BRAID MONODROMY AND TOPOLOGY OF PLANE CURVES

ENRIQUE ARTAL BARTOLO, JORGE CARMONA RUBER, AND JOSÉ IGNACIO COGOLLUDO AGUSTÍN

Abstract. In this paper we prove that braid monodromy of an affine plane curve determines the topology of a related projective plane curve.

Our purpose in this paper is to relate the topological embedding of algebraic curves o a refinement of a well-known invariant of curves such as braid monodromy. Roughly speaking, braid monodromy is defined for a triple $(\mathcal{C}, L, P)$, where $\mathcal{C} \subset \mathbb{P}^2$ is a curve, $L \subset \mathbb{P}^2$ is a line not contained in $\mathcal{C}$ and $P \in L$, as follows. Let us consider homogeneous coordinates $[x : y : z]$ such that $P = [0 : 1 : 0]$, $L = \{z = 0\}$ and $\mathcal{C} = \{f(x, y, z) = 0\}$. Let d be the y-degree of $\mathcal{C}$, i.e., $d = \deg_y(f(x, y, 1))$. The pencil $\mathcal{H}$ of lines passing through P (and different from L) is parametrized by $x \in \mathbb{C}$. By the theorem of continuity of roots, $\mathcal{H}$ determines a representation of a free group $\mathbb{F}$ on the braid group on d strings, which is called a braid monodromy of the triple $(\mathcal{C}, L, P)$. The free group $\mathbb{F}$ corresponds to the fundamental group of an r-punctured complex line, where the punctures come from the non-generic elements of $\mathcal{H}$ with respect to $\mathcal{C}$. The classical definition of braid monodromy refers to generic choices of L and P, e.g., $P \notin \mathcal{C}$ and L transversal to $\mathcal{C}$. In this work we will allow certain non-generic choices.

Braid monodromy is a strong invariant of plane curves. It is fair to say that the main ideas that lead to this invariant have already been used in the classic works of Zariski [12] and Van Kampen [6] to find the fundamental group of the complement of a curve. The first explicit definition of braid monodromy was made by Chisini using the terms algebraic braids and *fascio caratteristico*, see [5]. However, the main results and applications are due to B. Moishezon – see [11] and related references. This author uses braid monodromy to obtain a number of beautiful results in several papers, both as single author and with M. Teicher.

Date: May 6, 2002.

2000 *Mathematics Subject Classification.* 14D05 14H30 14H50 20F36.

Key words and phrases. Braid monodromy, plane curves.

First author is partially supported by PB1997-0284-C02-02 and BFM2001-1488-C02-02; second author is partially supported by PB1997-0284-C02-02 and BFM2001-1488-C02-01; third author is partially supported by PB1997-0284-C02-01 and BFM2001-1488-C02-02.

Later on, A. Libgober [8] proved that the homotopy type of $\mathbb{P}^2 \setminus (\mathcal{C} \cup L)$ is an invariant of its braid monodromy.

Braid monodromy determines a presentation of the fundamental group of $\mathbb{P}^2 \setminus (\mathcal{C} \cup L)$. Several invariants of the fundamental group have been found to be effective as invariants of curves, such as the Alexander polynomial, the Alexander module or the sequence of characteristic varieties [10]. A second group of effective invariants was described by A. Libgober in [9]. They depend on polynomial representations of the braid group and are invariants of the conjugation class of the image of the representation defining the braid monodromy.

By fixing a particular class of bases – called a *geometric basis* – of the aforementioned free group $\mathbb{F}$, braid monodromy may be represented by an r-tuple of braids. This sequence is not unique; there is a natural action of $B_d \times B_r$ on B_d^r such that the admissible r-tuples form an orbit of this action. Braid monodromies in the same orbit are called equivalent. Note that in the classical case, braid monodromies for different generic choices are equivalent. In general, it is not easy to decide whether two elements of B_d^r are in the same orbit or not, since these orbits are infinite. We attack this problem in another paper [1].

For curves having only nodes and cusps as singularities, Teicher and Kulikov [7] have recently proved that the diffeomorphism type of their embedding is determined by their braid monodromy. The general case has been solved independently by the second named author in [4].

The main purpose of this work is to prove a kind of converse of this result. We say that a triple $(\mathcal{C}, L, P)$ is horizontal if $f(x, y, 1)$ is monic as an element of $\mathbb{C}[x][y]$, according to the notation above. We denote by $\mathcal{C}^\varphi$ the union of $\mathcal{C}$ with the non-generic lines of $\mathcal{H}$ with respect to $\mathcal{C}$. We will prove the following:

Theorem 1. *Let $\mathcal{C}_1, \mathcal{C}_2 \subset \mathbb{P}^2$ be two curves, let L be a line not contained in $\mathcal{C}_1 \cup \mathcal{C}_2$ and let $P \in L$ such that $(\mathcal{C}_1, L, P)$ and $(\mathcal{C}_2, L, P)$ are horizontal triples. Let $F : \mathbb{P}^2 \to \mathbb{P}^2$ be an orientation-preserving homeomorphism such that:*

(i) *$F(P) = P$,*
(ii) *$F(L) = L$ preserving orientations, and*
(iii) *$F(\mathcal{C}_1^\varphi) = \mathcal{C}_2^\varphi$ preserving orientations.*

Then both triples have equivalent braid monodromies.

Remark 2. Note that by combinatorial considerations, it is often the case that a homeomorphism $F : (\mathbb{P}^2, \mathcal{C}_1^\varphi \cup L) \to (\mathbb{P}^2, \mathcal{C}_2^\varphi \cup L)$, that respects orientations everywhere, satisfies the conditions of Theorem 1.

Braid monodromy may also be useful in the study of the moduli of projective curves with prescribed degree and topological types of curve singularities. Let d be a positive integer and let $T_1, \ldots, T_r$ be topological types of curve singularities. Let us denote by $\Sigma(T_1, \ldots, T_r; d)$ the space of all projective plane curves of degree d with r singular points of type $T_1, \ldots, T_r$. Let $\mathscr{M}(T_1, \ldots, T_r; d)$ be the quotient of $\Sigma(T_1, \ldots, T_r; d)$ by the action of the projective group. Note that classical braid monodromy is an invariant of each connected component of these moduli spaces. For most non-generic choices of L and P, our refined definition of braid monodromy also provides an invariant of such connected components.

Example 3 ([1])**.** Any element of $\Sigma(\mathbb{E}_6, \mathbb{A}_7, \mathbb{A}_3, \mathbb{A}_2, \mathbb{A}_1; 6)$ is projectively equivalent to a curve $\mathcal{C}_\beta$, $\beta^2 = 2$, of equation $f_\beta(x,y,z) g_\beta(x,y,z) = 0$ where

$$\begin{aligned} f_\beta(x,y,z) &:= y^2 z^3 + (303 - 216\,\beta)\, y z^2 x^2 + (-636 + 450\,\beta)\, y z x^3 + \\ &\quad + (-234\,\beta + 331)\, y x^4 + (-18\,\beta + 27)\, z x^4 + (18\,\beta - 26)\, x^5, \\ g_\beta(x,y,z) &:= y + \left(\frac{10449}{196} - \frac{3645}{98}\,\beta\right) z + \left(-\frac{432}{7} + \frac{297}{7}\,\beta\right) x. \end{aligned} \tag{1}$$

Moreover, $\mathcal{C}_{\sqrt{2}}$ is not projectively equivalent to $\mathcal{C}_{-\sqrt{2}}$. Therefore, the moduli space $\mathscr{M}(\mathbb{E}_6, \mathbb{A}_7, \mathbb{A}_3, \mathbb{A}_2, \mathbb{A}_1; 6)$ consists of two points whose representatives, given in (1), have conjugate equations in the number field $\mathbb{Q}(\sqrt{2})$. The line $L = \{z = 0\}$ is the tangent line to the curves $\mathcal{C}_\beta$ at the $\mathbb{E}_6$-type point $P = [0:1:0]$. There are 5 non-transversal lines in $\mathcal{H}_\beta$; four of them pass through the singular points of type $\mathbb{A}_7$, $\mathbb{A}_3$, $\mathbb{A}_2$, $\mathbb{A}_1$, and the fifth one is an ordinary tangent to $\mathcal{C}_\beta$. Note that the curves $C^\varphi_{\sqrt{2}} \cup L$ and $C^\varphi_{-\sqrt{2}} \cup L$ are also conjugate in $\mathbb{Q}(\sqrt{2})$.

As an application of Theorem 1 we prove in [1] that there is no homeomorphism $F : (\mathbb{P}^2, C^\varphi_{\sqrt{2}} \cup L) \to (\mathbb{P}^2, C^\varphi_{-\sqrt{2}} \cup L)$. Let us assume such a homeomorphism F exists. As mentioned in Remark 2, by combinatorial reasons $F(P) = P$ and $F(L) = L$. By composing F with complex conjugation, if necessary, one can prove that F preserves orientations of $\mathbb{P}^2$ and of each irreducible component of the curves. Then, by Theorem 1, the triples $(\mathcal{C}_\beta, L, P)$, $\beta = \pm\sqrt{2}$, have equivalent braid monodromies. We can explicitly compute these braid monodromies and prove that they are not equivalent. This provides an example of a Zariski pair of conjugate curves in degree 11.

In §1, we set notations and definitions. In §2, we translate the definition of braid monodromy in terms of r-tuples of braids and in §3 we prove the main theorem.

1. Settings and definitions

Let G be a group and $a, b \in G$. For the sake of simplicity we use the following notation: $a^b := b^{-1}ab$, $[a,b] := a^{-1}b^{-1}ab$ and $b * a := bab^{-1}$. Given any set A we denote by Σ_A the group of all bijective maps from A onto itself. As usual we denote $\Sigma_{\{1,\dots,n\}}$ by Σ_n.

We also denote by $\mathbb{F}_n$ the free group on n generators, say $x_1, \dots, x_n$, and by B_n the braid group on n strings given by the following presentation:

$$\langle \sigma_1, \dots, \sigma_{n-1} : \ [\sigma_i, \sigma_j] = 1, \ |i-j| \geq 2, \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \ i = 1, \dots, n-2 \rangle .$$

There is a natural right action $\Phi : \mathbb{F}_n \times B_n \to \mathbb{F}_n$ such that

$$x_j^{\sigma_i} = \begin{cases} x_{i+1} & \text{if } j = i \\ x_{i+1} * x_i & \text{if } j = i+1 \\ x_j & \text{if } j \neq i, i+1. \end{cases} \tag{2}$$

The induced anti-homomorphism $B_n \to \operatorname{Aut} \mathbb{F}_n$ is injective. Identifying B_n with its image one has the following characterization, which may be found in any classic text on braids, e.g. [2] or [3]:

Proposition 1.1. *Let $\tau : \mathbb{F}_n \to \mathbb{F}_n$ be an automorphism. Then $\tau \in B_n$ if and only if $\tau(x_n \cdot \ldots \cdot x_1) = x_n \cdot \ldots \cdot x_1$ and there exists $\sigma \in \Sigma_n$ such that $\tau(x_i)$ is conjugate to $x_{\sigma(i)}$.*

Next we describe a geometrical interpretation of these definitions. All the results in this section appear in J. Birman's book [3]. Let us fix $\mathbf{y} := \{t_1, \dots, t_n\}$ a subset of $\mathbb{C}$ of exactly n elements. Let us also consider a *sufficiently big* closed geometric disk Δ such that $\mathbf{y}$ is contained in its interior. Let $*$ be a point on $\partial\Delta$.

Notation 1.2. We will denote by $\gamma \in \pi_1(\mathbb{C} \setminus \mathbf{y}; *)$ the homotopy class of the loop which surrounds the circle $\partial\Delta$ counterclockwise.

We recall:

Fact 1.3. *The group $\pi_1(\mathbb{C} \setminus \mathbf{y}; *)$ is isomorphic to $\mathbb{F}_n$.*

Definition 1.4. A $\mathbf{y}$-*special homeomorphism* is an orientation-preserving homeomorphism $\mathbb{C} \to \mathbb{C}$ such that $\mathbf{y}$ is fixed as a set, but not necessarily pointwise, and is the identity on $\overline{\mathbb{C} \setminus \Delta}$. A $\mathbf{y}$-*special isotopy* is an isotopy $H : \mathbb{C} \times [0,1] \to \mathbb{C} \times [0,1]$ with the above properties.

The set of $\mathbf{y}$-special homeomorphisms up to $\mathbf{y}$-special isotopy has a natural group structure and is denoted by $\mathbb{B}_{\mathbf{y}}$; if h_1, h_2 are $\mathbf{y}$-special homeomorphisms, then the product $[h_1] \cdot [h_2]$ of their classes is the class of the composition $h_1 \circ h_2$. According to Definition 1.4 there is a natural left action of $\mathbb{B}_{\mathbf{y}}$ on $\pi_1(\mathbb{C} \setminus \mathbf{y}; *)$.

Fact 1.5. *The group $\mathbb{B}_{\mathbf{y}}$ is isomorphic to B_n.*

Let V be the space of degree n monic polynomials in $\mathbb{C}[t]$ (V and $\mathbb{C}^n$ are naturally isomorphic as affine spaces). Let D be the discriminant space of V, i.e., the set of elements in V with multiple roots. The space D is an algebraic hypersurface of V. We will use the following notation: $X := V \setminus D$, $p(t) := \prod_{j=1}^n (t - t_j)$, $B_{\mathbf{y}} := \pi_1(X; p(t))$.

Remark 1.6. The space X can be naturally identified with the subsets of $\mathbb{C}$ having exactly n elements. From now on, X will denote, depending on the context, either $V \setminus D$ or the set $\{\mathbf{y} \subset \mathbb{C} \mid \#\mathbf{y} = n\}$ with quotient topology resulting from the n^{th}-symmetric product of $\mathbb{C}$.

By means of the theorem of continuity of roots, an element of $B_{\mathbf{y}}$ is identified with the homotopy class, relative to $\{0, 1\}$, of sets of paths $\{\gamma_1, \ldots, \gamma_n\}$, $\gamma_j : [0, 1] \to \mathbb{C}$, starting and ending at $\mathbf{y}$ and such that $\forall t \in [0, 1]$, $\{\gamma_1(t), \ldots, \gamma_n(t)\}$ is a set of n distinct points. The elements of $B_{\mathbf{y}}$ are called braids based at $\mathbf{y}$ and are represented as a set of non-intersecting paths in $\mathbb{C} \times [0, 1]$, as usual. Products are also defined in the standard way.

Since any orientation-preserving homeomorphism of $\mathbb{C}$ is isotopic to the identity, we can associate to any element of $\mathbb{B}_{\mathbf{y}}$ the braid representing the motion of $\mathbf{y}$ along the isotopy.

Fact 1.7. *The groups $\mathbb{B}_{\mathbf{y}}$ and $B_{\mathbf{y}}$ are naturally anti-isomorphic.*

Thus one can construct a right action

$$\Phi_{\mathbf{y}} : \pi_1(\mathbb{C} \setminus \mathbf{y}; *) \times B_{\mathbf{y}} \to \pi_1(\mathbb{C} \setminus \mathbf{y}; *).$$

In accordance with 2, exponential notation will be used to describe this action.

Example 1.8. Let us consider:

- $\mathbf{y}^0 = \{-1, \ldots, -n\}$,
- $\Delta = \{t \in \mathbb{C} |\ |t| \leq R\}$, $R \gg 0$,
- $* = R$.

We denote by Γ the polygonal path given by a union of segments Γ_1 (from $*$ to -1) and Γ_j, $j = 2, \ldots, n$ (from $-(j-1)$ to $-j$). Let δ_j be the path obtained by surrounding the circle of radius $\frac{1}{4}$ centered at $-j$ and starting at $-j + \frac{1}{4}$, $j = 1, \ldots, n$ counterclockwise. We will denote by δ_j^+ the upper half-circle and by δ_j^- the lower half one.

For any given $j = 1, \ldots, n$, one can define a path η_j from $*$ to $-j+\frac{1}{4}$ constructed from $\Gamma_1 \cup \cdots \cup \Gamma_j$ by replacing the sub-paths joining $-k+\frac{1}{4}$ and $-k-\frac{1}{4}$ with δ_k^+, $k = 1, \ldots, j-1$. We call $\nu_j \subset \Gamma_j$ the aforementioned subsegment contained in η_j.

The following set of paths based on $*$ will be useful: $x_j := \eta_j \cdot \delta_j \cdot (\eta_j)^{-1}$, $j = 1, \ldots, n$. It is well known that $x_1, \ldots, x_n$ is a basis of the free group $\pi_1(\mathbb{C} \setminus \mathbf{y}^0; *)$. Moreover, $\gamma = x_n \cdot \ldots \cdot x_1$.

Let us consider the group $B_{\mathbf{y}^0}$. For $j = 1, \ldots, n-1$, the braid σ_j will be defined as a set $\{\gamma_1^{(j)}, \ldots, \gamma_n^{(j)}\}$ of paths such that:

- $\gamma_k^{(j)}$ is constant with value $-k$ if $k \neq j, j+1$;
- $\gamma_j^{(j)}$ is the upper half-circle of radius $\frac{1}{2}$ starting at $-j$ and ending at $-(j+1)$.
- $\gamma_{j+1}^{(j)}$ is the lower half-circle of radius $\frac{1}{2}$ starting at $-(j+1)$ and ending at $-j$.

It is well known that $\sigma_1, \ldots, \sigma_{n-1}$ generate $B_{\mathbf{y}^0}$, providing a way to identify $B_{\mathbf{y}^0}$ and B_n. These identifications are equivariant with respect to the actions Φ and $\Phi_{\mathbf{y}^0}$. Hence the actions are identified as well.

Based on this example we will describe isomorphisms from $\pi_1(\mathbb{C} \setminus \mathbf{y}; *)$ to $\mathbb{F}_n$ and from $B_{\mathbf{y}}$ to B_n for any $\mathbf{y} \in X$, which are equivariant with respect to Φ and $\Phi_{\mathbf{y}}$. This can be achieved by defining braids with different ends.

Definition 1.9. Let $\mathbf{y}^1$ and $\mathbf{y}^2$ be subsets of $\mathbb{C}$ with exactly n elements – that is, $\#\mathbf{y}^1 = \#\mathbf{y}^2 = n$. A *braid from* $\mathbf{y}^1$ *to* $\mathbf{y}^2$ is a homotopy class, relative to $\{0,1\}$, of sets $\{\gamma_1, \ldots, \gamma_n\}$ of paths $\gamma_j : [0,1] \to \mathbb{C}$ starting at $\mathbf{y}^1$, ending at $\mathbf{y}^2$ and such that $\forall t \in [0,1]$, $\#\{\gamma_1(t), \ldots, \gamma_n(t)\} = n$. The sets $\mathbf{y}^1$ and $\mathbf{y}^2$ are usually referred to as the *braid ends*.

As above, no distinction will be made between a braid and its class representative. The set of all braids from $\mathbf{y}^1$ to $\mathbf{y}^2$ is denoted by $B(\mathbf{y}^1, \mathbf{y}^2)$. Note that $B_{\mathbf{y}} = B(\mathbf{y}, \mathbf{y})$ for any $\mathbf{y} \in X$. Given $\mathbf{y}^1, \mathbf{y}^2, \mathbf{y}^3 \in X$, there are natural operations

$$B(\mathbf{y}^1, \mathbf{y}^2) \times B(\mathbf{y}^2, \mathbf{y}^3) \to B(\mathbf{y}^1, \mathbf{y}^3), \quad B(\mathbf{y}^1, \mathbf{y}^2) \to B(\mathbf{y}^2, \mathbf{y}^1),$$

for the product and inverse of braids, which have the required natural properties that identify them with the fundamental groupoid of X. This fundamental groupoid will be called the *braid groupoid on n strings*.

Moreover, let us consider $\mathbf{y}^1, \mathbf{y}^2 \in X$ and let Δ be a geometric disk containing $\mathbf{y}^1 \cup \mathbf{y}^2$ in its interior.

Definition 1.10. A $(\mathbf{y}^2, \mathbf{y}^1)$-*special homeomorphism* is an orientation-preserving homeomorphism $f : \mathbb{C} \to \mathbb{C}$ such that f is the identity on $\mathbb{C} \setminus \Delta$ and $f(\mathbf{y}^1) = \mathbf{y}^2$. Analogously, $(\mathbf{y}^2, \mathbf{y}^1)$-*special isotopies* can be defined.

The quotient set of special isotopy classes of $(\mathbf{y}^2, \mathbf{y}^1)$-special homeomorphisms of $\mathbb{C}$ will be denoted by $\mathbb{B}(\mathbf{y}^2, \mathbf{y}^1)$. Considering the natural operations, one obtains a groupoid on X which is naturally anti-isomorphic to the braid groupoid. By choosing $* \in \partial\Delta$ as a base point, one has a natural mapping

$$\mathbb{B}(\mathbf{y}^2, \mathbf{y}^1) \times \pi_1(\mathbb{C} \setminus \mathbf{y}^1; *) \to \pi_1(\mathbb{C} \setminus \mathbf{y}^2; *)$$

which may be viewed as a groupoid action. It induces a right groupoid action

$$\pi_1(\mathbb{C} \setminus \mathbf{y}^1; *) \times B(\mathbf{y}^1, \mathbf{y}^2) \to \pi_1(\mathbb{C} \setminus \mathbf{y}^2; *),$$

which becomes $\Phi_{\mathbf{y}}$ when $\mathbf{y}^1 = \mathbf{y}^2 = \mathbf{y}$. This action will be described using exponential notation.

Let us fix two ends $\mathbf{y}^1, \mathbf{y}^2 \in X$ and a braid $\tau \in B(\mathbf{y}^1, \mathbf{y}^2)$. There are two natural isomorphisms

$$\begin{array}{rccc} \mathbb{I}_\tau : & B_{\mathbf{y}^1} & \to & B_{\mathbf{y}^2} \\ & \sigma & \mapsto & \tau^{-1} \cdot \sigma \cdot \tau \end{array}$$

and

$$\begin{array}{rccc} \Psi_\tau : & \pi_1(\mathbb{C} \setminus \mathbf{y}^1; *) & \to & \pi_1(\mathbb{C} \setminus \mathbf{y}^2; *) \\ & x & \mapsto & x^\tau. \end{array}$$

If $\mathbf{y}^1 = \mathbf{y}^2 = \mathbf{y}$, then $\tau \in B_{\mathbf{y}}$, $\mathbb{I}_\tau$ is the inner automorphism $\sigma \mapsto \tau^{-1} \cdot \sigma \cdot \tau$ and Ψ_τ is the automorphism $\Phi_{\mathbf{y}}(\bullet, \tau)$ induced by the right action $\Phi_{\mathbf{y}}$ with respect to τ. We summarize these results below.

Proposition 1.11. *Let $\mathbf{y}^0 = \{-1, \ldots, -n\}$ as in* example 1.8, *and let $\mathbf{y} \in X$. We assume that Δ is big enough so that $\mathbf{y} \subset \operatorname{Int}\Delta$. We identify $\pi_1(\mathbb{C} \setminus \mathbf{y}^0; *)$ with $\mathbb{F}_n$ and $B_{\mathbf{y}^0}$ with B_n.*

Then, for any $\tau \in B(\mathbf{y}^0, \mathbf{y})$ there is a canonical identification of $\pi_1(\mathbb{C} \setminus \mathbf{y}; *)$ with $\mathbb{F}_n$ and of $B_{\mathbf{y}}$ with B_n by means of Ψ_τ and $\mathbb{I}_\tau$, respectively. These identifications are equivariant with respect to Φ and $\Phi_{\mathbf{y}}$.

Moreover, two such identifications differ by an inner automorphism of B_n, with respect to a given $\hat{\tau} \in B_n$, and the corresponding automorphism of $\mathbb{F}_n$ is $\Phi(\bullet, \hat{\tau})$.

We recall an important definition from algebraic geometry.

Definition 1.12. Let Z be a connected projective manifold and let H be a hypersurface of Z. Let $* \in Z \setminus H$ and let K be an irreducible component of H. A homotopy class $\gamma \in \pi_1(Z \setminus H; *)$ is called a *meridian about K with respect to H* if γ has a representative δ satisfying the following properties:

(a) there is a smooth complex analytic disk $\Delta \subset Z$ transverse to H such that $\Delta \cap H = \{*'\} \subset K$ (transversality implies that $*'$ is a smooth point of H).

(b) there is a path α in $Z \setminus H$ from $*$ to $*'' \in \partial\Delta$.
(c) $\delta = \alpha \cdot \beta \cdot \alpha^{-1}$, where β is the closed path obtained by traveling from $*''$ along $\partial\Delta$ in the positive direction.

It is well known that any two meridians about K with respect to H are conjugate in $\pi_1(Z \setminus H; *)$. Moreover, the converse also holds.

In Example 1.8, x_j is a meridian about $-j$ in $\pi_1(\mathbb{C}\setminus\mathbf{y}^0; *)$ where $\mathbf{y}^0 = \{-1, \ldots, -n\}$, for $j = 1, \ldots, n$. Note that if $\mathbf{y}^1 \in X$ and $\tau \in B(\mathbf{y}^0, \mathbf{y}^1)$, then $x_1^\tau, \ldots, x_n^\tau$ are meridians about the points in $\mathbf{y}^1$, and of course they also form a free system of generators of the free group $\pi_1(\mathbb{C} \setminus \mathbf{y}^1; *)$.

Definition 1.13. Let us consider $\mathbf{y}$ a point of X, $\Delta \subset \mathbb{C}$ an oriented closed geometric disk containing $\mathbf{y}$ in its interior, and $*$ a point in $\partial\Delta$. Also, let $\gamma \in \pi_1(\mathbb{C} \setminus \mathbf{y}; *)$ be as in Notation 1.2. A *geometric basis* of the free group $\pi_1(\mathbb{C} \setminus \mathbf{y}; *)$ (*with respect to* Δ) is an ordered basis $\mu_1, \ldots, \mu_n$ such that:

(1) $\mu_1, \ldots, \mu_n$ are meridians of the points in $\mathbf{y}$;
(2) $\mu_n \cdot \ldots \cdot \mu_1 = \gamma$.

Remark 1.14. The curve γ is the inverse of a very special meridian about the point at infinity of $\mathbb{C}$. If the point $*$ had been chosen arbitrarily in $\mathbb{C} \setminus \mathbf{y}$, such a special meridian would not necessarily exist. If $\mu_1, \ldots, \mu_n$ is a basis of $\pi_1(\mathbb{C} \setminus \mathbf{y}; *)$ such that $\mu_n \cdot \ldots \cdot \mu_1$ is the inverse of a meridian of the point at infinity, we will say that $\mu_1, \ldots, \mu_n$ is a *pseudogeometric basis.*

The next proposition is a geometrical version of Artin's result in Proposition 1.1.

Proposition 1.15. *Under the conditions and notations of* Example 1.8 *the following holds:*

(i) *The basis $x_1, \ldots, x_n$ is geometric with respect to Δ.*
(ii) *For any $\tau \in B_n$, the basis $x_1^\tau, \ldots, x_n^\tau$ is geometric with respect to Δ.*
(iii) *Any geometric basis with respect to Δ is obtained as in* (ii)*.*

Let us denote by $\mathscr{G}(\Delta; \mathbf{y}^0)$ the set of all geometric bases with respect to Δ. The map Φ induces an action of B_n on $\mathbb{F}_n^n$ where $\mathscr{G}(\Delta; \mathbf{y}^0)$ becomes the orbit of $(x_1, \ldots, x_n)$. The group B_n acts freely on $\mathscr{G}(\Delta; \mathbf{y}^0)$. The next result is a direct consequence of propositions 1.11 and 1.15.

Corollary 1.16. *Let $\mathbf{y} \in X$, Δ and $*$ be as in* Definition 1.13. *Let us also assume that Δ contains $\mathbf{y}^0$ and $\mathbf{y}$ in its interior. Then the set $\mathscr{G}(\Delta; \mathbf{y})$ is an orbit under the action of $B_{\mathbf{y}}$ induced by $\Phi_{\mathbf{y}}$ on $\pi_1(\mathbb{C} \setminus \mathbf{y}; *)^n$.*

Moreover:

- *for any braid $\tau \in B(\mathbf{y}^0, \mathbf{y})$, the set $x_1^\tau, \dots, x_n^\tau$ is a geometric basis of the group $\pi_1(\mathbb{C} \setminus \mathbf{y}; *)$;*
- *any element of the orbit $\mathscr{G}(\Delta; \mathbf{y})$ can be obtained in this manner.*

2. Braid monodromy and fibered curves

As in the Introduction, we will consider triples $(\mathcal{C}, L, P)$ where $\mathcal{C} \subset \mathbb{P}^2$ is a curve, $L \subset \mathbb{P}^2$ is a line not contained in $\mathcal{C}$ and $P \in L$. For convenience we fix homogeneous coordinates $[x : y : z]$ such that $P = [0 : 1 : 0]$, $L = \{z = 0\}$ and $\mathcal{C} = \{f(x, y, z) = 0\}$. We identify $\mathbb{C}^2 = \mathbb{P}^2 \setminus L$ with affine coordinates (x, y) and we set $\mathcal{C}^{\text{aff}} := \mathcal{C} \cap \mathbb{C}^2$. The pencil $\mathcal{H}$ in the Introduction is determined by the projection $\pi : \mathbb{C}^2 \to \mathbb{C}$, $\pi(x, y) := x$. As a general notation for a curve $\mathcal{C} \subset \mathbb{P}^2$ we will use $X_\mathcal{C} := \mathbb{C}^2 \setminus \mathcal{C}^{\text{aff}} = \mathbb{P}^2 \setminus (\mathcal{C} \cup L)$.

Definition 2.1. A triple $(\mathcal{C}, L, P)$ is said to be *horizontal* if $\pi_{|\mathcal{C}^{\text{aff}}} : \mathcal{C}^{\text{aff}} \to \mathbb{C}$ is a proper map. The *degree of* $(\mathcal{C}, L, P)$ is the degree of $\pi_{|\mathcal{C}^{\text{aff}}}$, i.e., the number of pre-images of a regular value. The set of regular values of $\pi_{|\mathcal{C}^{\text{aff}}}$ will be denoted by $\mathbb{C}_\mathcal{C}$.

Let $(\mathcal{C}, L, P)$ be a horizontal triple of degree d. Note that we can assume that the equation $f(x, y, 1) = 0$ for $\mathcal{C}^{\text{aff}}$ is given by

$$f(x, y, 1) = y^d + f_{d-1}(x) y^{d-1} + \dots + f_1(x) y + f_0(x),$$

where $f_j(x) \in \mathbb{C}[x]$, $j = 0, 1, \dots, d-1$. The condition of being horizontal is equivalent to the non-existence of vertical asymptotes (vertical lines included). Note that the set of critical values is the set of zeros of the discriminant of f with respect to y, which is a polynomial $\mathfrak{D}(x) \in \mathbb{C}[x]$. Denoting this set of zeros by $\mathscr{D} := \{x_1, \dots, x_r\}$, one has $\mathbb{C}_\mathcal{C} = \mathbb{C} \setminus \mathscr{D}$. Therefore the special fibers of π are exactly the vertical lines $L_i := \pi^{-1}(x_i)$ of equation $x = x_i$, $i = 1, \dots, r$. The non generic elements of $\mathcal{H}$ are the projective closures $\bar{L}_i = L_i \cup \{P\}$ of L_i, $i = 1, \dots, r$. We will denote by $\mathscr{L}$ the union of these r projective lines.

Definition 2.2. Let $(\mathcal{C}, L, P)$ be a horizontal triple. Using the above notation, the *fibered curve* associated with the triple is the curve $\mathcal{C}^\varphi := \mathcal{C} \cup \mathscr{L}$.

The motivation behind this definition is the following. Let $\check{\mathcal{C}} := \pi_{|\mathcal{C}^{\text{aff}}}^{-1}(\mathbb{C}_\mathcal{C})$. Since $\pi_{|\mathcal{C}^{\text{aff}}}$ is proper, $\tilde{\pi} : \check{\mathcal{C}} \to \mathbb{C}_\mathcal{C}$ is a covering map, where $\check{\mathcal{C}}$ might not be connected. The mapping $\pi^\varphi : X_{\mathcal{C}^\varphi} \to \mathbb{C}_\mathcal{C}$ is a locally trivial fibration whose fiber is diffeomorphic to $\mathbb{C} \setminus \{1, \dots, d\}$. The polynomial f induces an algebraic mapping $\tilde{f} : \mathbb{C}_\mathcal{C} \to X$ defined as $\tilde{f}(x_0) := \{x = x_0\} \cap \mathcal{C}^{\text{aff}}$ – or equivalently as the polynomial $f(x_0, t) \in \mathbb{C}[t]$. Let us fix $* \in \mathbb{C}_\mathcal{C}$, a complex regular value on the boundary of a geometric disk Δ containing $\mathscr{D}$ in its interior. Let us denote by $\mathbf{y}^*$ the set of roots of the polynomial $f(*, t)$.

Definition 2.3. The homomorphism $\nabla_* : \pi_1(\mathbb{C}_{\mathcal{C}}; *) \to B_{\mathbf{y}^*}$ induced by $\tilde{f}$ is called a *braid monodromy* of the horizontal triple $(\mathcal{C}, L, P)$.

Remark 2.4. Note that ∇_* classifies the locally trivial fiber bundle π^{φ}.

Let us briefly explain how to construct this braid monodromy. Note that there is a particular class of bases for $\pi_1(\mathbb{C}_{\mathcal{C}}; *)$, namely the geometric bases with respect to Δ. Let us fix one of these bases, say $\mu_1, \ldots, \mu_r$. Let us choose a braid $\tau \in B(\mathbf{y}^0, \mathbf{y}^*)$ and consider the isomorphism $\mathbb{I}_\tau : B_n \to B_{\mathbf{y}^*}$. These considerations allow us to represent ∇_* by an element $(\tau_1, \ldots, \tau_r) \in B_d^r$ such that:

$$\tau_j := \mathbb{I}_\tau^{-1}(\nabla_*(\mu_j)), \quad \forall j = 1, \ldots, r.$$

Definition 2.5. We say that $(\tau_1, \ldots, \tau_r) \in B_d^r$ *represents the braid monodromy of* $(\mathcal{C}, L, P)$ if there exists:

- a geometric disk Δ containing $\{x_1, \ldots, x_r\}$ in its interior,
- an element $* \in \partial\Delta$ such that $\mathbf{y}^*$ is the set of roots of the polynomial $\tilde{f}(*) \in \mathbb{C}[t]$,
- a geometric basis $\mu_1, \ldots, \mu_r$ of $\pi_1(\mathbb{C}_{\mathcal{C}}; *)$ and
- a braid $\beta \in B(\mathbf{y}^*, \mathbf{y}^0)$

for which $\tau_j = \mathbb{I}_\beta(\nabla_*(\mu_j))$, $\forall j = 1, \ldots, r$.

Remark 2.6. Two natural right actions on B_d^r are related to the concept of r-tuples of braids representing the monodromy and the various choices we have made.

The first one is an action produced by B_r as follows:

$$(\tau_1, \ldots, \tau_r)^{\sigma_i} := (\tau_1, \ldots, \tau_{i-1}, \tau_{i+1}, \tau_{i+1}\tau_i\tau_{i+1}^{-1}, \tau_{i+2}, \ldots, \tau_r),$$

where $\sigma_i \in B_r$, $i = 1, \ldots, r-1$ is a canonical generator and $(\tau_1, \ldots, \tau_r) \in B_d^r$; this action of a braid in B_r is classically called a *Hurwitz move*. The second action is given by B_n and is defined by conjugation on each coordinate as follows:

$$(\tau_1, \ldots, \tau_r)^{\beta} := (\tau_1^{\beta}, \ldots, \tau_r^{\beta}),$$

where $\beta \in B_d$ and $(\tau_1, \ldots, \tau_r) \in B_d^r$. These actions commute with each other and hence they define a new action of $B_r \times B_d$. The first action represents a change of geometric basis. The second action represents both a change of the chosen braid in $B(\mathbf{y}^*, \mathbf{y}^0)$ and also a change of the base point.

This can be summarized as follows:

Proposition 2.7. *Let $(\tau_1, \ldots, \tau_r)$ be an r-tuple of braids representing the braid monodromy of $(\mathcal{C}, L, P)$. Then $(\tilde\tau_1, \ldots, \tilde\tau_r) \in B_d^r$ represents the braid monodromy of $(\mathcal{C}, L, P)$ if and only if $(\tilde\tau_1, \ldots, \tilde\tau_r)$ is in the orbit of $(\tau_1, \ldots, \tau_r)$ by the action of $B_r \times B_d$ on B_d^r.*

Remark 2.8. The choice of a geometric basis in the definition of braid monodromy can be relaxed in several ways.

(a) Let us assume that $(\tau_1, \ldots, \tau_r) \in B_d^r$ represents the braid monodromy of $(\mathcal{C}, L, P)$; note that cyclic permutations of r-tuples representing a braid monodromy are also representatives of such braid monodromy. In order to see this, first note that $(\tau_r, \tau_r * \tau_1, \ldots, \tau_r * \tau_{r-1})$ also represents such braid monodromy since it is obtained from the previous one by Hurwitz moves. Finally, conjugating by τ_r we obtain the desired cyclic permutation.
(b) If we choose an arbitrary base point $*$, we cannot use the term *geometric basis* anymore. The appropriate concept has been introduced in Remark 1.14: a basis $\mu_1, \ldots, \mu_r$ is a *pseudogeometric basis*, if μ_j is a meridian about $x_{\sigma(j)}$ for some $\sigma \in \Sigma_r$ and $(\mu_r \cdot \ldots \cdot \mu_1)^{-1}$ is a meridian about the point at infinity that can be obtained as $\alpha \cdot \delta \cdot \alpha^{-1}$ conjugate of δ, where δ is obtained by surrounding the boundary of a sufficiently large geometric disk Δ clockwise and α is a path in $\mathbb{C}_{\mathcal{C}}$ from $*$ to the boundary of Δ. Braid monodromies are sometimes easier to compute for pseudogeometric bases of $\pi_1(\mathbb{C}_{\mathcal{C}}; *)$ and they also provide r-tuples representing the braid monodromy.
(c) For technical reasons, we will sometimes consider the base point $*$ to be very close to a point in $\mathscr{D}$. Thus, starting from a geometric basis and considering a path joining $*$ with the base point in $\partial\Delta$, one obtains a pseudogeometric basis.

Definition 2.9. Two horizontal triples are said to have *equivalent braid monodromies* if they have the same representatives for their braid monodromies.

Remark 2.10. Note that this equivalence relation is finer than the one arising from curves with the *same* monodromy representation, since we have fixed pseudogeometric bases of $\pi_1(\mathbb{C}_{\mathcal{C}}; *)$.

3. Proof of Theorem 1

In this section, we develop some necessary results to prove Theorem 1.

Definition 3.1. Let $A, B \gg 0$. A polydisk $\tilde\Delta := \Delta_A \times \Delta_B \subset \mathbb{C}^2$ of multiradius (A, B) is *well adapted* to the triple $(\mathcal{C}, L, P)$ if $\mathscr{D} \subset \operatorname{Int} \Delta_A$ and

$$\{(x, y) \in \mathcal{C}^{\text{aff}} |\ x \in \Delta_A\} \subset \Delta_A \times \operatorname{Int} \Delta_B.$$

Lemma 3.2. *There exists a real number $A_0 > 0$ such that for any $A \geq A_0$ there exists $B_0(A) > 0$ satisfying that for any $B \geq B_0(A)$ the polydisk $\tilde{\Delta} := \Delta_A \times \Delta_B$ is well adapted to $(\mathcal{C}, L, P)$.*

We fix A, B such that $\tilde{\Delta} := \Delta_A \times \Delta_B \subset \mathbb{C}^2$ is well adapted to $(\mathcal{C}, L, P)$. Let us choose $*$ near a point in $\mathscr{D}$ and $\hat{*} \in \partial\Delta_B$. We will fix $(*, \hat{*})$ as the base point for the fundamental group of $X_{\mathcal{C}^\varphi}$.

Definition 3.3. Let μ be an element in $\pi_1(\mathbb{C}_\mathcal{C}; *) = \pi_1(\Delta_A \setminus \mathscr{D}; *)$. We say that $\alpha \in \pi_1(X_{\mathcal{C}^\varphi}; (*, \hat{*}))$ is a *suitable lifting* of μ if it is obtained from μ by the morphism j_* on the fundamental groups induced by the composition of the mappings $\Delta_A \setminus \mathscr{D} \to (\Delta_A \setminus \mathscr{D}) \times \{\hat{*}\} \hookrightarrow X_{\mathcal{C}^\varphi}$.

The image of j_* is the set of suitable liftings. The following proposition is the key point of the classical version of the Zariski-Van Kampen theorem. It is a consequence of the long exact sequence of homotopy.

Proposition 3.4. *Let $L^\varphi_* := (\pi^\varphi)^{-1}(*)$ be the fiber of π^φ over $*$ and let $g_1, \ldots, g_d$ be a geometric basis of $\pi_1(L^\varphi_*; (*, \hat{*}))$. Then both the elements $g_1, \ldots, g_d$ and a suitable lifting $\alpha_1, \ldots, \alpha_r$ of a pseudogeometric basis $\mu_1, \ldots, \mu_r$ of $\pi_1(\mathbb{C}_\mathcal{C}; *)$ may be chosen to obtain the following presentation of $\pi_1(X_{\mathcal{C}^\varphi}; (*, \hat{*}))$:*

$$\left\langle g_1, \ldots, g_d, \alpha_1, \ldots, \alpha_r \middle| g_j^{\tau_i} = \alpha_i^{-1} g_j \alpha_i, \quad i = 1, \ldots, r, \quad j = 1, \ldots, d \right\rangle, \tag{3}$$

where $\tau_i = \mathbb{I}_{\mathbf{y}^*}^{-1} \nabla_*(\mu_i)$.

Corollary 3.5. *Any presentation of $\pi_1(X_{\mathcal{C}^\varphi}; (*, \hat{*}))$ in terms of a geometric basis of $\pi_1(L^\varphi_*; (*, \hat{*}))$ and the suitable lifting $\alpha_1, \ldots, \alpha_r$ of a pseudogeometric basis of $\pi_1(\mathbb{C}_\mathcal{C}; *)$ determines the braid monodromy Φ of $(\mathcal{C}, L, P)$.*

We are going to prove that the topology of $(\mathbb{P}^2, \mathcal{C}^\varphi \cup L, L, P)$ not only determines the fundamental group of $\mathcal{C}^\varphi$ but also the presentation (3).

Let us set $G := \pi_1(X_{\mathcal{C}^\varphi}; (*, \hat{*}))$, $H := \pi_1(\mathbb{C}_\mathcal{C}; *)$ and $K := \ker \pi^\varphi_*$; they fit in the short exact sequence

$$1 \to K \longrightarrow G \xrightarrow{\pi^\varphi_*} H \to 1; \tag{4}$$

note that K is identified with $\pi_1(L^\varphi_*; (*, \hat{*}))$. This exact sequence splits by means of the morphism j_*; we define $\tilde{H} := j_*(H)$, which is the free group generated by $\alpha_1, \ldots, \alpha_r$. We will use this notation from now on.

Note that $G = K \ltimes \tilde{H}$ and conjugation by α_i on K reflects the action of the braid τ_i which is the image of μ_i by the braid monodromy.

Lemma 3.6. *The group K is a free group on d generators and it is a normal subgroup of G. Moreover, K is generated by the meridians of the components of $\mathcal{C}^{\text{aff}}$.*

This result is obvious from the presentation of G. Note that we can define K in terms of $\mathcal{C}^{\text{aff}}$ instead of π_*^φ. Let us denote $\gamma := g_d \cdot \ldots \cdot g_1 \in K$; this element does not depend on the choice of a given geometric basis in $\pi_1(L_*^\varphi; (*, \hat{*}))$. We will give a characterization of the geometric bases of K.

Lemma 3.7. *Let $\tilde{g}_1, \ldots, \tilde{g}_d$ be a basis of the free group K, such that $\tilde{g}_1, \ldots, \tilde{g}_d$ are meridians of $\mathcal{C}^{\text{aff}}$ and $\tilde{g}_d \cdot \ldots \cdot \tilde{g}_1 = \gamma$; then $\tilde{g}_1, \ldots, \tilde{g}_d$ is a geometric basis of K.*

Proof. Note that any element of G can be described in the form hk where $k \in K$ and $h \in \tilde{H}$. Let g be any meridian of $\mathcal{C}^{\text{aff}}$. Since every irreducible component of $\mathcal{C}^{\text{aff}}$ meets the line $x = *$, we know that g is conjugate to some g_i, i.e., $g = g_i^{hk}$ for some h, k as above. By the presentation (3), there are g_j and $k' \in K$ such that $g_i^h = g_j^{k'}$. Then g is conjugated to g_j in K.

We have proved that given $i = 1, \ldots, r$, one can choose $\sigma(i) \in \{1, \ldots, r\}$ such that $\tilde{g}_i$ is conjugated to $g_{\sigma(i)}$ in K. The condition $\tilde{g}_d \cdot \ldots \cdot \tilde{g}_1 = \gamma = g_d \cdot \ldots \cdot g_1$ implies that σ is a permutation.

By Artin's characterization of braids as automorphisms of the free group, see Proposition 1.1, we deduce that it is possible to go from $g_1, \ldots, g_d$ to $\tilde{g}_1, \ldots, \tilde{g}_d$ by Hurwitz moves. Hence $\tilde{g}_1, \ldots, \tilde{g}_d$ is also a geometric basis of K. $\square$

We also want to characterize suitable liftings.

Lemma 3.8. *Let $\mu \in H$ and let $\alpha \in G$ be its suitable lifting. The following properties hold for α:*

(a) *$\pi_*^\varphi(\alpha) = \mu$;*

(b) *the conjugation by α on K is a braid automorphism;*

(c) *if μ is a meridian of a point $x_i \in \mathscr{D}$, then α is a meridian of the line L_i.*

Moreover, if μ is a meridian of a point in $\mathscr{D}$ and $\beta \in G$ satisfies (a), (b) *and* (c)*, then $\beta = \alpha$.*

Proof. The three properties are immediate consequences of the definition of suitable liftings and the presentation of G.

To prove the last statement of the lemma, we consider a meridian μ of $x_i \in \mathscr{D}$ and β verifying (a), (b) and (c). By property (a), we know that $\beta = \alpha k$, where $k \in K$. By (b), we deduce that $\gamma = \gamma^\beta = \gamma^{\alpha k} = (\gamma^\alpha)^k = \gamma^k$. Then, $[k, \gamma] = 1$ in the free group K. Since γ cannot be a non-trivial power in K we deduce that $k = \gamma^m$, for some $m \in \mathbb{Z}$.

Let us consider the composition of epimorphisms

$$G \to \pi_1(X_{\mathcal{C}}; (*, \hat{*})) \to H_1(X_{\mathcal{C}}, \mathbb{Z}).$$

By condition (c), the image of α and β by this composition is trivial. Since the image of γ is non-trivial and $H_1(X_{\mathcal{C}}, \mathbb{Z})$ is a free abelian group, then $m = 0$ and hence $\alpha = \beta$. $\square$

Proof of Theorem 1. Note that $\mathscr{D}^{(1)}$ and $\mathscr{D}^{(2)}$ have the same cardinality, say r. We may assume that they are non-empty (otherwise there is nothing to prove).

Given any two triples $(\mathcal{C}_1, L, P)$ and $(\mathcal{C}_2, L, P)$ as in the statement we can consider the elements $g_1^{(1)}, \ldots, g_d^{(1)}$ and $\alpha_1^{(1)}, \ldots, \alpha_r^{(1)}$ described in Proposition 3.4. Let us fix polydisks $\tilde{\Delta}_1$ and $\tilde{\Delta}_2$ well adapted to $(\mathcal{C}_1, L, P)$ and $(\mathcal{C}_2, L, P)$, respectively, such that $F(\tilde{\Delta}_1) \supset \tilde{\Delta}_2$. We fix a point $*^{(1)}$ near the line $L_1^{(1)} = \{x = x_1^{(1)}\}$, where $x_1^{(1)} \in \mathscr{D}^{(1)}$. The short exact sequence (4) associated with $(\mathcal{C}_1, L, P)$ and the above choices is

$$1 \to K^{(1)} \longrightarrow G^{(1)} \xrightarrow{\pi_*^{\varphi(1)}} H^{(1)} \to 1. \tag{5}$$

By considering the conjugation action of $\alpha_1^{(1)}, \ldots, \alpha_r^{(1)}$ on $K^{(1)}$ we recover the braid monodromy of $(\mathcal{C}_1, L, P)$.

After reordering the elements of $\mathscr{D}^{(2)}$ we may assume that $F(L_1^{(1)}) = L_1^{(2)}$. We fix a point $*^{(2)}$ in $\tilde{\Delta}_2$ near the line $L_1^{(2)}$ and in the same fiber of $\pi^{\varphi(2)}$ as $F(*^{(1)})$. We choose a path joining $*^{(2)}$ and $F(*^{(1)})$ contained in a small punctured neighborhood of P inside $\bar{L}_1^{(2)}$. The homeomorphism F and this path produce an isomorphism $\rho : G^{(1)} \to G^{(2)}$. We can consider the short exact sequence for $G^{(2)}$:

$$1 \to K^{(2)} \longrightarrow G^{(2)} \xrightarrow{\pi_*^{\varphi(2)}} H^{(2)} \to 1. \tag{6}$$

Note that the hypotheses of the theorem imply that $F(\mathcal{C}_1) = \mathcal{C}_2$. Then, by Lemma 3.6 we deduce that $\rho(K^{(1)}) = K^{(2)}$.

Note that $\gamma^{(1)}$ is isotopic to the boundary $\tilde{\gamma}^{(1)}$ of a big geometric disk in $L_1^{(1)}$; this isotopy takes place in a small tubular neighborhood of $\tilde{\gamma}^{(1)}$ in $\mathbb{C}^2$. The condition $F(P) = P$ implies that $F(\tilde{\gamma}^{(1)})$ is isotopic to the boundary $\tilde{\gamma}^{(2)}$ of a big geometric disk in $L_1^{(2)}$; the isotopy occurs in a small tubular neighborhood of $\tilde{\gamma}^{(2)}$ in $L_1^{(2)}$. Let us denote by T a small tubular neighborhood of $\tilde{\gamma}^{(2)}$ in $\mathbb{C}^2$. Note that T has the homotopy type of $\mathbb{S}^1$ and that $\rho(\gamma^{(1)})$ is isotopic to $\gamma^{(2)}$ in T. Let us consider $T^* := T \setminus \mathcal{C}_2^{\varphi}$, which has the homotopy type of $\mathbb{S}^1 \times \mathbb{S}^1$. The image $\rho(\gamma^{(1)})$ is homotopic to the product of $\gamma^{(2)}$ and a power of a meridian of $L_1^{(2)}$ in T^*. Since $\gamma^{(1)} \in K^{(1)}$, one has $\rho(\gamma^{(1)}) \in K^{(2)}$ and $\rho(\gamma^{(1)}) = \gamma^{(2)}$.

Let us define

$$\rho(g_j^{(1)}) =: g_j^{(2)}, \; j = 1, \ldots, d,$$

$$\rho(\alpha_j^{(1)}) =: \alpha_j^{(2)},\ j = 1, \ldots, r$$

and

$$\pi_*^{\varphi(2)}(\alpha_j^{(2)}) =: \mu_j^{(2)} \in H^{(2)},\ j = 1, \ldots, r.$$

A priori, the elements with superscript (2) do not have a geometric meaning.

By Lemma 3.7, the orientation hypothesis and the fact that $g_d^{(2)} \cdot \ldots \cdot g_1^{(2)} = \gamma^{(2)}$, we have that $g_1^{(2)}, \ldots, g_d^{(2)}$ is a geometric basis of $K^{(2)}$. It is easily seen that $\mu_1^{(2)}, \ldots, \mu_r^{(2)}$ generate $H^{(2)}$. Since $H^{(2)}$ is a free group of rank r this is in fact a basis. Moreover, $\alpha_j^{(2)}$ is the suitable lifting of $\mu_j^{(2)}$, $j = 1, \ldots, r$, since it satisfies conditions (a), (b) and (c) of Lemma 3.8.

In order to prove that we can recover braid monodromy from the exact sequence (6) and the generators $g_i^{(2)}, \alpha_j^{(2)}$, $i = 1, \ldots, d$, $j = 1, \ldots, r$, it is enough to show that $\mu_1^{(2)}, \ldots, \mu_r^{(2)}$ is a pseudogeometric basis of $H^{(2)}$. Since $\alpha_j^{(2)}$ is a meridian of $L_j^{(2)}$, the set $\alpha_1^{(2)}, ..., \alpha_r^{(2)}$ is a basis of meridians. It only remains to be shown that $\mu^{(2)} := \mu_r^{(2)} \cdot \ldots \cdot \mu_1^{(2)}$ is a meridian of the point at infinity.

Let us consider the images of $\alpha_1^{(1)}, \ldots, \alpha_r^{(1)}$ in $\pi_1(X_{\mathscr{L}^{(1)}}; (*^{(1)}, \hat{*}^{(1)}))$. Since these elements are suitable liftings of elements in a pseudogeometric basis in $\pi_1(\mathbb{C}_{\mathcal{C}_1}; *^{(1)})$, it is easily seen that the image of the product $\alpha_r^{(1)} \cdot \ldots \cdot \alpha_1^{(1)}$ is a meridian of L. By the properties of the homeomorphism F, we deduce that the image of $\alpha_r^{(2)} \cdot \ldots \cdot \alpha_1^{(2)}$ in $\pi_1(X_{\mathscr{L}^{(2)}}; (*^{(2)}, \hat{*}^{(2)}))$ is also a meridian of L, and hence $\mu^{(2)}$ is a meridian of the point at infinity.

Let $M_1 \in (B_d)^r$ be the element representing the braid monodromy of $(\mathcal{C}_1, L, P)$ obtained by means of the exact sequence (5) and the generators $g_i^{(1)}, \alpha_j^{(1)}$, $j = 1, \ldots, d$, $j = 1, \ldots, r$. Analogously we obtain $M_2 \in (B_d)^r$, the element representing the braid monodromy of $(\mathcal{C}_2, L, P)$ obtained from the exact sequence (6) and the generators $g_i^{(2)}, \alpha_j^{(2)}$, $j = 1, \ldots, d$, $j = 1, \ldots, r$. Since ρ respects both the exact sequences and the generators, we deduce that $M_1 = M_2$. $\square$

References

[1] E. Artal Bartolo, J. Carmona and J.I. Cogolludo, *Effective invariants of braid monodromy*, Preprint, 2001.

[2] E. Artin, *Theory of braids*, Ann. of Math. (2) **48** (1947), 101–126.

[3] J. S. Birman, *Braids, links, and mapping class groups*, Princeton University Press, Princeton, N.J., 1974, Annals of Mathematics Studies, No. 82.

[4] J. Carmona, *thesis*, preprint.

[5] O. Chisini, *Una suggestiva rappresentazione reale per le curve algebriche piane*, Ist. Lombardo, Rend., II. Ser. **66** (1933), 1141–1155.

[6] E.R. van Kampen, *On the fundamental group of an algebraic curve*, Amer. J. Math. **55** (1933), 255–260.

[7] Vik. S. Kulikov and M. Teicher, *Braid monodromy factorizations and diffeomorphism types*, Izv. Ross. Akad. Nauk Ser. Mat. **64** (2000), no. 2, 89–120.

[8] A. Libgober, *On the homotopy type of the complement to plane algebraic curves*, J. Reine Angew. Math. **367** (1986), 103–114.

[9] ———, *Invariants of plane algebraic curves via representations of the braid groups*, Invent. Math. **95** (1989), no. 1, 25–30.

[10] ———, *Characteristic varieties of algebraic curves*, Applications of algebraic geometry to coding theory, physics and computation (Eilat, 2001), Kluwer Acad. Publ., Dordrecht, 2001, pp. 215–254.

[11] B. G. Moishezon, *Stable branch curves and braid monodromies*, L.N.M. 862, Algebraic geometry (Chicago, Ill., 1980), Springer, Berlin, 1981, pp. 107–192.

[12] O. Zariski, *On the problem of existence of algebraic functions of two variables possessing a given branch curve*, Amer. J. Math. **51** (1929), 305–328.

Departamento de Matemáticas, Campus Plaza de San Francisco s/n, E-50009 Zaragoza SPAIN

E-mail address: artal@posta.unizar.es

Departamento de Sistemas informáticos y programación, Universidad Complutense, Ciudad Universitaria s/n, E-28040 Madrid SPAIN

E-mail address: jcarmona@sip.ucm.es

Departamento de Matemáticas, Campus Plaza de San Francisco s/n, E-50009 Zaragoza SPAIN

E-mail address: jicogo@posta.unizar.es