 5.2. Let us consider in $\mathbb{P}^n$ the so-called *boolean arrangement* $\mathcal{H}$. It is the hypersurface whose irreducible components are the coordinate hyperplanes, i.e., its equation is $\prod_{j=0}^n x_j = 0$. Note that $\mathbb{P}^n \setminus \mathcal{H} \cong (\mathbb{C}^*)^n$ and hence its fundamental group is isomorphic to $\mathbb{Z}^n$. The plane sections of this arrangement coincide with the so-called *generic line arrangements* (with $n+1$ lines), i.e. arrangements of $n+1$ lines which intersect only at double points.

The rest of the section is devoted to other important example. Let us consider $(\mathbb{P}^1)^n$, i.e., the set of n-tuples of points in the Riemann sphere. There is a natural action of the symmetric group Σ_n on $(\mathbb{P}^1)^n$ by permutation of the entries.

PROPOSITION 5.3. *The quotient $(\mathbb{P}^1)^n \to (\mathbb{P}^1)^n/\Sigma_n$ is realized by the mapping $(\mathbb{P}^1)^n \to \mathbb{P}(V_n)$, where V_n is the vector space of homogeneous polynomials of degree n in two variables T, S and the map is given by*

$$([t_1 : s_1], \ldots, [t_n : s_n]) \mapsto \prod_{j=1}^n (s_j T - t_j S).$$

This map is generically $n! : 1$, but there are orbits with less elements which correspond to the big diagonal

$$\Delta_n := \bigcup_{1 \leq i < j \leq n} \{([t_1 : s_1], \ldots, [t_n : s_n]) \mid [t_i : s_i] = [t_j : s_j]\};$$

its image $D_n \subset \mathbb{P}(V_n) \equiv \mathbb{P}^n$ is the discriminant variety, i.e the variety of polynomials with multiple roots.

Note that $(\mathbb{P}^1)^n \setminus \Delta_n$ is identified with the tuples of n points in $\mathbb{P}^1$ which are pairwise distinct. Hence, $\pi_1((\mathbb{P}^1)^n \setminus \Delta_n)$ is identified with the homotopy classes of n-tuples of loops $\gamma_j : [0,1] \to \mathbb{P}^1$ such that we have $\#\{\gamma_1(t), \ldots, \gamma_n(t)\} = n$, $\forall t \in [0,1]$. This is the group of pure braids in the 2-sphere.

Using the interpretation of fundamental groups in the unramified covering $(\mathbb{P}^1)^n \setminus \Delta_n \to \mathbb{P}^n \setminus D_n$ we obtain that $\pi_1(\mathbb{P}^n \setminus D_n)$ is identified with the group of braids in the 2-sphere which admits an Artin presentation:

$$\left\langle \sigma_1, \ldots, \sigma_{n-1} \mid \underset{\text{if } i-j>1}{[\sigma_i, \sigma_j] = 1}, \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \prod_{i=1}^{n-1} \sigma_i \prod_{i=1}^{n-1} \sigma_{n-i} = 1 \right\rangle.$$

THEOREM 5.4 ([**26**]). *Let $\mathcal{C}_{2n-2}$ be a* maximal cuspidal rational curve*, i.e. a rational curve of degree $2n-2$ with $3(n-2)$ cusps (singularities with local equation $x^2-y^3=0$) and $2(n-2)(n-3)$ nodes (singularities with local equation $x^2-y^2=0$). Then, $\mathcal{C}_{2n-2}$ is a generic plane section of D_n.*

PROOF. The dual of $\mathcal{C}_{2n-2}$ is a nodal rational curve of degree n. This can be seen as follows. A generic plane section of D_n can be seen as

$$\{[\alpha:\beta:\gamma]\in\mathbb{P}^2 \mid \alpha F+\beta G+\gamma H\in D_n\}$$

for fixed generic $F,G,H\in V_n$. These points are exactly the tangent lines to the image of

$$[t:s]\mapsto[F(t,s):G(t,s):H(t,s)]$$

which is a ratonal nodal curve. Plücker formulæ do the rest for $\mathcal{C}_{2n-2}$. $\square$

EXERCISE 5.1. Compute directly $\mathcal{C}_4$ and $\mathcal{C}_6$. Determine the group $\pi_1(\mathbb{P}^2\setminus\mathcal{C}_4)$.

6. Further techniques: covers and Cremona transformations

Starting from simple examples, we are able to compute more complicated examples of fundamental groups of complement of curves.

Let $\sigma:\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a rational function. Let $\mathcal{C}_1,\mathcal{C}_2\subset\mathbb{P}^2$ curves such that $\sigma_|:\mathbb{P}^2\setminus\mathcal{C}_1\to\mathbb{P}^2\setminus\mathcal{C}_2$ is a well-defined unbranched covering. For simplicity, we will consider two examples:

- Birational maps, e.g the Cremona transformation $[x:y:z]\mapsto[yz:xz:xy]$ and $\mathcal{C}_1,\mathcal{C}_2$ contain the coordinate axes.
- Kummer covers: $[x:y:z]\overset{\rho_n}{\mapsto}[x^n:y^n:z^n]$

The use of Reidemeister-Schreier allows to compute complicated fundamental groups in simple ways.

EXAMPLE 6.1. Let us consider a curve $\mathcal{C}$ with four irreducible components: three lines in general position and a conic passing through the three triple points. Then, $\pi_1(\mathbb{P}^2\setminus\mathcal{C})$ is abelian as it is isomorphic through a Cremona transformation to the complement of an arrangement of four lines in general position, see Example 5.2.

PROPOSITION 6.2. *The restriction $\rho_{n|}:\mathbb{P}^2\setminus\{xyz=0\}\to\mathbb{P}^2\setminus\{xyz=0\}$ is a regular (Galois) cover with group $\mathbb{Z}/n\times\mathbb{Z}/n$. The meridian of the line $x=0$ (in the source) is the preimage of the n^{th}-power of a meridian of the same line in the target.*

Let us consider now a curve $\mathcal{C}$ with equation $F(x,y,z)=0$. We want to compute the fundamental group of the curve $\mathcal{C}_n$ with equation $0=F_n(x,y,z):=F(x^n,y^n,z^n)$. For the sake of simplicity, let us assume that $\mathcal{C}$ contains no axis. We will perform the following steps:

(K1) Compute $\pi_1(\mathbb{P}^n\setminus(\mathcal{C}\cup\{xyz=0\}))$ and detect meridians $\gamma_x,\gamma_y,\gamma_z$ of the axes.
(K2) Compute $G:=\pi_1(\mathbb{P}^n\setminus(\mathcal{C}\cup\{xyz=0\}))/\langle\gamma_x^n,\gamma_y^n,\gamma_z^n\rangle$.
(K3) Let $\sigma_n:G\to\mathbb{Z}/n\times\mathbb{Z}/n$ the morphism given by $\sigma_n(\gamma_x)=e_1$, $\sigma_n(\gamma_y)=e_2$, $\sigma_n(\gamma_z)=-e_1-e_2$ and $\sigma_n(\gamma)=0$ for any meridian of a component of $\mathcal{C}$.
(K4) $\pi_1(\mathbb{P}^n\setminus\mathcal{C}_n)=\ker\sigma_n$.

REMARK 6.3. If $\pi_1(\mathbb{P}^n\setminus(\mathcal{C}\cup\{xyz=0\}))$, then $\pi_1(\mathbb{P}^n\setminus\mathcal{C}_n)$ is abelian.

CHAPTER 2

Day #2ay 2

1. On previous episodes:

(1) The fundamental group $G := \pi_1(\mathbb{P}^2 \setminus \mathcal{C})$ of smooth curves is cyclic abelian of order the degree $\deg \mathcal{C}$.
(2) The group G can be generated by meridians around the irreducible components of $\mathcal{C}$.
(3) Group theory of finitely presented infinite groups is very complicated.

2. Milnor fibration and the link of a singularity

A local view.

THEOREM 2.1. *Let $p \in \mathcal{C}$ be a point on $\mathcal{C}$ and $\mathbb{B} := \mathbb{B}_\varepsilon(p)$ a small ball around p and $\mathbb{S} := \partial\mathbb{B}_\varepsilon(p)$, then $K := \mathbb{S} \cap \mathcal{C}$ is a smooth compact real 1-dimensional manifold called the link of p at $\mathcal{C}$.*

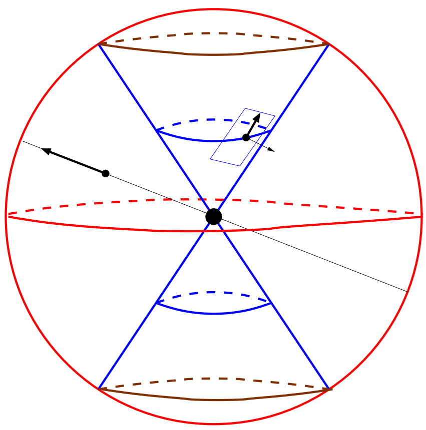
Moreover, the pair $(\mathbb{B}, \mathbb{B}\cap\mathcal{C})$ is homeomorphic to $(\operatorname{Cone}(\mathbb{S}), \operatorname{Cone}(K)) = (\mathbb{B}, \operatorname{Cone}(K))$, where $\operatorname{Cone}(\bullet)$ denotes the cone over $\bullet$.

PROOF. Consider the map $r(z_1, z_2) = r(x_1, y_1, x_2, y_2) = x_1^2 + y_1^2 + x_2^2 + y_2^2$. The set of critical points of $r|\mathcal{C}$ is given by the set of zeroes $\Sigma \subset \mathbb{C}^2$ of the 2×2 minors of

$$J := \begin{bmatrix} \partial r/\partial x_1 & \partial r/\partial y_1 & \partial r/\partial x_2 & \partial r/\partial y_2 \\ \partial f/\partial x_1 & \partial f/\partial y_1 & \partial f/\partial x_2 & \partial f/\partial y_2 \end{bmatrix}.$$

Since Σ is algebraic and r is constant on each component of Σ, the set of critical values $r(\Sigma)$ is finite. Hence, if $\varepsilon > 0$ is such that $r|_\Sigma > \varepsilon^2$, then $K := f^{-1}(0) \cap r^{-1}(\varepsilon^2)$ is a smooth compact real 1-dimensional manifold.

For the *moreover* part, consider a smooth *normalized* vector field $v(q)$ as in the picture:



and the differential equation $\frac{dq(t)}{dt} = v(q(t))$. It turns out that for any solution $q(t)$ one has $r(q(t)) = t + k$. For each $\bar{x} \in \mathbb{S}$ take $Q_{\bar{x}}(t)$ such that $r(Q_{\bar{x}}(t)) = t$ and $Q_{\bar{x}}(\varepsilon^2) = \bar{x}$.

The map $\bar{x} + (\varepsilon^2 - t)p \mapsto Q_{\bar{x}}(t)$, $t \in (0, \varepsilon^2]$ extends to a homeomorphism from $(\mathbb{B}, \mathrm{Cone}(K))$ to $(\mathbb{B}, \mathbb{B} \cap \mathcal{C})$.

□

COROLLARY 2.2. *The group* $\pi_1(\mathbb{S} \setminus K) = \pi_1(\mathbb{B} \setminus (\mathbb{B} \cap \mathcal{C}))$ *only depends on* $(\mathcal{C}, p)$.

3. Local fundamental group. Wirtinger presentation

THEOREM 3.1. *The singularity link* K *can be described as an* $m : 1$ *covering of* $\mathbb{S}^1_\varepsilon \subset \mathbb{C}$ *given by a link in the solid torus* $K_T := \{(\varepsilon e^{2\pi i t}, z_2) \mid f(\varepsilon e^{2\pi i t}, z_2) = 0\} \subset \mathbb{S}^1_\varepsilon \times \mathbb{D}_\varepsilon$, *where* $m := \mathrm{mult}_p(\mathcal{C})$.

Moreover,

$$\mathbb{S} \setminus K \cong \mathbb{S}_\infty \setminus K_T.$$

PROOF. Theorem 2.1 remains true for any metric equivalent to the euclidean metric, as long as the balls are topological manifolds (even with corners) and K avoids the corners. Hence it is true on the polydisk $\mathbb{B}_\infty = \mathbb{D}_\varepsilon \times \mathbb{D}_\varepsilon$, and $\mathbb{S}_\infty = \mathbb{D}_\varepsilon \times \mathbb{S}^1_\varepsilon \cup \mathbb{S}^1_\varepsilon \times \mathbb{D}_\varepsilon$. By the Weierstrass Preparation Theorem, one can assume that $f = z_2^m + a_{m-1}(z_1) z_2^{m-1} + \ldots + a_1(z_1) z_2 + a_0(z_1)$, after an analytic change of variables, where $a_k(0) = 0$. □

EXAMPLE 3.2. Consider $\mathcal{C} = \{F(x, y, z) = 0\} \subset \mathbb{P}^2$, where $f = zy^2 - x^3$. Note that $\mathrm{Sing}(\mathcal{C}) = \{[0 : 0 : 1]\}$. On the affine chart $\mathbb{C}^2 \cong U_z = \{[x : y : z] \mid z \neq 0\}$, $\mathcal{C}$ admits an equation $f(x, y) = y^2 - x^3$ in Weierstrass form.

The double cover referred to by Theorem 3.1 can be described as

$$K_T = \{(e^{2\pi i t}, y) \mid y^2 = e^{6\pi i t}, t \in [0, 1]\} = \{(e^{2\pi i t}, \pm e^{3\pi i t}) \mid t \in [0, 1]\} \subset \mathbb{S}^1_\varepsilon \times \mathbb{D}_\varepsilon$$

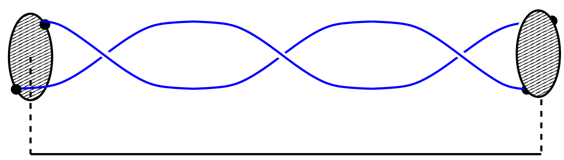
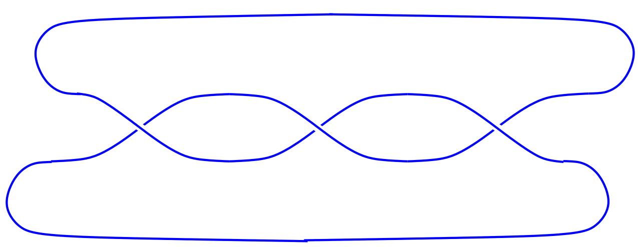


FIGURE 1. local knot K_T for $f(x, y) = y^2 - x^3$



DEFINITION 3.3. The group $\pi_1(\mathbb{S} \setminus K) = \pi_1(\mathbb{B} \setminus \mathcal{C})$ described above is called the *local fundamental group* of $(\mathcal{C}, p)$ and denoted $\pi_1^{\mathrm{loc}}(\mathcal{C}, p)$.

One can compute the local fundamental group using for instance the *Wirtinger presentation*.

EXAMPLE 3.4. In order to compute $G := \pi_1^{\mathrm{loc}}(\mathcal{C}, p)$ from Example 3.2

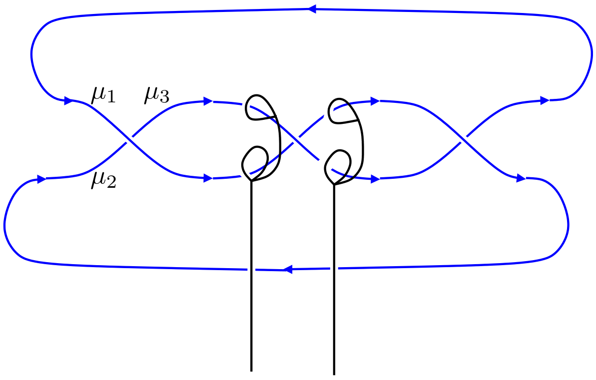


FIGURE 2. Wirtinger presentation of the $(2,3)$-knot (trefoil knot)

Hence μ_1, μ_2, μ_3 are generators of G and

$$\mu_1\mu_2 = \mu_3\mu_1 = \mu_2\mu_3$$

is a complete set of relations. Therefore $\mu_3 = \mu_1\mu_2\mu_1^{-1}$ and thus

$$G = \langle \mu_1, \mu_2 : \mu_1\mu_2\mu_1 = \mu_2\mu_1\mu_2 \rangle.$$

EXERCISE 3.1. Show that for any two-variable polynomial of order 2 can be rewritten (after a local change of coordinates in the local ring $\mathbb{C}[x,y]_{(x,y)}$) as $f(x,y) = y^2 - x^{k+1}$ for some $k > 0$. In other words, there exist $u, v \in \mathbb{C}[x,y]_{(x,y)}$ such that $\mathbb{C}[u,v]_{(u,v)} = \mathbb{C}[x,y]_{(x,y)}$ and $f = v^2 + u^{k+1}$. This local type of singularity is called $\mathbb{A}_k$-singularity.

Compute the local fundamental group G_k for $\mathbb{A}_k$-singularities. Determine whether or not $G_{k_1} \cong G_{k_2}$ for $k_1 \neq k_2$. What happens for $k = 1$? why?

4. The local geometric model, action of braids on a free group. Artin theorem

Let us go back to Example 3.2 and note that Figure 3.2 is in fact a braid. This is no coincidence, since Theorem 3.1 is another way to describe a braid on a disk, that is, as a finite topological covering of $\mathbb{S}^1$ in $\mathbb{S}^1 \times \mathbb{D}$ with respect to the projection on the first coordinate.

Recall that the braid group admits the following finite presentation

$$\mathbf{B}_m := \left\langle \sigma_1, \ldots, \sigma_{m-1} \;\middle|\; \underset{|i-j|>1}{\sigma_i\sigma_j = \sigma_j\sigma_i}, \quad \underset{1\leq k<m-1}{\sigma_k \cdot \sigma_{k+1} \cdot \sigma_k = \sigma_{k+1} \cdot \sigma_k \cdot \sigma_{k+1}} \right\rangle. \tag{3}$$

Any braid on m-strands on the disk acts (on the right) on the free group $\mathbb{F}_m$ of rank m as follows:

$$\mu_i^{\sigma_j} := \begin{cases} \mu_{i+1} & \text{if } j = i, \\ \mu_i \cdot \mu_{i-1} \cdot \mu_i^{-1} & \text{if } j = i-1, \\ \mu_i & \text{otherwise.} \end{cases} \tag{4}$$

EXERCISE 4.1. Compute $\mu_i^{\sigma_j^{-1}}$ and $(\mu_m\mu_{m-1}\ldots\mu_1)^{\sigma_i}$.

This action has a geometric interpretation

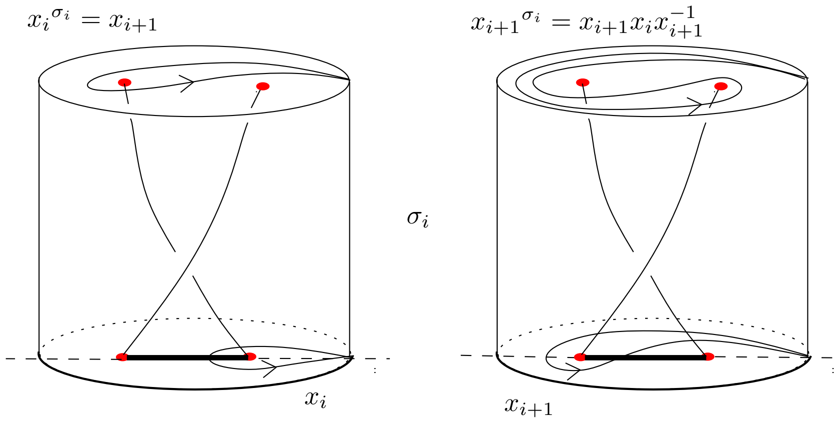


FIGURE 3. Geometric version of the action of $\mathbf{B}_m$ on $\mathbb{F}_m$.

Each braid thus produces an automorphism of the free group $\mathbb{F}_m$ satisfying:

$$\begin{aligned} \mu_i^\beta &= w(\bar\mu)x_{\tau(i)}w(\bar\mu)^{-1} \\ (\mu_m\mu_{m-1}\dots\mu_1)^\beta &= \mu_m\mu_{m-1}\dots\mu_1 \end{aligned} \tag{5}$$

for any braid $\beta \in \mathbf{B}_m$, where $w \in \mathbb{F}_m$ and τ is a permutation.

Moreover, this characterizes the braid group

THEOREM 4.1 (Artin). *The braid group $\mathbf{B}_m$ is isomorphic to the subgroup of automorphisms of the free group $\mathbb{F}_m$ satisfying* (5).

5. Artin presentation of the local fundamental group

THEOREM 5.1 (Artin presentation). *If $\beta \in \mathbf{B}_m$ represents the braid referred to in the discussion above, then*

$$\pi_1^{\text{loc}}(\mathcal{C}, p) = \langle \mu_1, \dots, \mu_m : \mu_i = \mu_i^{\sigma_j}, 1 \le i \le m, 0 \le j < m\rangle.$$

EXERCISE 5.1. Compute the Artin presentation of the link associated with the $\mathbb{A}_k$ singularity (see Exercise 3.1). Analogously, calculate an Artin presentation for the local fundamental group of $y^3 - x^5$ and the ordinary multiple point $y^m - x^m$.

6. Local vs. global fundamental groups

The inclusion $(\mathbb{B}, \mathcal{C}) \hookrightarrow (\mathbb{C}^2, \mathcal{C})$ induces a morphism $\pi_1^{\text{loc}}(\mathcal{C}, p) \to \pi_1(\mathbb{C}^2 \setminus \mathcal{C})$ which is not necessarily injective nor surjective.

However, in some cases one can be more specific. For instance,

PROPOSITION 6.1. *If $f(x, y)$ is a quasihomogeneous polynomial, then $\pi_1^{\text{loc}}(\mathcal{C}, p) \cong \pi_1(\mathbb{C}^2 \setminus \mathcal{C})$.*

PROOF. Assume f is a homogeneous polynomial of degree d. Consider the natural action of $\mathbb{C}^*$ on $\mathbb{C}^2$ given by $t \cdot (x, y) = (tx, ty)$. Note that $\mathcal{C} = \{f = 0\}$ is invariant by this action, hence there is a well-defined action on the complement $\mathbb{C}^2 \setminus \mathcal{C}$. The space $\mathbb{C}^2 \setminus \mathcal{C}$ is a deformation retract of $\mathbb{S} \setminus K$ by taking a point $z := (x, y)$ continuously to $\frac{1}{|z|} \cdot z \in \mathbb{S}$. Complete with Exercise 6.1. □

EXERCISE 6.1. Complete the proof of Proposition 6.1 assuming f is a quasi-homogeneous polynomial of degree d and weights $w := (p, q)$, that is $f(t^p x, t^q y) = t^d f(x, y)$. (**Hint:** follow the previous proof showing that The space $\mathbb{C}^2 \setminus \mathcal{C}$ is a deformation retract of $\mathbb{S}_w \setminus K_w$, where $\mathbb{S}_w^1$ is the weighted sphere, that is, $\mathbb{S}_w := \{(z_1, z_2) \mid |z_1|^q + |z_2|^p = 1\}$ and $K_w := \mathbb{S}_w \cap \mathcal{C}$. Finally, check that $\mathbb{S}_w \setminus K_w \cong \mathbb{S} \setminus K$).

EXERCISE 6.2. Compute the fundamental group of the global $\mathbb{A}_k$-singularity

$$\mathcal{C}_k := \{z(y^2 z^{k-1} - x^{k+1}) = 0\} \subset \mathbb{P}^2.$$

EXAMPLE 6.2. If $\mathcal{C} \subset \mathbb{C}^2$ is a nodal curve, and $p \in \mathcal{C}$ is a singular point, then $j_* : \pi_1^{\mathrm{loc}}(\mathcal{C}, p) \to \pi_1(\mathbb{C}^2 \setminus \mathcal{C})$ is injective if and only if $\mathcal{C}$ is a product of lines. Also j_* is surjective if and only if $\mathcal{C}$ is irreducible.

7. Global monodromy

Let us fix a curve $\bar{\mathcal{C}} \subset \mathbb{P}^2$ of degree d, a point $P_y \in \mathbb{P}^2$ and a line $\bar{L}_\infty$ such that $P_y \in \bar{L}_\infty$. We say that the curve is *horizontal* with respect to P_y if it does not contain any line through P_y; we assume $\bar{\mathcal{C}}$ to be horizontal. We consider a system of coordinates $[X : Y : Z]$ such that $P_y := [0 : 1 : 0]$ and $\bar{L}_\infty := \{Z = 0\}$. We identify $\mathbb{C}^2 \equiv \mathbb{P}^2 \setminus \bar{L}_\infty$ with affine coordinates $(x, y) \equiv [x : y : 1]$.

Let $F(x, y, z) = 0$ be a reduced equation of $\bar{\mathcal{C}}$, $k := \deg_y F$

$$F(x, y, z) = \sum_{j=0}^{k} \bar{a}_{d-j}(x, z) y^j, \quad \bar{a}_{d-k}(x, z) \neq 0, \quad \bar{a}_j \text{ homogeneous of degree } j,$$

normalized such that the coefficient of the term of higher degree of $\bar{a}_{d-k}(x, z)$ in x is 1. The fact that $\bar{\mathcal{C}}$ is horizontal is equivalent to $\gcd(F, \bar{a}_{d-k}) = 1$.

The pencil of lines through P_y is identified with $\mathbb{P}^1 \equiv \mathbb{C} \cup \{\infty\}$, where ∞ corresponds with $\bar{L}_\infty$. Following the previous notation the lines in the pencil are denoted by $\bar{L}_t := \{X - tZ = 0\}$. Let us restrict our attention to the affine part. Let $\mathcal{C} := \bar{\mathcal{C}} \cap \mathbb{C}^2$ and $L_t := \bar{L}_t \cap \mathbb{C}^2$; the line L_t has equation $x = t$ while $\mathcal{C}$ has equation $f(x, y) = 0$, where

$$f(x, y) := F(x, y, 1) = \sum_{j=0}^{k} a_{d-j}(x) y^j, \quad a_j(x) := \bar{a}_j(x, 1).$$

Let $\Delta := \{t \in \mathbb{C} \mid \#(L_t \cap \mathcal{C}) < k\}$; this is a finite set which contains the roots of $a_{d-k}(x)$ (if any) and the values t such that $L_t \not\pitchfork \mathcal{C}$. The set Δ is the zero locus of the product of $a_{d-k}(x)$ and the discriminant of $f(x, y)$ with respect to y.

Let $\Sigma_k(\mathbb{C}) := \{A \subset \mathbb{C} \mid \#A = k\}$ be a configuration space of $\mathbb{C}$; for any $A := \{x_1, \ldots, x_k\} \in \Sigma_k(\mathbb{C})$ the fundamental group $\pi_1(\Sigma_k(\mathbb{C}); A) =: \mathbf{B}(x_1, \ldots, x_k)$ is isomorphic to the braid group $\mathbf{B}_k$.

For the next sections we need to describe a canonical identification between $\mathbf{B}_k$ and $\mathbf{B}(x_1, \ldots, x_k)$; the group $\pi_1(\Sigma_k(\mathbb{C}); A)$ is identified with the homotopy classes of sets of arcs $\varphi_1, \ldots, \varphi_k : [0, 1] \to \mathbb{C}$ starting and ending in A and such that $\#\{\varphi_1(t), \ldots, \varphi_k(t)\} = k$, $\forall t \in [0, 1]$. Let us order the points of A, say $x_1, \ldots, x_k$ and consider a set I of simple segments A_i, $1 \leq i < k$, such that $\partial A_i = \{x_i, x_{i+1}\}$, $A_i \cap A_{i+1} = \{x_{i+1}\}$ and the other intersections are empty; such a collection I will be called a *diagram system* for $(x_1, \ldots, x_k)$. Then we associate to σ_i the braid which is constant for $x_1, \ldots, x_{i-1}, x_{i+2}, \ldots, x_k$ and performs a *half-twist* around A_i, that is, it exchanges the points x_i and x_{i+1} counterclockwise along $\partial N(A_i)$, where $N(A_i)$ is a (topological) disk of diameter A_i.

There is also a basis of the free group $\pi_1(\mathbb{C} \setminus A; x_0)$ if one chooses a simple edge A_0 from x_0 to x_1 intersecting $\bigcup_{i=1}^{k-1} A_i$ only at x_1. This basis $\mu_1, \ldots, \mu_k$ is obtained as follows: take small disks $\mathbb{D}_i$ centered at x_i and assume that their intersection with $A_{i-1} \cup A_i$ are diameters with ends x_i^-, x_i^+. Then μ_i is defined as follows: take a path α_i from x_0 to x_i^- running along $A_0 \cup \cdots \cup A_{i-1}$ outside the interior of the disk $\mathbb{D}_j$ and goes counterclockwise along $\partial \mathbb{D}_j$ from x_j^- to x_j^+, $1 \leq j \leq i$. Let β_i be the closed path obtained by running counterclockwise along $\partial \mathbb{D}_j$ with base point x_i^- and define $\mu_i := \alpha_i \cdot \beta_i \cdot \alpha_i^{-1}$.

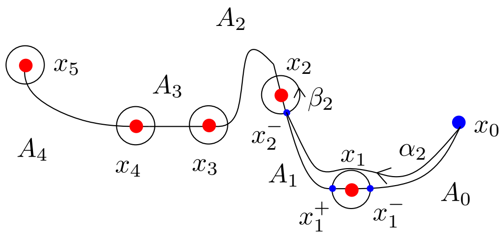


FIGURE 4. Diagram system, $k = 5$.

There are two important facts in these definitions; the element $\mu_\infty := (\mu_k \cdot \ldots \cdot \mu_1)^{-1}$ is a meridian of the point at infinity and μ_∞ is a fixed point by the action of $\mathbf{B}_k$. We say that $(\mu_1, \ldots, \mu_k)$ is an *ordered geometric basis* of $\pi_1(\mathbb{C} \setminus A; x_0)$. As a general notation, if G is a group and $\mathbf{x} := (x_1, \ldots, x_k) \in G^k$ we define the *pseudo-Coxeter element* of $\mathbf{x}$ as $c_{\mathbf{x}} := x_k \cdot \ldots \cdot x_1$.

After this digression, note that f defines a map $\tilde{f} : \mathbb{C} \setminus \Delta \to \Sigma_k(\mathbb{C})$.

DEFINITION 7.1. The *braid monodromy* of the triple $(\bar{\mathcal{C}}, P_y, \bar{L}_\infty)$ is the morphism

$$\nabla : \pi_1(\mathbb{C} \setminus \Delta; t_0) \to \mathbf{B}_k, \quad t_0 \in \mathbb{C} \setminus \Delta,$$

defined by $\tilde{f}$ on the fundamental group.

REMARK 7.2. Consider a geometric basis $(\gamma_1, \ldots, \gamma_r)$ of $\pi_1(\mathbb{C}\setminus\Delta; t_0)$ and let c_∞ be its pseudo-Coxeter element. Note that ∇ is determined by $(\nabla(\gamma_1), \ldots, \nabla(\gamma_r)) \in \mathbf{B}_k^r$ having as pseudo-Coxeter element $\nabla(c_\infty)$.

The braid monodromy measures the motions of the points of $\mathcal{C}$ along the affine lines L_t (identified with $\mathbb{C}$).

There are a lot of choices in order to obtain an element of $\mathbf{B}_k^r$ from $(\bar{\mathcal{C}}, P_y, \bar{L}_\infty)$. It is not hard to check that these choices are given by the orbits of an action of $\mathbf{B}_k \times \mathbf{B}_r$ on $\mathbf{B}_k^r$. The action of $\mathbf{B}_k$ is given by simultaneous conjugation. The action of $\mathbf{B}_r$ is defined as follows; let $h_1, \ldots, h_{r-1}$ an Artin system of generators of $\mathbf{B}_r$. Then, if $(\tau_1, \ldots, \tau_r) \in \mathbf{B}_k^r$, then:

$$(\tau_1, \ldots, \tau_r)^{h_i} := (\tau_1, \ldots, \tau_{i-1}, \tau_{i+1}, \tau_{i+1} \cdot \tau_i \cdot \tau_{i+1}^{-1}, \tau_{i+2}, \ldots, \tau_r); \tag{6}$$

h_i is called a *Hurwitz move*. In particular for a choice of $(\bar{\mathcal{C}}, P_y, \bar{L}_\infty)$ two objects are unique and well-defined: the conjugacy classes of the pseudo-Coxeter element and of the *monodromy group*, i.e., the group generated by $\nabla(\gamma_1), \ldots, \nabla(\gamma_r)$.

In light of the previous discussion, a braid monodromy ∇ of a triple $(\bar{\mathcal{C}}, P_y, \bar{L}_\infty)$ will sometimes be considered as a morphism (see Definition 7.1) or as a list of braids $(\nabla(\gamma_1), \ldots, \nabla(\gamma_r))$, where $\gamma_1, \ldots, \gamma_r$ is a geometric basis.

7.1. Generic braid monodromy.

We assume $P_y \notin \bar{\mathcal{C}}$, $\bar{L}_\infty \pitchfork \bar{\mathcal{C}}$ (i.e they intersect at d distinct points), and moreover for each $t \in \Delta$ there is exactly one point $P_t \in \bar{L}_t \cap \bar{\mathcal{C}}$ where the intersection is not transversal and satisfies

$$(\bar{\mathcal{C}} \cdot \bar{L}_t)_{P_t} = \begin{cases} 2 & \text{if } (\bar{\mathcal{C}}, P_t) \text{ is smooth,} \\ m_t & \text{if } (\bar{\mathcal{C}}, P_t) \text{ is singular,} \end{cases} \tag{7}$$

where m_t is the multiplicity of the germ $(\bar{\mathcal{C}}, P_t)$. In the singular case, it means that $\bar{L}_t$ is not in the tangent cone of $(\bar{\mathcal{C}}, P_t)$ and, in the smooth case, that P_t is not an inflection point.

DEFINITION 7.3. In this situation, ∇ is called a *generic braid monodromy*.

THEOREM 7.4 ([**6**]). *The Hurwitz class of a generic braid monodromy does not depend on P_y or L_∞. In fact, it is a topological invariant of the pair* $(\mathbb{P}^2, \bar{\mathcal{C}})$.

The proof of this result is far from the scope of this course. It is however necessary to point out that, after a continuous change of variables, the pencil of lines fails to be algebraic. This result requires a deep understanding of the topology of the embedding.

8. Duality in cohomology

Note that homology and cohomology dimensions for algebraic curves and their complements are in a way *combinatorially* determined by their degrees and local type of singularities.

PROPOSITION 8.1. *Let $\bar{\mathcal{C}}$ be a projective plane curve, then*

$$\begin{array}{l} H^2(X_{\mathcal{C}};\mathbb{C}) \cong H_1(\mathcal{C};\mathbb{C}), \text{ and} \\ H^1(X_{\mathcal{C}};\mathbb{C}) \cong H_2(\mathcal{C};\mathbb{C})/\mathbb{C}. \end{array}$$

Proof. Consider the following exact sequence of relative cohomology of pairs for the inclusion $X_{\mathcal{C}} \hookrightarrow \mathbb{P}^2$

$$\begin{array}{lcccccccl} & 0 & \stackrel{i^1}{\to} & H^1(X_{\mathcal{C}};\mathbb{C}) & \stackrel{\delta^2}{\to} & H^2(\mathbb{P}^2, X_{\mathcal{C}};\mathbb{C}) & \stackrel{j^2}{\to} & H^2(\mathbb{P}^2;\mathbb{C}) \cong \mathbb{C} & \stackrel{i^2}{\to} \\ (8) & & & & & & & & \\ & & \stackrel{i^2}{\to} & H^2(X_{\mathcal{C}};\mathbb{C}) & \stackrel{\delta^3}{\to} & H^3(\mathbb{P}^2, X_{\mathcal{C}};\mathbb{C}) & \stackrel{j^3}{\to} & H^3(\mathbb{P}^2;\mathbb{C}) \cong 0 & \end{array}$$

Let $\mathcal{TC}$ be a regular neighborhood of $\mathcal{C}$. Note that $X_{\mathcal{C}}$ has the homotopy type of $\mathbb{P}^2 \setminus \mathcal{TC}$. Hence, using excision and Lefschetz duality, one obtains

$$H^n(\mathbb{P}^2, X_{\mathcal{C}};\mathbb{C}) \stackrel{exc.}{\cong} H^n(\mathcal{TC}, \partial\mathcal{TC};\mathbb{C}) \stackrel{L}{\cong} H_{4-n}(\mathcal{TC};\mathbb{C}) \cong H_{4-n}(\mathcal{C};\mathbb{C}).$$

Thus, the sequence (8) becomes

$$0 \stackrel{i^1}{\to} H^1(X_{\mathcal{C}};\mathbb{C}) \stackrel{\delta^2}{\to} H_2(\mathcal{C};\mathbb{C}) \stackrel{j^2}{\to} \mathbb{C} \stackrel{i^2}{\to} H^2(X_{\mathcal{C}};\mathbb{C}) \stackrel{\delta^3}{\to} H_1(\mathcal{C};\mathbb{C}) \stackrel{j^3}{\to} 0.$$

Since $H_2(\mathcal{C};\mathbb{C})$ is non-trivial and j^2 is non-zero, j^2 is in fact surjective. Therefore

$$H^2(X_{\mathcal{C}};\mathbb{C}) \cong H_1(\mathcal{C};\mathbb{C}), \text{ and } H^1(X_{\mathcal{C}};\mathbb{C}) \cong H_2(\mathcal{C};\mathbb{C})/\mathbb{C}. \quad \square$$

NOTATION 8.2. Let Y be a topological space. In what follows we will denote by $h_i(Y)$ (resp. $h^i(Y)$) the dimension of the vector space $H_i(Y;\mathbb{C})$ (resp. $H^i(Y;\mathbb{C})$). Note that, by the Universal Coefficient Theorem, $h_i(Y) = h^i(Y)$.

PROPOSITION 8.3. *Let $\mathcal{C}$ be a curve, then*

$$h_2(X_{\mathcal{C}}) = (1-n) + \sum_{P \in \operatorname{Sing}\mathcal{C}} (r_P - 1) + 2\sum g(\mathcal{C}_i),$$

where r_P is the number of branches of $\mathcal{C}$ passing through P and $g(\mathcal{C}_i)$ is the topological genus of a normalization of the irreducible component $\mathcal{C}_i$.

PROOF. By Proposition (8.1), $h^2(X_{\mathcal{C}}) = h_1(\mathcal{C})$. On the other hand

$$h_2(\mathcal{C}) = n$$

and

$$\chi(\mathcal{C}) = \sum \chi(\hat{\mathcal{C}}_i) - \sum_{P \in \operatorname{Sing}\mathcal{C}} (r_P - 1) = \sum (2 - 2g(\mathcal{C}_i)) - \sum_{P \in \operatorname{Sing}\mathcal{C}} (r_P - 1).$$

Therefore

$$\begin{aligned} h_1(\mathcal{C}) &= 1 + h_2(\mathcal{C}) - \chi(\mathcal{C}) = \\ &= (n+1) - \textstyle\sum (2 - 2g(\mathcal{C}_i)) + \sum_{P \in \operatorname{Sing}\mathcal{C}} (r_P - 1) = \\ &= (1-n) + \textstyle\sum_{P \in \operatorname{Sing}\mathcal{C}} (r_P - 1) + 2\sum g(\mathcal{C}_i). \end{aligned}$$

$\square$

EXERCISE 8.1. Calculate $h_1(X_\mathcal{C})$ and $h_2(X_\mathcal{C})$ for all curves of degree at most 4.

9. Alexander invariants: Alexander polynomial, Alexander Module

For technical reasons we will assume $\mathcal{C}$ is transversal to the line at infinity and $G := \pi_1(X_\mathcal{C})$. Hence

$$G/G' = \mathbb{Z}^n,$$

where n is the number of irreducible components of the affine curve $\mathcal{C}$.

Consider $\varepsilon : G \to \mathbb{Z}$ epimorphism, and $K_\varepsilon := \ker(\varepsilon)$. Let $T_\varepsilon \in \operatorname{im}(\varepsilon)$ be a generator of the Galois group of the infinite cyclic cover $\tilde{X}_\varepsilon$.

DEFINITION 9.1. The *Alexander polynomial of $\mathcal{C}$ associated with ε* is defined as:

$$\Delta_{\mathcal{C},\varepsilon}(t) := \det((T_\varepsilon)_* - t\cdot \mathrm{Id}; H_1(\tilde{X}_\varepsilon;\mathbb{C})) = \det((T_\varepsilon)_* - t\cdot \mathrm{Id}; K_\varepsilon/K'_\varepsilon \otimes \mathbb{C})$$

A more group-theoretical interpretation of $\Delta_{\mathcal{C},\varepsilon}(t)$ can be given as follows: The group $G/K_\varepsilon = \mathbb{Z}$ acts on $K_\varepsilon/K'_\varepsilon = H_1(X_{\mathcal{C},\varepsilon};\mathbb{Z})$ by conjugation as follows

$$\begin{array}{cccc} *: & G/K_\varepsilon \times K_\varepsilon/K'_\varepsilon & \to & K_\varepsilon/K'_\varepsilon \\ & (\varepsilon(g),\bar{k}) & \mapsto & \varepsilon(g) * \bar{k} := \overline{g\cdot k\cdot g^{-1}}. \end{array}$$

Note that if $g' = gh_1$ ($h_1 \in K_\varepsilon$) and $k' = kh_2$ ($h_2 \in K'_\varepsilon$), then

$$\begin{aligned} \left(g'\cdot k'\cdot g'^{-1}\right)\left(g\cdot k^{-1}\cdot g^{-1}\right) &= \left((gh_1)\cdot(kh_2)\cdot(h_1^{-1}g^{-1})\right)\left(g\cdot k^{-1}\cdot g^{-1}\right) \\ &= g\cdot\left(h_1kh_2h_1^{-1}k^{-1}\right)\cdot g^{-1} \\ &= g\cdot\left((h_1k)\cdot h_2\cdot(h_1k)^{-1}\, h_1kh_1^{-1}k^{-1}\right)g^{-1} \in K'_\varepsilon. \end{aligned}$$

Hence "$*$" does not depend on the choice of g mod K_ε or k mod K'_ε.

This action endows $M^{\mathbb{Z}}_{\mathcal{C},\varepsilon} := H_1(X_{\mathcal{C},\varepsilon};\mathbb{Z})$ with a Λ_ε-module structure, where $\Lambda_\varepsilon := \mathbb{Z}[G/K_\varepsilon] \approx \mathbb{Z}[t^{\pm 1}]$. One can tensor such a module by a field $\mathbb{K} = \mathbb{Q},\mathbb{C},\mathbb{F}_p,\dots$ to obtain a module $M^{\mathbb{K}}_{\mathcal{C},\varepsilon}$ over $\Lambda^{\mathbb{K}}_\varepsilon = \mathbb{K}[t^{\pm 1}]$. Since G is finitely presented, $M^{\mathbb{K}}_{\mathcal{C},\varepsilon}$ is finitely generated as a $\Lambda^{\mathbb{K}}_\varepsilon$-module (by as many 1-cells as generators of G). The rings $\Lambda^{\mathbb{K}}_\varepsilon$ are principal ideal domains and hence one can define $\Delta^{\mathbb{K}}_{\mathcal{C},\varepsilon}(t)$ as the *order* of $M^{\mathbb{K}}_{\mathcal{C},\varepsilon}$. We recall that, if R is a principal ideal domain, the *order* of an R-module M, is defined as

$$\Delta := \begin{cases} 0 & \text{if } M \text{ has a free summand,} \\ 1 & \text{if } M = 0, \\ \prod_{i=1}^m \lambda_i & \text{if } M \approx \frac{R}{(\lambda_1)} \oplus \cdots \oplus \frac{R}{(\lambda_m)}. \end{cases} \tag{9}$$

Such a polynomial can be assumed to be unique by adding the extra condition $\lambda_i(0) = 1$. This is known as the *Alexander polynomial* of $\mathcal{C}$ associated with ε. In general, if $\mathbb{K} = \mathbb{Q}$ or $\mathbb{C}$, then the reference to the field will be omitted.

The *classical Alexander polynomial* (denoted $\Delta_\mathcal{C}(t)$) corresponds to the special case when $\mathbb{K} = \mathbb{Q}$, $\mathcal{C}_0$ is transversal to $\mathcal{C}_i$ for any $i = 1,\dots,r$, and ε is the epimorphism that sends any meridian γ_i around $\mathcal{C}_i$ to 1, except for $i = 0$, where $\varepsilon(\gamma_0) = -d$, where $d := \sum_{i=1}^r d_i$. We will refer to this morphism as the *trivial morphism.* If $\varepsilon(\gamma_i) \neq \pm 1$ for any $i = 0,1,\dots,r$ we will call ε a *non-coordinate* epimorphism. The *Oka polynomials* (denoted $\Delta_{\mathcal{C},\varepsilon}(t)$) correspond to $\mathbb{K} = \mathbb{Q}$, and a transversal $\mathcal{C}_0$ ([**20**]).

REMARK 9.2.

(1) Note that $M^{\mathbb{Z}}_{\mathcal{C},\varepsilon}$ is not necessarily a torsion module (see Exercise 9.1(1)).

(2) Also note that $M^{\mathbb{Z}}_{\mathcal{C},\varepsilon}$ depends only on $G = \pi_1(X_{\mathcal{C}})$ and ε. Hence one can associate an Alexander polynomial $\Delta_{G,\varepsilon}(t)$ to any finitely presented group G and epimorphism $\varepsilon : G/G' \to \mathbb{Z}$. In fact, such a polynomial corresponds to the Alexander polynomial of the CW-complex X_G associated with any finite presentation of G, and $\varepsilon : H_1(X_G;\mathbb{Z}) \to \mathbb{Z}$.

(3) Assume that

$$\left(\Lambda^{\mathbb{K}}_{\varepsilon}\right)^m \xrightarrow{A} \left(\Lambda^{\mathbb{K}}_{\varepsilon}\right)^n \to M^{\mathbb{K}}_{G,\varepsilon}$$

is a free resolution of $M^{\mathbb{K}}_{G,\varepsilon}$, where A is an $n \times m$ matrix with coefficients in $\Lambda^{\mathbb{K}}_{\varepsilon}$. Then $\Delta_{G,\varepsilon}(t)$ can also be defined as 0 if $m < n$, or as the greatest common divisor of all the minors of maximal order of A if $n \leq m$. From (2) above, n can be considered as the number of generators in a presentation of G.

A very useful remark on Alexander polynomials is the following:

Lemma 9.3. [**16**, Proposition 2.1] *Let $G \overset{\psi}{\twoheadrightarrow} H$ be an epimorphism of finitely presented groups and consider $\varepsilon_H : H/H' \twoheadrightarrow \mathbb{Z}$ another epimorphism. Then $\Delta^{\mathbb{K}}_{H,\varepsilon_H}$ divides $\Delta^{\mathbb{K}}_{G,\varepsilon_G}$, where $\varepsilon_G = \varepsilon_H \circ \psi_1$ and $\psi_1 : G/G' \to H/H'$ is induced by ψ.*

Proof. A presentation of H can be given from one of G just by adding a finite number of relations. Therefore from Remark 9.2(3), a presentation matrix for $M^{\mathbb{K}}_{H,\varepsilon_H}$ is the result of adding a finite number of columns to the presentation matrix of $M^{\mathbb{K}}_{G,\varepsilon_G}$. Therefore the ideal generated by the minors of maximal order of $M^{\mathbb{K}}_{G,\varepsilon_G}$ is contained in the one of $M^{\mathbb{K}}_{H,\varepsilon_H}$. □

This situation appears in a natural way when an equisingular family of curves $\{C_t\}_{t\in(0,\delta]}$ degenerates into a reduced curve $\mathcal{C}_0$.

Proposition 9.4. *Under the above conditions there is an epimorphism of fundamental groups*

$$\pi_1(X_{\mathcal{C}_0}) \overset{j}{\twoheadrightarrow} \pi_1(X_{\mathcal{C}_\delta}).$$

Hence $\Delta^{\mathbb{K}}_{\mathcal{C}_\delta,\varepsilon_1}$ divides $\Delta^{\mathbb{K}}_{\mathcal{C}_0,\varepsilon_2}$, where $\varepsilon_2 = \varepsilon_1 \circ j_1$ as in Lemma 9.3.

Proof. A proof of the first part can be found in [**12**, Corollary §3 (3.2)]. The second part is an immediate consequence of Lemma 9.3. □

Exercise 9.1.

(1) Consider a family of n lines $\mathcal{C}_t := \ell_{t,1} \cup \cdots \cup \ell_{t,n}$, $t \in (0,1]$ in general position degenerating into n lines $\mathcal{C}_0 := \ell_1 \cup \cdots \cup \ell_n$ passing through a common point. If ε is the trivial morphism $\varepsilon(\gamma_i) = 1$, $i = 1,\ldots,n-1$ and $\varepsilon(\gamma_n) = 1-n$. Check that

$$\begin{aligned} &M^{\mathbb{K}}_{\mathcal{C}_t,\varepsilon} = 0 && \Rightarrow \quad \Delta^{\mathbb{K}}_{\mathcal{C}_t,\varepsilon}(t) = 1 \\ &M^{\mathbb{K}}_{\mathcal{C}_0,\varepsilon} = \left(\Lambda^{\mathbb{K}}_{\varepsilon}\right)^{n-2} \oplus \left(\Lambda^{\mathbb{K}}_{\varepsilon}/(t-1)\right)^{\binom{n-2}{2}} && \Rightarrow \quad \Delta^{\mathbb{K}}_{\mathcal{C}_0,\varepsilon}(t) = \begin{cases} 0 & \text{if } n > 2 \\ 1 & \text{if } n = 2 \end{cases} \end{aligned} \tag{10}$$

(2) Consider the three-cuspidal quartic $\mathcal{C}_1$ and a generic line $\mathcal{C}_0$. The fundamental group of $\mathcal{C} := \mathcal{C}_0 \cup \mathcal{C}_1$ has a presentation

$$\langle a, b : aba = bab, \quad [a, a^2b^2] = [b, a^2b^2] = 1\rangle. \tag{11}$$

Note that there is basically only one possible morphism ε, the abelianization morphism, which we will omit in the notation. Check that

$$M^{\mathbb{K}}_{\mathcal{C}} = \frac{\mathbb{K}[t^{\pm 1}]}{(3, t^2 - t + 1)}$$

and hence

$$\Delta^{\mathbb{K}}_{\mathcal{C}}(t) = \begin{cases} (t+1)^2 & \text{if } \mathrm{Char}(\mathbb{K}) = 3 \\ 1 & \text{otherwise.} \end{cases}$$

Since the three-cuspidal quartic is dual to a nodal cubic, we know it has a bitangent, say ℓ_0. The fundamental group of $\mathcal{C}' := \ell_0 \cup \mathcal{C}_1$ has the following presentation (see [**21**, Example 4.5(3)])

$$\langle a_1, a_2, a_3, a_4 : a_i a_{i+1} a_i = a_{i+1} a_i a_{i+1} \ (i = 1, .., 3), \quad a_2 a_4 = a_1 a_2 \rangle. \tag{12}$$

Check that

$$M^{\mathbb{K}}_{\mathcal{C}'} = \frac{\mathbb{K}[t^{\pm 1}]}{(t^2 - t + 1)} \oplus \frac{\mathbb{K}[t^{\pm 1}]}{(t^2 - t + 1)}$$

and hence

$$\Delta^{\mathbb{K}}_{\mathcal{C}'}(t) = (t^2 - t + 1)^2.$$

Note that if $\mathrm{Char}(\mathbb{K}) = 3$, then $(t^2 - t + 1) = (t+1)^2$, and hence $\Delta^{\mathbb{K}}_{\mathcal{C}'}(t) = (t+1)^4$.

The geometrical interpretation of the classical Alexander polynomial is given as follows (see [**22**]). The polynomial $C_1 \cdot \ldots \cdot C_n$ defines a non-isolated singularity at the origin of $\mathbb{C}^3$. The monodromy of the Milnor fiber defines an automorphism on the H_1 and the classical Alexander polynomial is the characteristic polynomial of the monodromy of the Milnor fiber.

Theorem 9.5. [**20**, Theorem 43] *The Alexander polynomial of $\mathcal{C}$ with respect to the epimorphism $\varepsilon : H_1(X_{\mathcal{C}}) \twoheadrightarrow \mathbb{Z}$ ($\varepsilon_i \geq 0$, $i = 1, \ldots, n$) is equal to the characteristic polynomial of the monodromy $h_* : H_1(F) \to H_1(F)$ where F is the Milnor fiber of the polynomial $C_1^{\varepsilon_1} \cdot \ldots \cdot C_n^{\varepsilon_n}$.*

Since the monodromy has a finite order, this implies the following.

Corollary 9.6. *All the zeroes of the Alexander polynomial $\Delta_{\mathcal{C},\varepsilon}(t)$ of a curve $\mathcal{C}$ with respect to an epimorphism ε are roots of unity.*

Alexander polynomials depend on the local type of singularities of $\mathcal{C}$. To describe this dependency we will consider $L_1, \ldots, L_s$ the local links of the singularities of the affine part $\mathcal{C} := \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_n$ and L_∞ the link at infinity, that is, the intersection of $\mathcal{C}$ with the boundary of a tubular neighborhood of the line at infinity $\mathcal{C}_0$. The inclusion $\mathbb{S}^3 \setminus L_k \hookrightarrow X_{\mathcal{C}}$ induces a map $\pi_1(\mathbb{S}^3 \setminus L_k) \to \pi_1(X_{\mathcal{C}})$. Therefore ε also induces epimorphisms $\pi_1(\mathbb{S}^3 \setminus L_k) \twoheadrightarrow \mathbb{Z}$. The Alexander polynomials associated with such maps will be called local Alexander polynomials and denoted by $\Delta_{L_k,\varepsilon}$ for simplicity.

This dependency can be described for classical Alexander polynomials.

Theorem 9.7 ([**16**]). *The Alexander polynomial $\Delta_{\mathcal{C},\varepsilon}$ of $\mathcal{C}$ divides both the product of the local Alexander polynomials $\prod_{k=1}^{s} \Delta_{L_k,\varepsilon}(t)$ and $\Delta_{L_\infty,\varepsilon}(t)$.*

10. Quasi-projectivity, Zariski pairs

10.1. Quasi-projectivity.

Definition 10.1. A group is called *(quasi)-projective* if it is the fundamental group of a (quasi)-projective variety.

J.-P. Serre [**24**] posed the following question:

Problem 10.1. Classify (quasi)-projective groups.

As is well-known, every finitely presented group G is the fundamental group of a smooth, compact, connected 4-dimensional manifold.

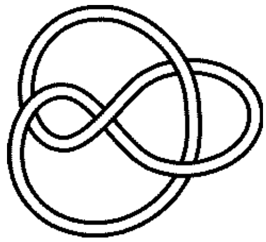


FIGURE 5. Knot 4_1

THEOREM 10.2 (Fox-Neuwirth 1962). *Both* $\mathbf{P}_n$ *and* $\mathbf{B}_n$ *are quasi-projective groups, but not projective.*

For instance $\mathbf{B}_n$ is not projective since $b_1(\mathbf{B}_n) = 1$ odd.

THEOREM 10.3 (Brieskorn 1971). *If* W_Γ*, is finite, then* G_Γ*, is quasi-projective.*

Other than that there are only necessary conditions for a group to be (quasi)-projective.

One of them refers to their Alexander polynomial having only roots of unity as their zeroes.

EXERCISE 10.1. Prove that the fundamental group of the complement of the knot 4_1 shown in Figure 10.1 is not quasi-projective.

10.2. Zariski pairs.

Theorem 9.7 reveals that the fundamental group G of a plane curve $\mathcal{C}$ depends on its singularities. However, these don't determine G.

EXAMPLE 10.4. Consider the space of sextics

$$\Sigma_6 := \left\{ \sum_{i+j+k=6} a_{ijk} X^i Y^j Z^k = 0 \mid a_{ijk} \in \mathbb{C} \right\} \rightleftarrows [a_{ijk}] \subset \mathbb{P}^N,$$

$N = \binom{8}{2} - 1 = 27$ and inside Σ_6 consider the Zariski closure of the subspace of sextics with six cusps $\Sigma_{6,0,6} \subset \Sigma_6$ which is a space of dimension 15.

Kummer cover of a conic tangent to two lines. Its fundamental group is:

$$G' := \langle e, \ell_1, \ell_2 : [e\ell_1 e^{-1}, \ell_2] = 1, (\ell_1 e)^2 = (e\ell_1)^2, (\ell_2 e)^2 = (e\ell_2)^2, [\ell_2\ell_1, e] = 1 \rangle$$

Using Reidemeister-Schreier for the surjection

$$\begin{array}{ccc} G' & \to & \mathbb{Z}_3^2 \\ e & \mapsto & (0,0) \\ \ell_1 & \mapsto & (1,0) \\ \ell_2 & \mapsto & (0,1) \end{array}$$

one obtains the following presentation:

$$G'' = \langle e_{i,j}, \ell_{1,i,j}, \ell_{2,i} : \begin{array}{l} e_{i,j+2} = e_{i,j+1}^{-1} e_{i,j} e_{i,j+1}, \\ e_{i+2,j} = e_{i+1,j}^{-1} e_{i,j} e_{i+1,j}, \\ \ell_{1,i,j} = e_{i,j}^{-1} e_{i,0} e_{i+1,0} e_{i+1,j} \\ e_{i+1,j+1} = e_{i,j} \end{array} \quad i,j \in \mathbb{Z}_3 \rangle,$$

where $e_{i,j} := (\ell_1^i \ell_2^j) e (\ell_1^i \ell_2^j)^{-1}$, $\ell_{1,i,j} = (\ell_1^i \ell_2^j) \ell_1 (\ell_1^{i+1} \ell_2^j)^{-1}$, and $\ell_{2,i} = \ell_1^i \ell_2^2 \ell_1^{-i}$.

After elliminating the meridians around the preimages of the lines, that is, $\ell_{1,i,j}^2 = \ell_{1,2,0} = 1$, $\ell_{2,i} = 1$, and $(\ell_1 e^2 \ell_2)^3 = 1$ one obtains

$$G = \langle e_{0,0}, e_{1,0} : e_{0,0} e_{1,0} e_{0,0} = e_{1,0} e_{0,0} e_{1,0}, (e_{0,0} e_{1,0})^3 = 1 \rangle.$$

Substituting $y = e_{0,0}e_{1,0}$ and $x = e_{1,0}e_{0,0}e_{1,0}$ one obtains

$$G = \langle x, y : x^2 = y^3 = 1\rangle \cong \mathbb{Z}_2 * \mathbb{Z}_3.$$

There is another posible construction of a sextic in $\Sigma_{6,0,6}$ as follows. Take a smooth cubic and three tangent lines through inflexion points. One has to check that tangencies are lines in general position, for instance: Consider the (smooth) Fermat cubic $u^3 + v^3 + w^3$. Let ξ be a primitive sixth root of unity. The inflexion points and tangencies of this cubic are shown in Table 10.4. Namely, the points $p_{i,j}$ denote the 9 inflexion points of the Fermat cubic. The lines $t_{i,j}$ are tangent to such a cubic at $p_{i,j}$. Finally, the rows are arranged so that the lines $t_{i,1}, t_{i,2}, t_{i,3}$ are concurrent for each $i = 1, 2, 3$. The first column describes the intersection of such concurrent tangent lines.

TABLE 1.

$\cap t_{1,j} = \{[0:1:0]\}$	$p_{1,1} := [1:0:-1]$	$p_{1,2} := [\xi:0:-1]$	$p_{1,3} := [\xi^2:0:-1]$
	$t_{1,1} := \{u+w\}$	$t_{1,2} := \{u+\xi w\}$	$t_{1,3} := \{u+\xi^2 w\}$
$\cap t_{2,j} = \{[0:0:1]\}$	$p_{2,1} := [1:-1:0]$	$p_{2,2} := [\xi:-1:0]$	$p_{2,3} := [\xi^2:-1:0]$
	$t_{2,1} := \{u+v\}$	$t_{2,2} := \{u+\xi v\}$	$t_{2,3} := \{u+\xi^2 v\}$
$\cap t_{3,j} = \{[1:0:0]\}$	$p_{3,1} := [0:1:-1]$	$p_{3,2} := [0:\xi:-1]$	$p_{3,3} := [0:\xi^2:-1]$
	$t_{3,1} := \{v+w\}$	$t_{3,2} := \{v+\xi w\}$	$t_{3,3} := \{v+\xi^2 w\}$

Consider the Kummer cover κ_2 of order 2 ramified along $t_{1,1}$, $t_{1,2}$, and $t_{2,1}$, that is, $[u:v:w] \mapsto [t_{1,1}^2 : t_{1,2}^2 : t_{2,1}^2]$. The preimage of $\mathcal{C}_3$ under κ_2 is a sextic with six cusps which are the preimages of the inflexion points $P_{1,1}$, $P_{1,2}$, and $P_{2,1}$. Since $t_{1,1}$, $t_{1,2}$, and $t_{1,3}$ are concurrent lines at a point $[0:1:0]$ which is totally ramified (i.e. it has only one preimage), the preimage of $t_{1,3}$ decomposes in a product of two lines, say ℓ_1 and ℓ_2. Also note that ℓ_1 and ℓ_2 are bitangent lines through the inflexion points in the preimage of $P_{1,3}$.

One can use Reidemeister-Schreier again to check that the fundamental group of this sextic is abelian.

DEFINITION 10.5. Two curves $\mathcal{C}_1, \mathcal{C}_2 \subset \mathbb{P}^2$ with the same combinatorics (same number of irreducible components, sames degrees, same singularities) but non-homeomorphic embeddings $((\mathbb{P}^2, \mathcal{C}_1) \not\cong (\mathbb{P}^2, \mathcal{C}_2))$ form a *Zariski pair*.

Since $\pi_1(\mathbb{P}^2 \setminus \mathcal{C})$ is an invariant of the pair $(\mathbb{P}^2, \mathcal{C})$, the two curves from Example 10.4 form a Zariski pair.

THEOREM 10.6. *If $\mathcal{C}_1$ and $\mathcal{C}_2$ can be deformed into one another in the same equisingular stratum, then $(\mathbb{P}^2, \mathcal{C}_1) \not\cong (\mathbb{P}^2, \mathcal{C}_2)$.*

COROLLARY 10.7. *The space of sextics with six cusps $\Sigma_{6,0,6}$ is not connected.*

The isomorphism problem for finitely generated groups is in general intractable. This is why the search for other invariants such as Alexander polynomials is so important.

EXAMPLE 10.8. Consider the space $\mathcal{M}$ of sextics with the following combinatorics:

(1) $\mathcal{C}$ is a union of a smooth conic $\mathcal{C}_2$ and a quartic $\mathcal{C}_4$.
(2) $\mathrm{Sing}(\mathcal{C}_4) = \{P, Q\}$ where Q is a cusp of type $\mathbb{A}_4$ and P is a node of type $\mathbb{A}_1$.
(3) $\mathcal{C}_2 \cap \mathcal{C}_4 = \{Q, R\}$ where Q is a $\mathbb{D}_7$ on $\mathcal{C}$ and R is a $\mathbb{A}_{11}$ on $\mathcal{C}$.

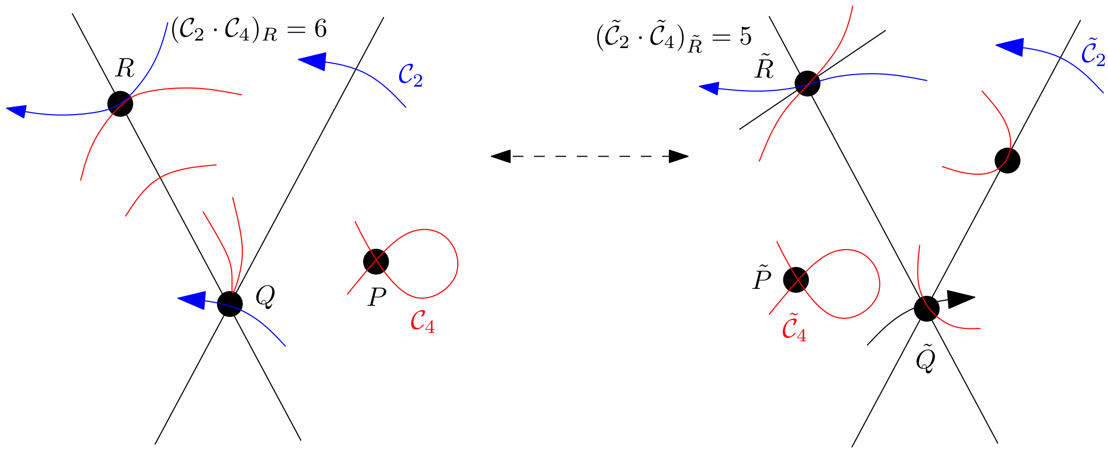


FIGURE 6. Cremona transformation.

Performing a degenerated Cremona transformation based on $2Q$ and R, the problem is equivalent to finding a nodal cubic $\tilde{\mathcal{C}}_4$ and a smooth conic $\tilde{\mathcal{C}}_2$ intersecting in two singular points of types $\mathbb{A}_9$ and $\mathbb{A}_1$.

Assuming $\tilde{\mathcal{C}}_4$ has equation $xyz + x^3 - y^3$, one can consider the following parametrization

$$\begin{array}{rccc} \varphi: & \mathbb{C} & \to & \tilde{\mathcal{C}}_4 \\ & t & \mapsto & [t : t^2 : t^3 - 1]. \end{array}$$

Note that $\varphi|_{\mathbb{C}^*} : \mathbb{C}^* \to \operatorname{Reg}(\tilde{\mathcal{C}}_4)$ is a group isomorphism from the multiplicative group $\mathbb{C}^*$ to the set of regular points on the cubic $\tilde{\mathcal{C}}_4$ whose geometric group structure has the inflexion point $\varphi(1) = [1 : 1 : 0]$ as unity.

Let t_1, t_2 and t_3 denote the parameters corresponding to $\tilde{R}$, $\mathbb{A}_1$ and $\tilde{Q}$ respectively. One has the following relations given by E_1, $\tilde{\mathcal{C}}_2$ and E_2:

$$\begin{aligned} t_1 t_2^2 &= 1 \\ t_1^5 t_2 &= 1 \\ t_2 t_3^2 &= 1. \end{aligned}$$

This implies that t_1 is a ninth root of 1, say α, and $t_2 = \alpha^4$. Therefore α must also be a primitive ninth root of unity. That leaves us with two possibilities for t_3, namely, $t_3 = \pm\alpha^7$. The solution $-\alpha^7$ (resp. $+\alpha^7$) corresponds to the case where the tangent line to $\tilde{\mathcal{C}}_4$ at $\tilde{R}$ passes (resp. doesn't pass) through $\tilde{Q}$. One can obtain equations for two sextics $\mathcal{C}_6^{(1)} = \mathcal{C}_4^{(1)} \cup \mathcal{C}_2^{(1)}$ and $\mathcal{C}_6^{(2)} = \mathcal{C}_4^{(2)} \cup \mathcal{C}_2^{(2)}$ satisfying the properties stated above and an extra property: there exists a conic $\tilde{\ell}$ –the inverse image of ℓ– passing through R and Q such that $\operatorname{mult}_R(\tilde{\ell}, \mathcal{C}_2^{(i)}) = \operatorname{mult}_R(\tilde{\ell}, \mathcal{C}_4^{(i)}) = 3$, $\operatorname{mult}_Q(\tilde{\ell}, \mathcal{C}_2^{(i)}) = 1$ and $\operatorname{mult}_Q(\tilde{\ell}, \mathcal{C}_4^{(i)}) = 3 + i$.

Note that, by construction, $\mathcal{C}_6^{(1)}$ and $\mathcal{C}_6^{(2)}$ belong to different components of $\mathcal{M}$. Moreover, if we consider the action of $PGL(3,\mathbb{C})$ on $\mathcal{M}$, then $\mathcal{M}/PGL(3,\mathbb{C})$ consists of exactly two points having representatives $\mathcal{C}_6^{(1)}$ and $\mathcal{C}_6^{(2)}$.

Special affine equations for these curves are shown below. The affine coordinates are (y,z); the line at infinity is tangent to the type $\mathbb{D}_7$ point, which is the base point of the pencil of vertical lines $y =$ constant:

$$f_1(y,z) := \left((y+3)\,z + \frac{3y^2}{2}\right)\left(z^2 - \left(y^2 + \frac{15}{2}\,y + \frac{9}{2}\right)z - 3\,y^3 - \frac{9y^2}{4} + \frac{y^4}{4}\right)$$

for $\mathcal{C}_6^{(1)}$ and

$$f_2(y,z) := \left(\left(y+\frac{1}{3}\right)z-\frac{y^2}{6}\right)\left(z^2-\left(y^2+\frac{9y}{2}+\frac{3}{2}\right)z+\frac{y^4}{4}+\frac{3y^2}{4}\right)$$

for $\mathcal{C}_6^{(2)}$.

In the future we will refer to $\mathcal{C}^{(i)}$ $i=1,2$ as the union of the sextic curve $\mathcal{C}_6^{(i)}$ and a transversal line $\mathcal{C}_0$, where $\mathbb{C}^2 = \mathbb{P}^2 \setminus \mathcal{C}_0$ and $X^{(i)} = \mathbb{P}^2 \setminus \mathcal{C}^{(i)}$.

PROPOSITION 10.9. *The fundamental groups $G^{(i)} := \pi_1(X^{(i)})$ have the following presentations*

$$G^{(1)} = \langle e_1, e_2 : [e_2, e_1^2] = 1, (e_1e_2)^2 = (e_2e_1)^2, [e_1, e_2^2] = 1\rangle$$
$$G^{(2)} = \langle e_1, e_2 : [e_2, e_1^2] = 1, (e_1e_2)^2 = (e_2e_1)^2\rangle.$$

EXERCISE 10.2. Check that $G^{(1)}$ and $G^{(2)}$ are not isomorphic by calculating their Alexander invariant.

$$M_{G^{(1)}} = 0, \qquad M_{G^{(2)}} = \mathbb{Z}.$$

Also, check that their Alexander polynomials are both trivial.

11. Orbifold techniques

The understanding of projective (or Kähler) fundamental groups can also be approached by the study of possible morphisms of the manifold onto curves of high genus. This is the de Franchis-Beauville-Catanese-Castelnuovo method ([**9, 5, 7, 1**]). This is a typical result of this type.

THEOREM 11.1 (Catanese [**7**]). *Let X be a compact Kähler manifold, and assume that its fundamental group admits a non-trivial homomorphism ψ to the fundamental group Π_g of a compact Riemann surface of genus $g \geq 2$, with kernel K. Then the following conditions are equivalent:*

(1) ψ is induced by an irrational pencil of genus g without multiple fibres.
(2) ψ is surjective and its kernel K is finitely generated.

For plane algebraic curves (or quasi-projective groups), a similar type of results can be given when considering morphisms onto curves of lower genus, but with an additional structure named orbicurves.

DEFINITION 11.2. An *orbicurve* is a complex orbifold of dimension equal to one, i.e. a smooth complex curve with a finite collection R of points (called *the orbifold points*) with a multiplicity assigned to each point in R. The complement to R is called *the regular part* of the orbifold. An orbicurve $\mathcal{C}$ is called a *global quotient* if there exists a finite group G and a manifold C such that $\mathcal{C}$ is the quotient of C by G with standard orbifold structure.

DEFINITION 11.3. Let $\mathcal{C}$ be a global orbifold quotient and ρ a character. Let R be the set of orbifold points and $C \setminus R \to \mathcal{C} \setminus \mathcal{R}$ be the quotient map with the covering group G. The integer

$$d(\rho) = \dim\{v \in H^1(C \setminus R, \mathbb{C}) \mid g \cdot v = \rho(g)v, \quad g \in G\} \tag{13}$$

is called the *depth* of a character ρ of the orbicurve $\mathcal{C}$.

If ρ is of finite order, then $d(\rho)$ has a nice topological interpretation.

THEOREM 11.4 (Sakuma Formula). *Let Y be a finite covering of X associated with the group of covering transformations G, then*

$$b_1(Y) = b_1(X) + \sum_{\rho \in \mathrm{Hom}(G,\mathbb{C}^*)} d(\rho).$$

EXAMPLE 11.5. Let $\mathbb{C}_{n,n}$ be the orbifold supported on $\mathbb{C}$ with two orbifold points of multiplicity n. We shall identify $\mathbb{C}$ with $\mathbb{P}^1 \setminus \{[1:1]\}$ so that the orbifold points correspond to $[0:1], [1:0]$. This is the global quotient of a smooth curve C by the cyclic group $\mathbb{Z}/n$ where C is the complement in $\mathbb{P}^1$ to the set $S := \{[\xi_n^i : 1] \mid i = 0, 1 \ldots, n-1\}$ of n points (here ξ_n is a primitive root of unity of degree n) and the global quotient map is the restriction on the complement to S of the map $\mathbb{P}^1 \to \mathbb{P}^1$ given by $z \mapsto z^n$. We have $\pi_1^{\text{orb}}(\mathbb{C}_{n,n}) = \mathbb{Z}/n * \mathbb{Z}/n$. Note that $\text{Hom}(\mathbb{Z}/n * \mathbb{Z}/n, \mathbb{C}^*) = \mu_n \times \mu_n$, where μ_n is the multiplicative cyclic group of order n. Consider a character $\rho = (\zeta, \zeta^{-1}) \in \mu_n \times \mu_n$ where ζ is a primitive root of unity. It follows that if $\pi_1(\mathbb{P}^1 \setminus \{[1:0], [1:1], [0:1]\}) \to \pi_1^{\text{orb}}(\mathbb{C}_{n,n})$ is the canonical surjection (in the above identification of $\mathbb{C}$ and $\mathbb{P}^1$ so that the point at infinity corresponds to $[1:1]$), then the pullback of ρ takes values $\zeta, 1, \zeta^{-1}$ on generators corresponding to $[1:0], [1:1], [0:1]$. In particular the covering space corresponding to such ρ is $\mathbb{P}^1 \setminus S$ and the dimension of the ρ-eigenspace is equal to one.

DEFINITION 11.6. Let $\mathcal{X}$ be a quasi-projective manifold and $\mathcal{C}$ be an orbicurve. A holomorphic map ϕ between $\mathcal{X}$ and the underlying $\mathcal{C}$ complex curve is called an *orbifold pencil* if the index of each orbifold point p divides the multiplicity of each connected component of the fiber $\phi^*(p)$ over p.

REMARK 11.7. Note that this definition implies that if Γ_i is the boundary of a small disk normal to $\phi^{-1}(p_i)$ at its smooth point then $\phi(\Gamma_i)$ belongs to the subgroup of $\pi_1(\mathcal{C} \setminus p_i)$ generated by $\gamma_i^{m(p_i)}$. In particular an orbifold pencil induces the map $\pi_1(\mathcal{X}) \to \pi_1^{\text{orb}}(\mathcal{C})$.

One has the following result regarding orbifold morphisms.

PROPOSITION 11.8 ([**3**, Proposition 1.5]). *Let $\rho : X \to S$ define an* orbifold morphism $X \to S_{\bar{m}}$. *Then φ induces a morphism $\varphi_* : \pi_1(X) \to \pi_1^{\text{orb}}(S_{\bar{m}})$. Moreover, if the generic fiber is connected, then φ_* is surjective.*

Using this technique one can show the following result needed in the sequel and which we shall prove for completeness (cf. [**13**, Ch 2, Theorem 2.3]).

PROPOSITION 11.9. *The number of multiple members in a primitive pencil of plane curves (with no base components) is at most two.*

DEFINITION 11.10. A *quasi-toric relation of type* (p, q, r) is a sextuple $\mathcal{R}_{\text{qt}}^{(p,q,r)} := (F_1, F_2, F_3, h_1, h_2, h_3)$ of non-zero homogeneous polynomials in $\mathbb{C}[x, y, z]$ satisfying the following functional relation

$$h_1^p F_1 + h_2^q F_2 + h_3^r F_3 = 0. \tag{14}$$

The *support* of a quasi-toric relation $\mathcal{R}_{\text{qt}}^{(p,q,r)}$ as above is the zero set $\mathcal{C} := \{F_1 F_2 F_3 = 0\}$. In this context, we may also refer to $\mathcal{C}$ as a curve that *satisfies (or supports) a quasi-toric relation* of type (p, q, r).

THEOREM 11.11. *For any irreducible plane curve $\mathcal{C} = \{F = 0\}$ whose only singularities are nodes and cusps the following statements are equivalent:*

(1) $X = \mathbb{P}^2 \setminus \mathcal{C}$ admits a holomorphic map onto the orbicurve $\mathbb{P}^1_{(2,3,6)}$.
(2) F admits a quasi-toric relation.
(3) $\Delta_{\mathcal{C}}(t)$ is not trivial ($\Delta_{\mathcal{C}}(t) \neq 1$).

Moreover, the set of quasi-toric relations of $\mathcal{C}$ $\{(f, g, h) \in \mathbb{C}[x, y, z]^3 \mid f^2 + g^3 + h^6 F = 0\}$ has a group structure and it is isomorphic to $\mathbb{Z}^{2q}$, where $\Delta_{\mathcal{C}}(t) = (t^2 - t + 1)^q$. Also, $\mathcal{C}$ admits a finite number of primitive quasi-toric relations iff $q = 1$.

EXERCISE 11.1. Apply Theorem 11.11 to the Zariski sextic.

12. Applications: MacLane and Rybnikov's example

CHAPTER 3

Day #3ay 3

1. Outline

- the braid monodromy revisited
- the Zariski van Kampen theorem
- fundamental groups of discriminant complements

2. Braid group invariant

DEFINITION 2.1 (Definition of Br M, the braid monodromy group). Given a space X of simple monic polynomials p_x of degree k, then

$$\operatorname{Br} M_X \quad = \quad \nabla\pi_1(X),$$

where the map to the configuration space of $\mathbb{C}$,

$$x \mapsto p_x^{-1}(0) \in \Sigma_k(\mathbb{C})$$

induces the braid monodromy on fundamental groups

$$\nabla : \pi_1 X \to \mathbb{B}_k = \pi_1 \Sigma_k(\mathbb{C})$$

REMARKS 2.1.

- simple polynomials are those which have number of zeroes equal to the degree (without multiplicities)
- local problem, i.e. X punctured disc, solvable by Newton-Puiseux.
- global problem: how to fit local solutions together

3. Affine divisor complements

3.1. A typical situation.

- V affine space of polynomials,
- $X \subset V$ Zariski-open subset of simple polynomials

3.2. Plane curve situation.

If $p(x,y) = F(x,y,1)$ is the dehomogenization of a plane curve equation, then with $V = \mathbb{C}$ the x-axis, $X = \mathbb{C} \setminus \Delta$, we get back our study case.

For fundamental groups we get nothing new:

THEOREM 3.1 (Zariski-Lefschetz Theorem). *Given a generic line L in V, there is a natural embedding*

$$L \times \mathbb{C} \hookrightarrow V \times \mathbb{C}$$

which induces an isomorphism of fundamental groups

$$\pi_1(L \times \mathbb{C} \setminus p^{-1}(0)) \quad \cong \quad \pi_1(V \times \mathbb{C} \setminus p^{-1}(0))$$

But the transition from generic to non-generic lines can be studied and exploited for computation.

REMARK 3.2. We start now with the argument leading to a presentation of the knot group of a curve (the fundamental group of its complement).

PROPOSITION 3.3. *If $D = D_1 \cup D'$ is a divisor in a complex manifold M with D_1 irreducible, then*

$$\pi_1(M \setminus D) \quad \to \quad \pi_1(M \setminus D')$$

induced by the natural embedding is surjective and the kernel is normally generated by any meridian around D_1.

PROOF. Surjectivity follows from the general position argument, that every path can be homotoped off a codimension two subset.

Each meridian represents a trivial element in $\pi_1(M \setminus D')$, since it comes with a disc transversal to D_1 and disjoint from D'. This disc serves to get a homotopy to the trivial path.

An element in $\pi_1(M \setminus D)$ can be represented by the image in $M \setminus D$ of the boundary of a disc. If the element belongs to the kernel the map extends to the disc as a map to $M \setminus D'$ and can be chosen transversal to D_1 by a general position argument.

Take a geometric basis (of meridians) in the disc with respect to the finitely many preimages of D_1. Then they map to meridians of D_1 and their ordered product is homotopic to the path we started with.

The claim now follows since all meridians are freely homotopic. □

PROPOSITION 3.4. *Given a locally trivial fibration $\pi : M \to X$ with connected fibre M_x and with a section, ie. a map $s : X \to M$ with $\pi \circ s = id_X$, then there is a homotopy exact sequence*

$$\pi_2(X, x) \to \pi_1(M_x, m_0) \to \pi_1(M, m_0) \to \pi_1(X, x) \to 1$$

and s induces a splitting of the surjection.

REMARK 3.5. The Hopf fibration

$$S^3 \to S^2$$

has no section. Neither has a connected topological cover of degree $m > 1$.

LEMMA 3.6. *There is a continuous section to the fibration*

$$s : X \to X \times \mathbb{C} \setminus p^{-1}(0)$$

which extends continuously to

$$s : V \to V \times \mathbb{C} \setminus p^{-1}(0)$$

if the leading coefficient of p is non-vanishing on V.

PROOF. Immediate from the fact that the modulo of all zeroes is bounded in terms of continuous function depending on the coefficients and the reciprocal of the leading coefficient. □

LEMMA 3.7. *If the leading coefficient does not vanish, the lift via s of a meridian of a divisor $S \subset V \setminus X$ is a meridian of the preimage $S \times \mathbb{C}$.*

REMARK 3.8. If the leading coefficient vanishes along a component S_1 of S, then the lift via s of a meridian of S_1 is not a meridian of the preimage $S_1 \times \mathbb{C}$.

THEOREM 3.9 (Zariski-van Kampen Theorem [**15**]). *Let $\mathcal{D} = p^{-1}(0)$. Suppose the leading coefficient of p is constant. Let the braid group $\mathrm{Br}\, M_X$ be generated by $\{\beta_1, ..., \beta_r\} \subset \mathbb{B}_k$, then $\pi_1(V \times \mathbb{C} \setminus \mathcal{D})$ is finitely presented as*

$$\langle t_1, ..., t_k \,|\, t_i^{-1} t_i^{\beta_j}, i \le k, j \le r \rangle.$$

We recall the Hurwitz action:

$$t_i^{\sigma_j} = \begin{cases} t_{i+1} & \text{if } j = i, \\ t_i t_{i-1} t_i^{-1} & \text{if } j = i-1, \\ t_i & \text{else.} \end{cases}$$

Proof. First we employ the locally trivial fibration on $X \times \mathbb{C} \setminus \mathcal{D}$.

Since the fibre fundamental group is free of rank k it is immediate that the boundary map $\pi_2 \to \pi_1$ in the homotopy sequence is trivial.

The section provides a semi-direct product structure:

$$\pi_1(X \times \mathbb{C} \setminus \mathcal{D}) \cong \pi_1(\{x_0\} \times \mathbb{C} \setminus p_x^{-1}(0)) \ltimes \pi_1(X)$$

where the two groups are subgroups by the embedding resp the section and the second act on the former by braid automorphisms.

This gives a presentation relying on $\pi_1(X) = \langle a_j | \mathcal{R} \rangle$:

$$\pi_1(X \times \mathbb{C} \setminus \mathcal{D}) \cong \langle t_1, ..., t_k, a_1, \dots, a_{r'} |\, a_j^{-1} t_i^{-1} a_j t_i^{\nabla a_j}, i \leq k, j \leq r', \mathcal{R} \rangle.$$

The map $\pi_1(V \times \mathbb{C} - \mathcal{D}) \to \pi_1(X \times \mathbb{C} \setminus \mathcal{D})$ is surjective with kernel generated by a_j. Thus

$$\pi_1(V \times \mathbb{C} \setminus \mathcal{D}) \cong \langle t_1, ..., t_k |\, t_i^{-1} t_i^{\nabla a_j}, i \leq k, j \leq r' \rangle.$$

and the claim follows by replacing the generators of $\operatorname{Br} M_X$. □

Corollary 3.10. *In the projective plane curve case with* $(0:1:0) \notin F^{-1}(0)$

$$\pi_1(\mathbb{C} \times \mathbb{C} - F^{-1}(0)) \cong \langle t_1, ..., t_k |\, t_k \dots t_1,\ t_i^{-1} t_i^{\beta_j}, i \leq k, j \leq r, \rangle.$$

Proof. The map $\pi_1(\mathbb{C} \times \mathbb{C} - F^{-1}(0)) \to \pi_1(\mathbb{P}^2 - F^{-1}(0))$ is surjective with kernel generated by the meridian $t_k \dots t_1$ of the line at infinity. □

Remark 3.11. A careful choice of generators β_j can reduce the number of relations.

In case all braid monodromies ∇a_j are conjugate of σ_1^2 or σ_1^3 (or other powers) – e.g. in case of a curve with cusps and nodes only – the number of relation reduces substantially due to the following:

Lemma 3.12. *Suppose* $\sigma = \beta^{-1} \sigma_1^2 \beta \in \mathbf{B}_k$ *(resp.* $\sigma = \beta^{-1} \sigma_1^3 \beta \in \mathbf{B}_k$*), then the normal subgroup generated by* $t_i^{-1} t_i \sigma$*,* $i = 1, ..., n$*, is equal to the normal subgroup generated by*

$$(t_1 t_2 t_1^{-1} t_2^{-1})^\beta \quad \big(\ \textit{resp.}\ (t_1 t_2 t_1 t_2^{-1} t_1^{-1} t_2^{-1})^\beta \big).$$

Proof. Replace the generators t_i by generators t_i^β, the normal subgroup is generated by

$$\big(t_i^\beta\big)^{-1} t_i^{\beta\sigma} = (t_i^{-1})^\beta t_i^{\sigma_1^2 \beta} = \big(t_i^{-1} t_i^{\sigma_1^2}\big)^\beta \quad \left(\text{resp. } \big(t_i^{-1} t_i^{\sigma_1^3}\big)^\beta\right).$$

This yields trivial relations for $i > 2$. For $i = 1, 2$ they are explicitly computed using the Artin action of σ_1 on free generators:

$$(t_1^{-1} t_2 t_1 t_2^{-1})^\beta, (t_2^{-1} t_2 t_1 t_2 t_1^{-1} t_2^{-1})\beta \quad \big(\text{resp.} (t_1^{-1} t_2^{t_1^{-1} t_2^{-1}})^\beta, (t_2^{-1} t_1^{t_2^{-1} t_1^{-1} t_2^{-1}})\beta \big).$$

Since both arguments are conjugate to $t_1 t_2 t_1^{-1} t_2^{-1}$, (resp. $t_1 t_2 t_1 t_2^{-1} t_1^{-1} t_2^{-1}$) or its inverse, the claim follows. □

4. Discriminant knot group of some Brieskorn-Pham singularities

Let f be a polynomial on the affine space $\mathbb{C}^k$ with an isolated singularity in 0.

REMARK 4.1. The discriminant complement of such spaces may be Eilenberg-MacLane spaces of their fundamental groups.

Our approach applies, since we get natural spaces of polynomials.

4.1. Basic setup.

A holomorphic function f on a complex affine space, (function germ is more precise but too cumbersome), is studied by means of versal unfoldings, e.g. given by a function

$$F(x,z,u) = f(x) - z + \sum g_i u_i, \qquad x \in \mathbb{C}^k, z, u_i \in \mathbb{C},$$

where monomials g_i generate additively the local algebra of function germs up to elements in the Jacobian ideal of f.

In case of a semi-universal unfolding the unfolding dimension n is given by the Milnor number and we get a diagram

$$\begin{array}{ccccccc} (z,u_1,...,u_{n-1}) & \in & \mathbb{C}^n & \supset & \mathcal{D} & = & \{(z,u)\,|\,F^{-1}_{z,u}(0) \text{ is singular }\} \\ \downarrow & & {}^{p}\!\downarrow & & & & \\ (u_1,...,u_{n-1}) & \in & \mathbb{C}^{n-1} & \supset & \mathcal{B} & = & \{u\,|\,F_{0,u} \text{ is not Morse }\} \end{array}$$

The restriction $p|_{\mathcal{D}}$ to the *discriminant* $\mathcal{D}$ is a finite map, the branch set coincides with the *bifurcation set* $\mathcal{B}$.

We consider polynomials

$$f(x,y) = x^3 + y^{\ell+1}.$$

4.2. Unfolding spaces in singularity theory:

- $V_f = f + \mathbb{C}[x_1,...,x_n]_{trunc}$, an *unfolding space* of

  $$f = x_1^{d_1} + x_2^{d_2} + \cdots + x_n^{d_n}, \text{ (Brieskorn-Pham polynomial)}$$
- u_ν, the coefficients of monomials $g_\nu = \prod_i x_i^{\nu_i}$ are coordinates.
- get $P \in \mathbb{C}[u_\nu][z]$, monic in z, by eliminating x_i from

  $$F = f(x_1,...,x_n) - z + \sum u_\nu \prod x_i^{\nu_i} \;=\; \tfrac{\partial}{\partial x_i} F \;=\; 0.$$
- $X_f = \{u \in V_f \,|\, P_u \text{ simple}\}$ the *bifurcation complement.*
- $\operatorname{Br} M_{X_f}$ is an invariant of f.

REMARK 4.2. Due to a genericity argument, unfolding over $\mathbb{C}[x_1,...,x_n]_{\deg\le 2}$ is sufficient.

4.3. Hefez Lazzeri basis.

The generic function is Morse in the unfolding of f by linear terms

$$F(x,y,a,b) \quad := \quad x^3 - 3ax + y^{\ell+1} - \frac{\ell+1}{\ell} by,$$

The function $\tilde{f} = F(x,y,1,1)$ has critical values $z_i = 1 + y_i, z_{\ell+i} = -1 + y_i$, where the y_i are the ℓ solutions to $y^\ell = 1/\ell$ ordered by increasing argument.

The geometric basis $\{t_i, 1 \le i \le 2\ell\}$ for $\tilde{f}$ [**17**], can be understood from Figure 1. Over a generic line in unfolding space, we get a plane curve $\mathcal{C}$, the plane section of $\mathcal{D}$, such that the restricted fibre bundle is given as the complement of the vertical lines through singular points of $\mathcal{C}$.

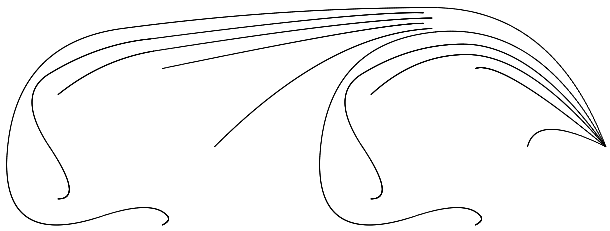


FIGURE 1. Hefez Lazzeri system in case $\ell = 5$

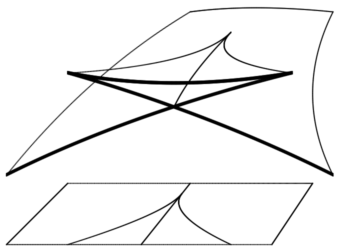


FIGURE 2. plane section of discriminant

These are of two kinds (as opposed to the uniqueness of a Morse singularity), the ordinary node and the ordinary cusp, corresponding to the two distinct strata in $\mathcal{D}$ of codimension one.

Opposed to that a generic line in the linear unfolding $F_{a,b}$ gives a curve with more general singularities.

It neither gives the correct braid monodromy group nor has isomorphic knot group.

4.4. Versal braid monodromy.

Versal braid monodromy serves to find the braid monodromy of versal unfoldings from computable data of suitable non-versal subunfoldings.

Let a one-parameter family of functions f_λ be given with f_0 tame, ie. critical values may only coincide for non-degenerate critical points, and f_λ Morse for $\lambda \neq 0$.

So there is an associated family p_λ of *discriminant polynomials* characterized by being univariate, monic and of constant degree, having only simple zeroes for $\lambda \neq 0$ and

$$p_\lambda(u) = 0 \quad \Longleftrightarrow \quad \exists x : \operatorname{grad}_x f_\lambda(x) = 0,\ f_\lambda(x) = u.$$

Let v_j denote the roots of p_0. Then for $\varepsilon > 0$ and $0 < \delta \ll \varepsilon$ sufficiently small, the discriminant complement $Y = \mathbb{C} \times D_\delta \setminus p_\lambda^{-1}(0)$ is trivialisable over the disc D_δ in the complement of $\cup_j B_\varepsilon(v_j)$.

In any fibre Y_λ, $0 < |\lambda| < \delta$, we assign a group of mapping classes choosing generators – for each v_j – supported on the punctured disc $D_j = Y_\lambda \cap B_\varepsilon(v_j)$, the *local punctured fibre* associated to v_j:

In case that v_j is a multiple root of p_0, which is the image of a single critical point c_j of f_0, we assign the braid monodromy group for the germ of f_0 at c_j consisting of mapping classes supported on D_j.

In case v_j is the image of non-degenerate critical points of f, we choose the group of mapping classes of D_j which fix the punctures and thus correspond to pure braids.

Given a family of functions f_λ which are generically Morse and generically tame in the bifurcation set $\mathcal{B}$ of non-Morse functions, we locally assign groups of mapping classes to each tame function using local slices to $\mathcal{B}$.

The *versal braid monodromy group* is then defined to be generated by all such classes identified by topological trivialisations along all possible paths with classes in a reference fibre. It is determined as a subgroup of an abstract braid group $\mathbb{B}_n$ upon choice of a geometric basis for the reference fibre consisting of n paths.

The identification along paths can be simplified:

PROPOSITION 4.3. *Suppose the family f_λ of polynomials is parameterized by a disc. Then the versal braid monodromy group is generated by elements obtained from generators of all locally assigned groups via identification along paths of a geometric basis associated to the parameters of bifurcation.*

PROPOSITION 4.4 (Key Property). *Over a line in the unfolding space corresponding to tame polynomials only, the versal braid monodromy group coincides with* $\mathrm{Br}\, M_{X_f}$

4.5. Versal braid monodromy computation.

We need the versal braid monodromy associated to singularities of type A_ℓ:

PROPOSITION 4.5 (Looijenga [**19**], Catanese-Wajnryb [**8**]). *The braid monodromy of $f = x^{\ell+1}$ is generated by*

$$\sigma_i^3, \sigma_{i,j}^2, |i-j| \geq 2$$

with respect to the Hefez-Lazzeri geometric basis associated to the polynomial $\tilde{f} = x^{\ell+1} - 3(\ell+1)x$.

To gain some geometric insight we translate this result back into a statement on mapping classes of a reference fibre.

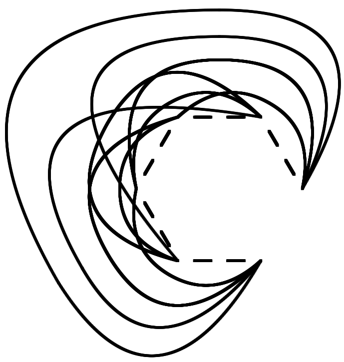


FIGURE 3. Generators of local braid monodromy

A set of mapping classes which generate the braid monodromy in the punctured fibre corresponding to the function $x^{\ell+1} - (\ell+1)x$, with punctures thus at the ℓ-th roots of ℓ, is given by

- the *cusp-twists* on the dashed arcs joining consecutive punctures,
- the *full twists* on arcs joining non-consecutive punctures in the complement of the inscribed polygon and the open cone defined by the first and the last puncture.

With this notion of versal braid monodromy group we are able to show:

Proposition 4.6. *The braid monodromy group of a singularity given by $f(x) = x^3 + y^{\ell+1}$ is generated by the versal braid monodromy groups of the tame families*

$$f_a(x,y) = x^3 - 3ax + y^{\ell+1} - \tfrac{\ell+1}{\ell}y, \quad g_b(x,y) = x^3 - 3x + y^{\ell+1} - \tfrac{\ell+1}{\ell}by$$

This Proposition is also valid in higher dimension, see [**17**].

4.6. Tameness.
Degenerate critical points are detected by the vanishing of the Hessian, which happens only at $a = 0$ resp. $b = 0$. But there it can be shown by a short computation that the critical values are in bijection with the critical points.

4.7. Central idea of proof.
The combination of the families is a topologically equivalent to the family over a generic line in the base of a the linear unfolding.

Fast forward of the remaining steps of the proof

- The versal braid monodromy of the second family.
- A suitable choice of generators for the braid monodromy group.

Theorem 4.7 ([**18**]). $\operatorname{Br} M_{X_f}$ *of the polynomial $x^3 + y^{\ell+1}$ is generated by:*

- σ_{ij}^2 *in case* ${}_i\cdot \quad \cdot_j$
- σ_{ij}^3 *in case* ${}_i\cdot\!-\!\!\cdot_j$,
- $\sigma_{ij}^{\pm 2}\sigma_{ik}^2\sigma_{ij}^{\mp 2}$ *in case* ${}_i \underset{j}{\triangledown} {}_k$ *($\pm = \varepsilon_{ijk}$ antisymmetric).*

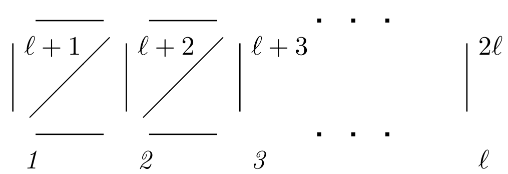


Figure 4. Dynkin diagram of $x^3 + y^{\ell+1}$

The braids σ_{ij} are the so-called band generators of the braid group.

5. Fundamental groups

Theorem 5.1. *For $\tilde{X} = \{u \in V_f \mid P(u,0) \neq 0\}$, the discriminant complement:*

$$\pi_1 \quad \cong \quad \left\langle t_i,\, i \in I \;\middle|\; \begin{array}{rcll} t_i t_j &=& t_j t_i, & \text{for } {}_i\cdot \quad \cdot_j \\ t_i t_j t_i &=& t_j t_i t_j, & \text{for } {}_i\cdot\!-\!\!\cdot_j, \\ t_i^{\varepsilon} t_j t_i^{-\varepsilon} t_k &=& t_k t_i^{\varepsilon} t_j t_i^{-\varepsilon}, & \text{for } {}_i \underset{j}{\triangledown} {}_k \end{array} \right\rangle.$$

(ε_{ijk} antisymmetric)

5.1. Combinatorial structure.
The Dynkin diagram is naturally associated to the geometry of the generic smooth fibre of f:

Vanishing cycles provide a basis for the middle homology and are in bijection to the vertices, edges (in our case) are in bijection to non-zero intersection (in fact -1).

A given geometric basis is naturally acted on by the braid group $\mathbb{B}_{2\ell}$. Elements in the braid monodromy group act trivially on the Dynkin diagram, since by the theorem of van Kampen they act trivially on the discriminant knot group.

It remains open, whether the braid monodromy group is the whole stabiliser group – at least it is a subgroup with a generating set of appealing simplicity.

6. Open questions

Finally we yield to the temptation to list a few problems and questions which seem now to come into reach of our curiosity:

(1) What results should be expected in the case of arbitrary singularities? Gabrièlov [**14**] developed a general method which in principle yields Dynkin diagrams for arbitrary hypersurface singularities by an induction on the codimension. These diagrams feature simple edges and triangles only, but neither edges of higher multiplicity nor cycles which are not subdivided into triangles.

We hope that an induction for braid monodromy can be found in analogy to the induction in the work of Gabrièlov [**14**].

(2) If we impose a relation on two singularities, how are the invariants related? We would like to understand the equivalence relation on singularities which is detected by the braid monodromy or the discriminant knot group. Also there is some evidence that adjacency induces injective maps on both invariants.

(3) How does combinatorial group theory apply to our new presentations? They arise in a natural setting generalising the standard presentations of Artin-Brieskorn groups of finite type. Recently there has been a surge of activities in combinatorial group theory thanks to the new ideas and techniques centering around the concept of Garside groups, [**10**]. In this framework the question should be addressed whether there exists a finite dimensional $K(\pi, 1)$. It could well prove to become a major ingredient to settle the question of asphericity of the discriminant complement, cf. Thom [**25**].

(4) Can we obtain a better understanding of the various monodromy groups in singularity theory? Our groups form the domain of such monodromy homomorphisms, e.g. algebraic, geometric or the recently proposed symplectic monodromy, [**2, 23**]. A more detailed study of the kernel and of presentations for the image groups thus seems promising.

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