Géométrie algébrique/Algebraic Geometry

Invariants of the complement to rational arrangements[1]

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Abstract-*In this note we give a summarized description of two topological invariants of the embedding of a rational arrangement in the projective complex plane, namely, the cohomology ring of the complement and the characteristic varieties.*

Invariants du complément des rationnelle arrangements

Résumé-*Dans cette note, nous donnons une description de deux invariants topologiques du plongement d'un arrangement rationnel dans le plan projectif complexe: L'anneau de cohomologie du complément et les variétés caractéristiques.*

Version française abrégée – Soit $\mathcal{R}$ une courbe algébrique dans le plan projectif complexe $\mathbf{P}^2$. Considérons $\mathcal{R} = \mathcal{C}_0 \cup \mathcal{C}_1 \cup ... \cup \mathcal{C}_n$ sa décomposition en composantes irréductibles. Nous supposons que chaque courbe $\mathcal{C}_i = \{[x:y:z] \in \mathbf{P}^2 \mid C_i(x,y,z) = 0\}$ est rationnelle, i.e., sa résolution est isomorphe à $\mathbf{P}^1$. Pour simplifier, nous supposons que $\mathcal{C}_0$ est une droite transverse à la courbe $\mathcal{C}_1 \cup ... \cup \mathcal{C}_n$. Le but de cette note est d'étudier des invariants topologiques du couple $(\mathbf{P}^2, \mathcal{R})$.

Nous commençons par l'anneau de cohomologie de l'espace $X = \mathbf{P}^2 \setminus \mathcal{R}$ à coefficients dans $\mathbf{C}$, noté $H^\bullet(X, \mathbf{C})$. Cet invariant a été classiquement étudié dans le cas d'arrangements d'hyperplans dans $\mathbf{P}^r$ (ou $\mathbf{C}^{r+1}$) par V.I. Arnold [2], H. Brieskorn [4] et P. Orlik-L. Solomon [11]. Notons que le cas $r = 2$ est un cas particulier du nôtre.

Nous introduisons quelques notations pour pouvoir décrire l'invariant $H^\bullet(X, \mathbf{C})$. Soit d_i le degré de la composante irréductible $\mathcal{C}_i$ et soit $\sigma_i := d\ \log(C_i/C_0^{d_i})$ une 1-forme méromorphe associée à $\mathcal{C}_i$ où $1 \leq i \leq n$. Si $P \in \mathbf{P}^2$, nous notons $\Delta_P(\mathcal{C})$ l'ensemble des branches locales de la courbe $\mathcal{C}$ en P. Si $P \notin \mathcal{C}$ alors $\Delta_P(\mathcal{C}) = \emptyset$ et si $P \in \mathcal{C} \setminus Sing(\mathcal{C})$ alors $\#\Delta_P(\mathcal{C}) = 1$. Si $\mathcal{C}$ a plusieurs branches en P, pour chaque paire $\delta, \delta' \in \Delta_P(\mathcal{C})$ distinctes, nous notons $(\delta, \delta')_P$ la multiplicité d'intersection.

Théorème 1. *Il existe un ensemble de 2-formes*

$$v_2(\mathcal{R}) := \{\psi_P^{\delta,\delta'}\}_{\substack{P \in Sing(\mathcal{C}_i \cup \mathcal{C}_j), \\ \delta \in \Delta_P(\mathcal{C}_i), \delta' \in \Delta_P(\mathcal{C}_j)}} \cup \{\psi_\infty^{i,k_i}\}_{k_i \in \{1,\ldots,d_i-1\}}$$

où $i,j \in \{1,...,n\}$ et un ensemble de 1-formes $v_1(\mathcal{R}) := \{\sigma_i\}_{i \in \{1,\ldots,n\}}$ tel que $H^\bullet(X, \mathbf{C})$ soit engendré par $v(\mathcal{R}) := v_1(\mathcal{R}) \cup v_2(\mathcal{R})$ et la famille suivante donne un ensemble complet de relations:

$$\sigma_i \wedge \sigma_j = \sum_{\substack{P \in \mathcal{C}_i \cap \mathcal{C}_j, \\ \delta \in \Delta_P(\mathcal{C}_i), \delta' \in \Delta_P(\mathcal{C}_j)}} (\delta, \delta')_P\ \psi_P^{\delta,\delta'} + d_j \Sigma_{k_i=1}^{d_i} \psi_\infty^{i,k_i} - d_i \Sigma_{k_j=1}^{d_j} \psi_\infty^{j,k_j},$$

$$\sigma_i \wedge \sigma_j \wedge \sigma_k = 0,$$

$$\psi_P^{\delta,\delta'} = -\psi_P^{\delta',\delta},$$

$$\psi_P^{\delta,\delta'} \wedge \varphi = \varphi \wedge \psi_P^{\delta,\delta'} = 0 \quad \forall \varphi \in v(\mathcal{R}),$$

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$$\sigma_i \wedge \varphi = \varphi \wedge \sigma_i = 0 \quad \forall \varphi \in v_2(\mathcal{R})$$

et

$$\psi_P^{\delta_1,\delta_2} + \psi_P^{\delta_2,\delta_3} + \psi_P^{\delta_3,\delta_1} = 0$$

$$\forall P \in \mathcal{C}_i \cap \mathcal{C}_j \cap \mathcal{C}_k, \delta_1 \in \Delta_P(\mathcal{C}_i), \delta_2 \in \Delta_P(\mathcal{C}_j) \text{ et } \delta_3 \in \Delta_P(\mathcal{C}_k).$$

L'existence de la droite $\mathcal{C}_0$ est une condition technique qui n'impose de vraies restrictions. Si $\mathcal{C}_0$ n'est pas une composante de $\mathcal{R}$, le résultat est le sous-anneau de $H^*(\mathbf{P}^2 \setminus (\mathcal{R} \cup \mathcal{C}_0), \mathbf{C})$ engendré par $d_j\sigma_i - d_i\sigma_j$ où $i, j \in \{1, ..., n\}$ et par $\psi_{P_1}^{\delta_1,\delta_2'} + \psi_{P_2}^{\delta_2,\delta_3'} + ... + \psi_{P_k}^{\delta_k,\delta_1'}$ où δ_i, δ_i' sont branches locales de la même composante irréductible.

Le deuxième type d'invariants que nous étudions sont les variétés caractéristiques qui ont été introduites par A.Libgober [10]. On peut les définir comme suit. Soit $\tilde{X}$ le revêtement universel abelien de $X = \mathbf{P}^2 \setminus \mathcal{R}$. Le groupe d'automorphismes du revêtement $\tilde{X} \to X$ s'identifie à $\pi_1(X)/\pi_1(X)' = H_1(X) = \gamma_1\mathbf{Z} \oplus ... \oplus \gamma_n\mathbf{Z}$ où γ_i est un méridien de la composante $\mathcal{C}_i$. Le premier espace vectoriel d'homologie $M = H_1(\tilde{X}, \mathbf{C})$ est muni d'une structure de $\mathbf{C}[\mathbf{Z}^n]$-module. L'anneau $\Lambda := \mathbf{C}[\mathbf{Z}^n]$ étant noetherien, nous pouvons considérer les idéaux de Fitting $F_k(M)$ du Λ-module M construits à partir d'une présentation finie de M. Dans ces conditions, on dit que la variété caractéristique de profondeur k de X est la sous-variété du tore de dimension n, $\mathrm{Spec}\Lambda = (\mathbf{C}^*)^n$, définie par

$$\mathrm{Char}_k(X) := \mathrm{Supp}_\Lambda(\Lambda/F_k(M)). \tag{1}$$

Il est évident que $\mathrm{Char}_k(X)$ est un invariant topologique du couple $(\mathbf{P}^2, \mathcal{R})$. En fait, il s'agit d'un invariant du groupe fondamental $\pi_1(X)$.
Pour mieux comprendre la structure de ces variétés, nous pouvons les relier avec certains espaces de représentations unidimensionnels de $\pi_1(X)$. Plus précisément, posons

$$V_k(X) := \{\rho \in Hom(\pi_1(X), \mathbf{C}^*) = (\mathbf{C}^*)^n \mid \dim H^1(X, \rho) \geq k\} \tag{2}$$

où $H^1(X, \rho)$ représente la cohomologie de X à coefficients dans le système local ρ. D'après [10], nous avons $\mathrm{Char}_k(X) \cong V_k(X)$. Mieux, la structure des variétés $V_k(X)$ est bien connue, car d'après [1] $V_k(X)$ est une réunion finie de sous-tores (multipliés par un élément de torsion) de dimension plus grande où égale à zéro.
Notre but est de décrire $\mathrm{Char}_k^{\bar{1}}(X)$, i.e., la réunion des composantes irréductibles de $\mathrm{Char}_k(X)$ de dimension positive passant par l'origine. Pour ce faire nous définissons une famille de complexes d'espaces vectoriels paramétrée par $H^1(X, \mathbf{C})$. Pour chaque $\omega \in H^1(X, \mathbf{C})$, nous avons un complexe $(A^\bullet, \omega)$ que nous appelons Complexe Généralisé d'Aomoto qui ne dépend que de la combinatoire de la courbe $\mathcal{R}$ (voir [3]).
Nous notons que chaque $\omega \in H^1(X, \mathbf{C})$ définit un système local noté ρ_ω qui est obtenu à partir de l'exponentielle. Posons

$$\mathcal{V}_k(X) := \{\omega \in H^1(X, \mathbf{C}) \mid \dim H^1(A^\bullet, \omega) \geq k\};$$

alors, nous pouvons énoncer et montrer:

THÉORÈME 2. *Les ensembles $\mathcal{V}_k(X)$ sont des réunions finies de sous-espaces linéaires de dimension $k+1$ et chacun est le revêtement universel d'une composante de $\mathrm{Char}_k^{\bar{1}}(X)$, via l'application qui envoie chaque ω à la représentation ρ_ω.*

En particulier, les composantes de $\mathrm{Char}_k^{\bar{1}}(X)$ ne dépendent que de la combinatoire de la courbe $\mathcal{R}$.

Let $\mathcal{R}$ be an algebraic curve in the complex projective plane $\mathbf{P}^2$. Let us consider $\mathcal{R} = \mathcal{C}_0 \cup \mathcal{C}_1 \cup ... \cup \mathcal{C}_n$ its decomposition in irreducible components. We assume each curve $\mathcal{C}_i := \{[x : y : z] \in \mathbf{P}^2 \mid C_i(x, y, z) = 0\}$ is rational, that is, its function field is isomorphic to $\mathbf{C}(t)$. We will refer to $\mathcal{R}$ as a rational arrangement, which is a generalization of the case where all components are lines, known as a line arrangement. We will also refer to an algebraic hypersurface whose irreducible components are hyperplanes as a hyperplane arrangement. For simplicity, we may assume that $\mathcal{C}_0$ is a line transversal to the curve $\mathcal{C}_1 \cup ... \cup \mathcal{C}_n$. Our purpose is to study topological invariants of the pair $(\mathbf{P}^2, \mathcal{R})$.

A first type of invariant is the cohomology ring $H^\bullet(X, \mathbf{C})$ of the complement $X = \mathbf{P}^2 \backslash \mathcal{R}$ with coefficients in $\mathbf{C}$. This invariant has been classically studied in the case of hyperplane arrangements in $\mathbf{P}^r$ (or $\mathbf{C}^r$) by V.I. Arnold [2], H.Brieskorn [4], P.Orlik and L.Solomon [11]. Note that the case $r = 2$ is a particular case of ours.

To describe this invariant we first fix a resolution $\pi : \overline{X} \longrightarrow \mathbf{P}^2$ of the divisor $(\mathcal{R})$ in $\mathbf{P}^2$ so that $\overline{\mathcal{R}} := (\pi^*\mathcal{R})_{red}$ is a divisor with normal crossings. Then we consider the sheaves $\mathcal{A}^k(\log\langle\overline{\mathcal{R}}\rangle)$ of C^∞-logarithmic k-forms with poles along $\overline{\mathcal{R}}$. By the de Rham type Theorem for logarithmic sheaves [8] we know that

$$H^k(\mathcal{A}^\bullet(\log\langle\overline{\mathcal{R}}\rangle), d) \cong H^k(X, \mathbf{C}).$$

In the case of hyperplane arrangements it is well known that $H^k(X, \mathbf{C})$ is generated by logarithmic 1-forms on $\overline{X}$ with respect to $\overline{\mathcal{R}}$. For rational arrangements this is in general no longer true. In fact, $H^k(X, \mathbf{C})$ is generated by 1-forms if and only if $\mathcal{R}$ is a line arrangement. However, one has the following important result

THEOREM 3. *Any cohomology class of differential logarithmic forms on $\overline{X}$ with respect to $\overline{\mathcal{R}}$ contains a holomorphic representative. Moreover, such a representative is the pull-back by π of a meromorphic form on $\mathbf{P}^2$ with poles along $\mathcal{R}$.*

A presentation for $H^\bullet(X, \mathbf{C})$ is shown in THEOREM 1. Observe that in the case of line arrangements there are no forms of type ψ^*_∞ and the given presentation coincides with the classical one given by Orlik and Solomon [11]. Note also that the cohomology ring depends only on the combinatorial type of the arrangement (see [3] for a definition). Rational arrangements are the only type of arrangements whose cohomology ring of the complement is generated by global meromorphic forms.

The second type of invariant we want to focus on consists of a family of algebraic varieties. An alternative definition to (1) is the following

$$\mathrm{Char}_k(X) := \mathrm{Supp}_\Lambda(\wedge^k H_1(\tilde{X}, \mathbf{C})) \subset \mathrm{Spec}\Lambda = (\mathbf{C}^*)^n \tag{3}$$

(see [9]).
From (1) it is immediate that in the case of irreducible curves the first characteristic variety $\mathrm{Char}_1(X)$ coincides with the set of zeroes of the Alexander polynomial. In the non-irreducible case, the first characteristic variety will hardly ever be a hypersurface. From (3) one can deduce (see [9]) that a closed point m_ρ is in $\mathrm{Char}_k(X)$ if and only if $\dim H_1(X,\rho) \geq k$, where $H_1(X,\rho)$ denotes the homology of X with coefficients in the following local system defined by $\rho = (\rho_1, ..., \rho_n)$

$$\begin{array}{rccc} H_1(X) = & \gamma_1\mathbf{Z} \oplus ... \oplus \gamma_n\mathbf{Z} & \longrightarrow & \mathbf{C}^* \\ & \gamma_i & \mapsto & \rho_i \end{array}$$

By means of a duality theorem [6] and the fact that $Hom(H_1(X), \mathbf{C}^*) = Hom(\pi_1(X), \mathbf{C}^*)$ one has that $\mathrm{Char}_k(X)$ coincides – except maybe at the origin $\overline{1} = (1, ..., 1)$ – with the cohomology support loci $V_i(X)$ defined in (2). It is known from D.Arapura ([1]) that $V_k(X)$ is a finite union of translated tori of dimension greater or equal to zero. We will denote by $V_k^{\overline{1}}(X)$ the components of $V_k(X)$ of positive dimension passing through the origin $\overline{1}$. In what follows we will study more closely the varieties $V_k^{\overline{1}}(X)$, where X is the complement to a rational arrangement in the complex projective plane $\mathbf{P}^2$. We will define a family of complexes $(A^\bullet, d_\omega)$, called the Generalized Aomoto Complexes (GAC), parametrized by $H^1(X, \mathbf{C})$ as follows. We put

$$A^k := H^k(W_k, d)$$

where W_k denotes the vector space of global differential logarithmic k-forms on $\overline{X}$ with respect to $\overline{\mathcal{R}}$. The differential depends on the parameter $\omega \in H^1(X, \mathbf{C}) \cong H^1(W_1, \mathbf{C})$ as follows

$$\begin{array}{rccc} d_\omega: & A^k & \longrightarrow & A^{k+1} \\ & \varphi & \mapsto & \varphi \wedge \omega \end{array}$$

One has the following

THEOREM 4. *For any $\omega \in H^1(X, \mathbf{C})$, the corresponding GAC $(A^\bullet, d_\omega)$ is isomorphic to the complex*

$$0 \longrightarrow H^0(X, \mathbf{C}) \xrightarrow{\wedge\omega} H^1(X, \mathbf{C}) \xrightarrow{\wedge\omega} H^2(X, \mathbf{C}) \longrightarrow 0$$

defined using the ring structure of $H^\bullet(X, \mathbf{C})$. Moreover, if ω satisfies certain non-resonance conditions, then

$$H^k(A^\bullet, d_\omega) \cong H^k(X, \rho_\omega)$$

where

$$\begin{array}{rccc} \rho_\omega : & H_1(X) & \longrightarrow & \mathbf{C}^* \\ & \gamma_i & \mapsto & exp(2\pi\sqrt{-1}\omega(\gamma_i)). \end{array}$$

An analogue to this result for hyperplane arrangements was proved by H. Esnault, V. Schechtman and E. Viehweg [7].
We can therefore define the following stratification of the space of parameters $H^1(X, \mathbf{C}) = Hom(H_1(X), \mathbf{C})$ of GAC's

$$\mathcal{V}_k(X) := \{\omega \in H^1(X, \mathbf{C}) \mid \dim H^1(A^\bullet, \omega) \geq k\}$$

We have the main result stated in THÉORÈM 2, namely, the components of $\mathcal{V}_k(X)$ are the universal covering of the components of $V_k^{\overline{1}}(X) = \mathrm{Char}_k^{\overline{1}}(X)$. An analogue to this result for the case of line arrangements was proved by Libgober [9]. These linear subspaces can also be regarded as tangent to the components of $\mathrm{Char}_k^{\overline{1}}(X)$ at the origin (see D.Cohen and A.Suciu [5]).
As a corollary, from THÉORÈM 1 and THEOREM 4 one has that the stratification $\mathcal{V}_k(X)$ is determined by the combinatorial type of the arrangement and hence so are the $\mathrm{Char}_k^{\overline{1}}(X)$'s.

As an example of an application of characteristic varieties to the study of the topology of the complement, we can prove the following

PROPOSITION. *A sextic with six cusps on a conic cannot degenerate to a sextic with six tacnodes.*

Proof. One can easily check that the only possible sextic with six tacnodes is a union of three conics intersecting pairwise in two tacnodes. Let us denote such a sextic by $\mathcal{C}_{3,2}$. One has

$$\mathrm{Char}_1(\mathbf{P}^2 \setminus \mathcal{C}_{3,2}) = \{(1,-1,1), (-1,1,-1), (-1,-1,1)\} \cup$$

$$\cup \{(1,t,-t^{-1}), (t,1,-t^{-1}), (t,-t^{-1},1)\}$$

On the other hand, if we denote by $\mathcal{C}_6$ a sextic with six cusps on a conic, one has

$$\mathrm{Char}_1(\mathbf{P}^2 \setminus \mathcal{C}_6) = \{\zeta, \overline{\zeta}\},$$

where ζ is a primitive sixth root of unity. If $\mathcal{C}_6$ degenerated into $\mathcal{C}_{3,2}$ then one would have an embedding of characteristic varieties via the surjection of fundamental groups

$$\pi_1(\mathbf{P}^2 \setminus \mathcal{C}_{3,2}) \twoheadrightarrow \pi_1(\mathbf{P}^2 \setminus \mathcal{C}_6)$$

but

$$\{(\zeta,\zeta,\zeta),(\overline{\zeta},\overline{\zeta},\overline{\zeta})\} \not\subset \mathrm{Char}_1(\mathbf{P}^2 \setminus \mathcal{C}_{3,2})$$

and hence the statement follows. $\square$

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