Topological Invariants of the Complement to Arrangements of Rational Plane Curves

José Ignacio Cogolludo-Agustín

Author address:

Department of Mathematics, Statistics and Computer Science, University of Chicago, 851 S. Morgan St. Chicago, IL 60607, USA.

Current address: Departamento de Álgebra, Facultad de CC.Matemáticas, Universidad Complutense de Madrid, Avda. Ciudad Universitaria s/n, 28040 Madrid, Spain.

E-mail address: `jicogo@eucmos.sim.ucm.es`

Contents

Abstract

In the present work we analyze two topological invariants of an embedding of an arrangement of rational plane curves in the projective complex plane, namely, the cohomology ring of the complement and the characteristic varieties. Our main result states that the cohomology ring of the complement to a rational arrangement is generated by logarithmic 1 and 2-forms and its structure depends on a finite number of invariants of the curve (its combinatorial type). This theorem generalizes similar results for arrangements of lines by Brieskorn and Orlik-Solomon. We also define a family of complexes (Generalized Aomoto Complexes) that play an important role in determining the characteristic varieties. They are equivalent to purely combinatorial complexes that were already defined for hyperplane arrangements (Aomoto Complexes). The resonance varieties of such complexes allow one to study the cohomology support loci of rank one local systems on the complement of a curve. In particular, we prove that the irreducible subgroups of the characteristic varieties of a rational arrangement are fully determined by its combinatorial data.

Received by the editor Jan 10, 1999.

2000 *Mathematics Subject Classification.* 32S50, 14Q05, 14B05, 14F45, 14H30, 14H45, 55N25, 32Q55 and 32S20.

Key words and phrases. Characteristic varieties, rational arrangements, plane curves, singularities.

This work was partially supported by CAICYT PB97-0284-C02-01.

Introduction

The present work is devoted to the study of topological invariants of the pair $(\mathbb{P}^2, \mathcal{R})$, where $\mathcal{R}$ is a reduced algebraic curve (possibly singular) in the complex projective plane $\mathbb{P}^2$ whose irreducible components are all rational. Such curves will be called rational arrangements. Although the term "topological invariant" means "invariant up to homeomorphism", all the invariants presented here are in fact invariants of the homotopy type of the complement.

In this paper, we will focus on three important invariants of the pair $(\mathbb{P}^2, \mathcal{R})$, namely, the fundamental group of the complement $X_{\mathcal{R}} := \mathbb{P}^2 \setminus \mathcal{R}$, the cohomology ring of $X_{\mathcal{R}}$, and the sequence of characteristic varieties of $\mathcal{R}$ – defined below. The aim of this paper is to find a presentation for the cohomology ring of $X_{\mathcal{R}}$ and develop a method to calculate special components of the characteristic varieties.

The motivation for using rational arrangements is two-fold. On one hand, rational arrangements are a natural generalization of line arrangements, for which the invariants mentioned above can be effectively calculated. On the other hand, the class of rational arrangements is of interest not only in its own right, but also because it can provide information about more general plane curves. For instance, degeneration techniques were applied by Zariski [**54**] to calculate the fundamental group of the complement of curves that degenerate to certain rational curves.

The fundamental group, cohomology ring and characteristic varieties are defined for an arbitrary algebraic plane curve, say $\mathcal{C}$. From now on, we will denote by $X_{\mathcal{C}}$ the complement of $\mathcal{C}$ in the complex projective plane $\mathbb{P}^2$. In order to give a more precise description of the problems we are interested in and the results we obtain, let us recall some basic results about these invariants.

Fundamental group of the complement, $\pi_1(X_{\mathcal{C}})$. It is one of the most widely used topological invariants of the pair $(\mathbb{P}^2, \mathcal{C})$. Techniques for computing a finite presentation of these groups were first carried out by Zariski ([**56**] Chapter 8) and van Kampen [**31**] in the early 30's. These invariants are sensitive to algebraic and geometrical properties of the curve $\mathcal{C}$ (such as degree or position of singularities). However, the way this information is encoded is not well understood. Also, the practical complexity of using finitely presented groups makes this invariant difficult to work with.

Cohomology ring of the complement, $H^*(X_{\mathcal{C}})$. Its structure is well known for hyperplane arrangements. A presentation was first found by V.I.Arnold [**2**] for hyperplane braid arrangements and generalized by Brieskorn ([**9**], Lemma 5) to complexifications of real hyperplane arrangements. Finally, Orlik-Solomon [**44**] not only extended the presentation to general hyperplane arrangements, but also described it combinatorially in terms of a finite invariant called the **intersection lattice** of the arrangement. For the particular case of line arrangements one has the following.

THEOREM 0.1. *Let $\mathcal{L}$ be a line arrangement with components ℓ_0, ℓ_1, ..., ℓ_n, where each ℓ_i is the zero set of a linear form l_i. Then the logarithmic 1-forms*

$$\omega_i = \frac{1}{2\pi\sqrt{-1}}\ d\ \log\left(\frac{l_i}{l_0}\right) \quad i = 1, ..., n$$

generate the cohomology ring $H^*(X_{\mathcal{L}})$. Moreover, $H^*(X_{\mathcal{L}})$ is isomorphic to the subalgebra generated by the ω_i's in the algebra of meromorphic forms.

As mentioned above, the cohomology ring $H^*(X_{\mathcal{L}})$ depends only on the so called intersection lattice of $\mathcal{L}$, which consists of a finite number of geometrical data of $\mathcal{L}$ also called the **combinatorial type of $\mathcal{L}$**. The combinatorial type of a line arrangement is a partially ordered set whose elements are the components of $\mathcal{L}$ and the singular points of $\mathcal{L}$. The partial order is given by the relation belonging (i.e. $P < \ell \Leftrightarrow P \in \ell$). A presentation of $H^*(X_{\mathcal{L}})$ can be given as follows:

$$H^*(X_{\mathcal{L}}) \cong \langle \sigma_i, \sigma_{jk} : \begin{array}{l} \sigma_i \wedge \sigma_j = \sigma_{ij}, \\ \sigma_i \wedge \sigma_{jk} = \sigma_{ij} \wedge \sigma_{kl} = 0 \\ \sigma_{i_1 i_2} + \sigma_{i_2 i_3} + \sigma_{i_3 i_1} = 0 \end{array} \rangle, \tag{0.1}$$

where $i, j, k \in \{1, ..., n\}$ and $\ell_{i_1} \cap \ell_{i_2} \cap \ell_{i_3} \neq \emptyset$.

Sequence of characteristic varieties of $\mathcal{C}$, $\{\mathrm{Char}_k(\mathcal{C})\}_{k\in\mathbb{N}}$. This sequence of algebraic varieties was first defined by Libgober in [**37**]. It was originally used to obtain information about all abelian covers of the complex projective plane $\mathbb{P}^2$ ramified along the curve $\mathcal{C}$. Characteristic varieties can be defined as follows.

DEFINITION 0.2. Let us denote by π_1 the fundamental group of the complement of the curve $\mathcal{C}$ in $\mathbb{P}^2$. The quotient $M_{\mathbb{Z}} := \pi_1/\pi_1$ has a natural structure of $\mathbb{Z}[\pi_1/\pi_1']$-module – c.f. Section 1.2. Tensoring both the group ring and module by $\mathbb{C}$, $M_{\mathbb{C}} := M_{\mathbb{Z}} \otimes \mathbb{C}$ becomes a $\mathbb{C}[\pi_1/\pi_1']$-module. **The k-th characteristic variety** is the support of the k-th exterior power of the module $M_{\mathbb{C}}$, that is, the set of prime ideals $\mathfrak{p}$ of $\mathbb{C}[\pi_1/\pi_1']$ such that $\wedge^k M_{\mathbb{C}}$ becomes trivial when localizing at $\mathfrak{p}$. We will denote the k-th characteristic variety of $\mathcal{C}$ by $\mathrm{Char}_k(X_{\mathcal{C}})$ or simply by $\mathrm{Char}_k(\mathcal{C})$.

Note that $\mathrm{Char}_k(\mathcal{C})$ is a subvariety of

$$\mathrm{Spec}\ \mathbb{C}[\pi_1/\pi_1'] \cong \mathrm{Spec}\ \mathbb{C}[\mathbb{Z}^n \oplus \mathbb{Z}_{d'}], \tag{0.2}$$

where $n+1$ is the number of irreducible components of the arrangement $\mathcal{C}$ and d' is the greatest common divisor of their degrees. The right-hand side of (0.2) is the disjoint union of d' tori defined by

$$\mathbb{T}_n^{d'} := \{(t_1, ..., t_n, t_{n+1}) \in (\mathbb{C}^*)^{n+1} \mid \ t_{n+1}^{d'} = 1\}.$$

We will often refer to this algebraic group simply as $\mathbb{T}$ when no ambiguity is likely to arise.

By definition, the varieties $\mathrm{Char}_k(\mathcal{C})$ are invariants of the fundamental group $\pi_1(X_{\mathcal{C}})$ and hence, topological invariants of the pair $(\mathbb{P}^2, \mathcal{C})$. These invariants come from knot theory, where one simply has to consider $\pi_1(S^3 \setminus K)$ as π_1 in the above definition −c.f. [**27**]. In the case of a link L, it follows directly from the definition that the first characteristic variety $\mathrm{Char}_1(L)$ is the set of zeroes of the multivariable Alexander polynomial. In the case of irreducible curves, the set of zeroes of the one-variable Alexander polynomial coincides with $\mathrm{Char}_1(\mathcal{C})$. In the non-irreducible case, although the Alexander ideal is no longer principal, $\mathrm{Char}_1(\mathcal{C})$ still coincides with its set of zeroes. It will be seen later that $\mathrm{Char}_k(\mathcal{C})$ are finite unions of smooth varieties.

It is known that if $\mathcal{L}$ happens to be a line arrangement, then all components of $\mathrm{Char}_k(\mathcal{L})$ passing through the origin $\overline{1} = (1, 1, ..., 1) \in \mathbb{T}$ can be computed from the combinatorial type of $\mathcal{L}$ (c.f. Cohen-Suciu [**16**] and independently Libgober [**37**]).

The key ingredient that allows one to compute these subvarieties is their relation to the cohomology support loci of rank one local systems on $X_{\mathcal{C}} = \mathbb{P}^2 \setminus \mathcal{C}$. One can define this loci as

$$\Sigma_k(X_{\mathcal{C}}) := \{\rho \in \mathrm{Hom}(\pi_1/\pi_1', \mathbb{C}^*) \mid \dim H^1(X_{\mathcal{C}}, \rho) \geq k\}.$$

Hironaka [**29**] and Libgober [**37**] have studied the relation between $\Sigma_k(X_{\mathcal{C}})$ and the characteristic variety $\mathrm{Char}_k(\mathcal{C})$. The main result in this direction states that there is a bijection between the closed points of $\mathrm{Char}_k(\mathcal{C})$ and $\Sigma_k(X_{\mathcal{C}})$ −c.f. Proposition 1.13. The space $\Sigma_k(X_{\mathcal{C}})$ has been studied in a more general context by Arapura ([**1**], Theorem 1.6) based on the studies by Green and Lazarsfeld [**23**]. Arapura's structure theorem for the stratification $\Sigma_\bullet$ of the space of rank one representations applied to our case reads as follows:

THEOREM 0.3. *Let $\mathcal{C}$ be an algebraic curve in $\mathbb{P}^2$. Then, there exists a finite number of torsion characters $\rho_r \in H^1(X_{\mathcal{C}}, \mathbb{C}^*)$, unitary characters ρ_s, and surjective holomorphic maps $f_r : X_{\mathcal{C}} \to \mathcal{C}_r = \mathbb{P}^1 \setminus \{p_1^r, ..., p_{i_r}^r\}$ that can be extended to holomorphic maps on a smooth compactification of $X_{\mathcal{C}}$ (i.e. $\overline{f}_r : \overline{X}_{\mathcal{C}} \to \mathbb{P}^1$) such that*

$$\Sigma_k(X_{\mathcal{C}}) = \bigcup_r \rho_r f_r^* H^1(\mathcal{C}_r, \mathbb{C}^*) \cup \bigcup_s \rho_s \quad .$$

In particular, this theorem[1] shows that any characteristic variety is a finite union of tori translated by the torsion points ρ_r.

Let $\mathcal{L}$ be a line arrangement and $\rho = (\rho_1, ..., \rho_n) \in \mathrm{Hom}(\pi_1(X_{\mathcal{L}}), \mathbb{C}^*)$ a local system on $\pi_1(X_{\mathcal{L}})$. Such a local system defines a global logarithmic 1-form $\omega \in H^0(X_{\mathcal{L}}, \mathbb{C})$ by

$$\omega_\rho = \sum_{i=1}^n \alpha_i \ d \ \log\left(\frac{l_i}{l_0}\right), \quad \text{and } \rho_i = \exp(2\pi\sqrt{-1}\alpha_i).$$

An important tool in the study of cohomology with coefficients in ρ is the Aomoto complex $(H^\bullet, \omega_\rho)$ defined by

$$\begin{aligned} 0 \longrightarrow H^0(X_{\mathcal{L}}, \mathbb{C}) \xrightarrow{\wedge\omega_\rho} H^1(X_{\mathcal{L}}, \mathbb{C}) \xrightarrow{\wedge\omega_\rho} H^2(X_{\mathcal{L}}, \mathbb{C}) \longrightarrow \ldots \\ \ldots \xrightarrow{\wedge\omega_\rho} H^r(X_{\mathcal{L}}, \mathbb{C}) \longrightarrow 0 \end{aligned} \tag{0.3}$$

[1] In [**1**], the theorem is proved only for $k = 1$. An extension of this theorem to any Σ_k can be found in [**39**] Theorem 5.1.

This construction is in fact valid for general hyperplane arrangements. In [**20**], Esnault, Schechtman and Viehweg showed that, under certain conditions on ω_ρ, the cohomology of the complex $(H^\bullet, \omega_\rho)$ is isomorphic to the cohomology of $X_{\mathcal{L}}$ with coefficients in ρ.

Note that the family of Aomoto complexes is parametrized by $H^1(X_{\mathcal{L}}; \mathbb{C})$. Such a space admits a stratification given by the cohomology loci of $(H^\bullet, \omega)$, as follows

$$\mathcal{V}_k(\mathcal{L}) := \{\omega \in H^1(X_{\mathcal{L}}; \mathbb{C}) \mid \dim H^1(H^\bullet, \omega) \geq k\}.$$

The stratum $\mathcal{V}_k = \mathcal{V}_k(\mathcal{L})$ is called the **k-th resonance variety** of $\mathcal{L}$.

Answering a conjecture by Falk [**21**], Cohen and Suciu [**16**] and independently Libgober [**37**] proved that the support cohomology loci of the complexes (0.3) are finite unions of linear subspaces.

Both approaches refer to the components of $\operatorname{Char}_k(\mathcal{L})$ passing through the trivial character. Such a variety will be denoted by $\operatorname{Char}_k^{\bar{1}}(\mathcal{L})$. Cohen and Suciu proved that $\mathcal{V}_k(\mathcal{L})$ can be identified with the tangent cone of $\operatorname{Char}_k^{\bar{1}}(\mathcal{L})$ at the origin, whereas Libgober proved that $\mathcal{V}_k(\mathcal{L})$ is the universal cover of $\operatorname{Char}_k^{\bar{1}}(\mathcal{L})$. Moreover, Libgober and Yuzvinsky [**38**] proved that $\mathcal{V}_k(\mathcal{L})$ has dimension $k+1$.

Subsequent generalizations of Falk's conjecture are proved in [**13**] and [**36**].

Our first, and probably most technical, result is an analogue of Brieskorn's Lemma (c.f. Theorem 0.1) for rational arrangements. We prove that the cohomology ring of the complement to any rational arrangement $\mathcal{R}$ is generated by global holomorphic 1 and 2-forms on the complement of $\mathcal{R}$. In fact, it can be deduced from our results in Chapter 2 that rational arrangements are the only type of arrangements for which this is the case. The structure of the cohomology ring is described in Theorems 2.46 and 2.47, and can be summarized as follows:

THEOREM 0.4. *Let $\mathcal{R} = \mathcal{C}_0 \cup \mathcal{C}_1 \cup \ldots \cup \mathcal{C}_n$ be a rational arrangement where $\mathcal{C}_0$ is a transversal line at infinity. The set of local branches of $\mathcal{R}$ at P will be denoted by $\Delta_P(\mathcal{R})$. Then there exists a set of 1-forms*

$$v_1(\mathcal{R}) = \{\sigma_i\}_{i \in \{1,\ldots,n\}}$$

and a set of 2-forms

$$v_2(\mathcal{R}) = \{\psi_P^{\delta,\delta'}\}_{\substack{P \in \operatorname{Sing}(\mathcal{C}_i \cup \mathcal{C}_j),\\ \delta \in \Delta_P(\mathcal{C}_i), \delta' \in \Delta_P(\mathcal{C}_j)}} \cup \{\psi_\infty^{i,k_i}\}_{k_i \in \{1,\ldots,d_i-1\}}, \quad i,j \in \{1,\ldots,n\},$$

such that the ring $H^*(X_{\mathcal{R}}; \mathbb{C})$ is generated by $v(\mathcal{R}) = v_1(\mathcal{R}) \cup v_2(\mathcal{R})$. Moreover, the following is a complete system of relations

$$\sigma_i \wedge \sigma_j = \sum_{\substack{P \in \mathcal{C}_i \cap \mathcal{C}_j,\\ \delta \in \Delta_P(\mathcal{C}_i), \delta' \in \Delta_P(\mathcal{C}_j)}} (\delta,\delta')_P \ \psi_P^{\delta,\delta'} + d_j \sum_{k_i=1}^{d_i} \psi_\infty^{i,k_i} - d_i \sum_{k_j=1}^{d_j} \psi_\infty^{j,k_j},$$

$$\sigma_i \wedge \sigma_j \wedge \sigma_k = 0,$$

$$\psi_P^{\delta,\delta'} = -\psi_P^{\delta',\delta},$$

$$\psi_P^{\delta,\delta'} \wedge \varphi = \varphi \wedge \psi_P^{\delta,\delta'} = 0 \quad \forall \varphi \in v(\mathcal{R}),$$

$$\sigma_i \wedge \varphi = \varphi \wedge \sigma_i = 0 \quad \forall \varphi \in v_2(\mathcal{R})$$

and

$$\psi_P^{\delta_1,\delta_2} + \psi_P^{\delta_2,\delta_3} + \psi_P^{\delta_3,\delta_1} = 0$$

$$\forall P \in \mathcal{C}_i \cap \mathcal{C}_j \cap \mathcal{C}_k, \delta_1 \in \Delta_P(\mathcal{C}_i), \delta_2 \in \Delta_P(\mathcal{C}_j) \text{ and } \delta_3 \in \Delta_P(\mathcal{C}_k),$$

where $(\delta, \delta')_P$ *denotes the multiplicity of intersection of* δ *and* δ' *at* P.

A detailed description of $v(\mathcal{R})$ can be found in Chapter 2. Also note that the relations listed in Theorem 0.4 do not refer to the forms themselves, but to their cohomology classes. The main tool to compare cohomology classes is the Poincaré residue operator, introduced in Preliminaries.

The main difference between the cohomology ring structure of rational arrangements and line arrangements is that the graded algebra $H^*(X_{\mathcal{R}}; \mathbb{C})$ is in general not generated by $H^1(X_{\mathcal{R}}; \mathbb{C})$. Moreover, $H^*(X_{\mathcal{R}}; \mathbb{C})$ is generated by $H^1(X_{\mathcal{R}}; \mathbb{C})$ if and only if $\mathcal{R}$ is a line arrangement.

In Chapter 3, a family of complexes of vector spaces $(A^\bullet, \omega)$ is associated with each rational arrangement. This is the family of Generalized Aomoto Complexes (GAC's). Such families are parametrized by $H^1(X_{\mathcal{R}}; \mathbb{C})$ and generalize the classical Aomoto Complexes (0.3) in a natural way.

Our second main theorem states that the cohomology of $(A^\bullet, \omega)$ is determined by the combinatorial type of $\mathcal{R}$ –c.f. Definition 3.2. This result is a consequence of the following. First, the cohomology of $(A^\bullet, \omega)$ is an invariant of the cohomology ring $H^*(X_{\mathcal{R}}; \mathbb{C})$. Second, the combinatorial type of a rational arrangement determines the incidence map of $\mathcal{R}$, the multiplicities of intersection of local branches at singular points, and the degrees of the irreducible components of $\mathcal{R}$. Hence, the combinatorial type of a rational arrangement determines its cohomology ring.

Our third result, given in Chapter 4, relates GAC's and cohomology of local systems on $X_{\mathcal{R}}$. We prove the following

Theorem 0.5. *Let* $\mathcal{R}$ *be a rational arrangement, and let* ω *be a complex linear 1-form, viewed as a complex homomorphism of* $H_1(X_{\mathcal{R}})$. *Let* ρ_ω *be the local system obtained by composing* ω *with the exponential map. With this notation,*

$$H^*(A^\bullet, \omega) \cong H^*(X_{\mathcal{R}}, \rho_\omega)$$

on the complement of a discrete set of 1-forms ω.

This theorem is a generalization –in dimension 2– of the result by Esnault, Schechtman and Viehweg [**20**] mentioned above.

One can stratify the space of parameters of the GAC's $(A^\bullet, \omega)$ using the resonance varieties

$$\mathcal{V}_k(\mathcal{R}) = \{\omega \in H^1(X_{\mathcal{R}}; \mathbb{C}) \mid \dim H^1(A^\bullet, \omega) \geq k\}.$$

With this notation we state our last main result.

Theorem 0.6. *The resonance varieties* $\mathcal{V}_k(\mathcal{R})$ *are a finite union of linear spaces of dimension* $k+1$. *Moreover, there is a bijection between the irreducible components of* $\mathcal{V}_k(\mathcal{R})$ *and* $\operatorname{Char}^{\overline{1}}_k(\mathcal{R})$.

Summarizing, one has the following

Corollary 0.7. *Let* $\mathcal{R}$ *be a rational arrangement. Then the space* $\operatorname{Char}^{\overline{1}}_k(\mathcal{R})$ *is completely determined by the combinatorial type of the arrangement.*

It is important to note that these generalizations of similar results for hyperplane arrangements do not apply in other contexts, such as arrangements of real planes in $\mathbb{R}^4$ (c.f. [**40**]) or links in S^3 (c.f. [**41**]).

Due to their relatively simple computability and dependency on fundamental groups, characteristic varieties can be used as fine invariants for a number of detection-and-distinction tasks in the study of pairs $(\mathbb{P}^2, \mathcal{C})$. For instance, in Chapter 5, we find Zariski pairs using characteristic varieties. This method detects Zariski pairs that no other method does – other than the direct comparison of fundamental groups. Characteristic varieties also give us restrictions on possible degenerations of curves. As an example, we find that a particular curve, a degeneration of the family of sextics with six cusps not on a conic, cannot be a degeneration of the family of sextics with six cusps on a conic. We also study the effect of birational transformations of $\mathbb{P}^2$ on $\mathrm{Char}_k(\mathcal{C})$. Finally, we state some open questions about characteristic varieties and rational or non-rational arrangements that we hope will enhance the study of these interesting invariants.

An announcement of this paper has already appeared in [**14**].

Acknowledgements

Thanks to Anatoly Libgober for his patience and willingness to transmit his broad knowledge. Thanks to the University of Illinois that gave me the opportunity to work with a great group of faculty and graduate students. Thanks to Enrique Artal and family for their support. A very special thanks to Mireya.

CHAPTER 1

Preliminaries

In this section we want to set the definitions and notation that will be particularly useful throughout this work.

1.1. Cohomology with coefficients in a local system

A local system of rank n on a topological space X is a representation of the fundamental group $\pi_1(X)$ of X on $\mathbb{C}^n$ –that is, a homomorphism from $\pi_1(X)$ to the general linear group $\mathrm{GL}(n,\mathbb{C})$. We recall that a vector bundle has a flat structure, or simply is flat, if it admits a flat connection –that is, a connection whose curvature vanishes. This is equivalent to the following –see, for instance, [**32**] Proposition 1.2.5. A vector bundle $\mathcal{V}$ of rank n over X is flat if the transition functions for an open cover $\{(V_\alpha, s_{V_\alpha})\}_{\alpha\in\Lambda}$ are constant. Flat vector bundles have the property that parallel transport along paths only depends on the homotopy class of the path. Parallel transport of flat vector bundles can be described as a collection $\{\mathcal{V}_x \mid x \in X\}$ of n-dimensional vector spaces and homomorphisms

$$h_\lambda : \mathcal{V}_x \longrightarrow \mathcal{V}_y$$

for each $\lambda \in \pi_1(X; x, y)$ satisfying:

i) if λ is homotopically trivial in $\pi_1(X, x)$ then $h_\lambda = id_{\mathcal{V}_x}$

ii) if $\lambda \in \pi_1(X; x, y)$ and $\mu \in \pi_1(X; y, z)$ then $h_{\lambda*\mu} = h_\mu \circ h_\lambda$.

where $\pi_1(X; x, y)$ is the set of homotopy classes of paths λ in X so that $\lambda(0) = x$, $\lambda(1) = y$ –that is, the set of morphisms from y to x of the fundamental groupoid of X.

Any local system ρ of rank n defines a flat vector bundle $\mathcal{V}_\rho$ of rank n over X as follows. Let $* \in X$ be a base point and $\lambda_x \in \pi_1(X; *, x)$ a path for each $x \in X$. One can define $\mathcal{V}_\rho = X \times \mathbb{C}^n$ where the trivialization functions are given by

$$h_{\lambda_x} = 1_{\mathbb{C}^n}$$

and

$$h_\mu = \rho(\lambda_x * \mu * \lambda_y^{-1}) \in \mathrm{Hom}(\mathcal{V}_{\rho,x}, \mathcal{V}_{\rho,y})$$

for any $\mu \in \pi_1(X; x, y)$. Conversely, a flat vector bundle gives rise to a representation

$$\rho : \pi_1(X, *) \longrightarrow \mathrm{Aut}(\mathbb{C}^n) = \mathrm{GL}(n, \mathbb{C})$$

for any $* \in X$ –see, for instance, [**32**] for a classical reference.

From a more topological point of view, one can reconstruct $\mathcal{V}_\rho$ as follows. Let $\tilde{X}_u$ be the universal covering of X and $\tilde{\mathcal{V}}_\rho = \tilde{X}_u \times \mathbb{C}^n$ the trivial bundle of rank n over $\tilde{X}_u$. The fundamental group $\pi_1(X)$ can be identified with the group of fiber automorphisms –or deck transformations– of $\tilde{X}_u$. Therefore $\pi_1(x)$ acts on $\tilde{X}_u$ –this action will be denoted by $\lambda \cdot x$, where $\lambda \in \pi_1(x)$ and $x \in \tilde{X}_u$. Note that $\mathbb{C}^n$ inherits a structure of $\mathbb{Z}[\pi_1(X)]$-module via ρ, where

$$\lambda \cdot u = \rho(\lambda)u, \tag{1.1}$$

–we will write $\mathbb{C}^n_\rho$ when referring to the space $\mathbb{C}^n$ with the module structure induced by ρ. Then

$$\mathcal{V}_\rho = \tilde{\mathcal{V}}_\rho / \sim_\rho$$

where $(x,u) \sim_\rho (x',u')$ if and only if there exists $\lambda \in \pi_1(X)$ such that $\lambda \cdot x = x'$ and $\lambda \cdot u = u'$.

Now one can define homology and cohomology with coefficients in a local system ρ. Let X be a CW-complex and $C_p(X)$ the free abelian groups generated by the set of p-cells of X. Then $\tilde{X}_u$ naturally inherits a structure of CW-complex such that the action of $\pi_1(X)$ on $\tilde{X}_u$ can be passed over $C_p(\tilde{X}_u)$. Moreover, $C_p(\tilde{X}_u)$ is a free $\pi_1(X)$-module and the quotient of $C_p(\tilde{X}_u)$ by this action is isomorphic to $C_p(X)$. Thus, after choosing a basis, one can write $C_p(\tilde{X}_u)$ as $C_p(X) \otimes \mathbb{Z}[\pi_1(X)]$. Let us define

$$C_p(X, \mathcal{V}_\rho) = C_p(X,\rho) := C_p(\tilde{X}_u) \otimes_{\pi_1(X)} \mathbb{C}^n_\rho$$

and $\delta_{\rho,p} := \tilde{\delta}_p \otimes id_{\mathbb{C}^n_\rho}$, where $\tilde{\delta}_p$ is the boundary operator on $C_p(\tilde{X}_u)$.

Definition 1.1. The **homology of X with coefficients in the local system ρ** is the homology of the complex $C_p(X,\rho)$, that is,

$$H_p(X,\rho) := H_p(C_*(X,\rho), \delta_{\rho,*}).$$

One can also define the cochain complex $C^p(X,\rho) := \mathrm{Hom}_{\pi_1(X)}(C_p(\tilde{X}_u), \mathbb{C}^n_\rho)$ and the coboundary operator δ^p_ρ satisfying

$$\begin{array}{ccc} C_{p+1}(\tilde{X}_u) & \xrightarrow{\delta^p_\rho(h_p)} & \mathbb{C}^n_\rho \\ e_{p+1} & \mapsto & h_p(\tilde{\delta}_{p+1} e_{p+1}), \end{array}$$

where $h_p \in C^p(X,\rho)$. In this dual situation one has the following

Definition 1.2. The **cohomology of X with coefficients in the local system ρ** is the homology of the complex $C^p(X,\rho)$, that is,

$$H^p(X,\rho) := H_p(C^*(X,\rho), \delta^*_\rho).$$

Note that if X is a finite CW-complex, then $C_p(X)$ is a finitely generated group. Subsequently, $C_p(X,\rho)$ is a finitely generated $\mathbb{Z}[\pi_1(X)]$-module and hence

$$C^p(X,\rho) \cong \mathrm{Hom}_{\pi_1(X)}(C_p(\tilde{X}_u), \mathbb{Z}[\pi_1(X)]) \otimes_{\pi_1(X)} \mathbb{C}^n_\rho \cong$$

$$\cong C^p(\tilde{X}_u) \otimes_{\pi_1(X)} \mathbb{C}^n_\rho. \tag{1.2}$$

This isomorphism of vector spaces becomes an isomorphism of complexes if we think of the right-most term in (1.2) as a complex with coboundary operator $\tilde{\delta}^* \otimes id_{\mathbb{C}^n_\rho}$. Hence

$$H^p(X,\rho) \cong H_p(C^*(\tilde{X}_u) \otimes_{\pi_1(X)} \mathbb{C}^n_\rho). \tag{1.3}$$

In some specific situations the complex $C_p(X,\rho)$ is isomorphic to a simpler one. For instance, that is the case if ρ factors through a group G as follows

$$\pi_1(X) \overset{j}{\twoheadrightarrow} G \xrightarrow{\rho'} \mathrm{GL}(n,\mathbb{C}),$$

where $\rho = \rho' \circ j$. Let's denote by $\tilde{X}_G$ the cover of X with Galois group G corresponding to the projection j. Then

$$C_p(\tilde{X}_u) \otimes_{\pi_1(X)} \mathbb{C}^n_\rho \cong C_p(\tilde{X}_G) \otimes_G \mathbb{C}^n_{\rho'} \cong (\mathbb{C}^n)^{\dim C_p(X)}$$

as vector spaces, and their boundary operators commute. Therefore

$$H_p(X, \rho) \cong H_p(C_*(\tilde{X}_G) \otimes_G \mathbb{C}^n_{\rho'}).$$

Note that any rank one ($n = 1$) local system is abelian, hence it factors through $H_1(X, \mathbb{Z})$. Therefore, homology and cohomology with coefficients in rank one local systems can be calculated using the abelian cover, $\tilde{X}_{ab}$, of X. By the Universal Coefficient Theorem –c.f. [**12**] Ch. XVI, Th. 8.4– homology with coefficients in a local system can be calculated as the abutment of the following spectral sequence

$$H_p(H_1(X), H_q(\tilde{X}_{ab}; \mathbb{C})_\rho) \Rightarrow H_{p+q}(X, \rho). \tag{1.4}$$

In (1.4), $H_q(\tilde{X}_{ab}; \mathbb{C})_\rho$ denotes the q-th homology of

$$C_*(\tilde{X}_{ab}; \mathbb{C}^n) = C_*(\tilde{X}_{ab}) \otimes \mathbb{C}^n$$

with the $H_1(X)$-module structure given by the diagonal action

$$g\ (e \otimes \alpha) = g \cdot e \otimes \rho(g^{-1})\alpha,$$

where $g \in H_1(X)$, $e \in C_q(\tilde{X}_{ab})$ and $\alpha \in \mathbb{C}_\rho$. Using the 5-term exact sequence of low-dimensional homology groups one obtains

$$\begin{aligned} H_2(X, \rho) \longrightarrow H_2(H_1(X), \rho) \longrightarrow H_1(\tilde{X}_{ab}; \mathbb{C})_\rho \otimes_{H_1(X)} \mathbb{Z} \longrightarrow \\ \longrightarrow H_1(X, \rho) \longrightarrow H_1(H_1(X), \rho) \longrightarrow 0. \end{aligned} \tag{1.5}$$

In case ρ is not trivial, one has $H_i(H_1(X), \rho) = 0$, hence (1.5) gives

$$H_1(\tilde{X}_{ab}; \mathbb{C})_\rho \otimes_{H_1(X)} \mathbb{Z} \cong H_1(X, \rho).$$

Note also that

$$H_1(\tilde{X}_{ab}; \mathbb{C})_\rho \otimes_{H_1(X)} \mathbb{Z} \cong H_1(\tilde{X}_{ab}; \mathbb{C}_\rho) \otimes_{H_1(X)} \mathbb{Z}$$

as vector spaces, and

$$H_1(\tilde{X}_{ab}; \mathbb{C}_\rho) \otimes_{H_1(X)} \mathbb{Z} = H_1(\tilde{X}_{ab}; \mathbb{C}) \otimes_{\mathbb{C}[H_1(X)]} \mathbb{C}_\rho.$$

Summarizing, one has the following

PROPOSITION 1.3. *Let X be a finite CW-complex and ρ a non-trivial rank one local system on X, then*

$$H_1(X, \rho) = H_1(\tilde{X}_{ab}; \mathbb{C}) \otimes_{\mathbb{C}[H_1(X)]} \mathbb{C}_\rho.$$

Finally, we state a duality theorem for local systems on finite CW-complexes.

PROPOSITION 1.4 ([**17**] p. 14). *If X is a finite CW-complex, then*

$$H^p(X, \rho^*) \cong H_p(X, \rho)^*$$

where ρ^* is the dual fiber bundle to ρ.

Later on, in Chapter 4, another method to calculate cohomology with coefficients in a local system will be described.

1.2. Characteristic varieties

Hironaka [**29**] and Libgober [**34**] introduced a new topological invariant of the complement to algebraic plane curves: the sequence of characteristic varieties. In this section, characteristic varieties are defined, first for general noetherian modules and then for finitely presented groups. The main purpose of this section is to relate characteristic varieties of the complement X of a curve to the cohomology of X with coefficients in rank one local systems. Our main reference is [**37**].

Let R be a commutative noetherian ring with unity and M a finitely generated R-module. One has a finite free presentation for M

$$\phi: \quad R^m \quad \longrightarrow \quad R^n$$

where

$$M = \text{coker } \phi.$$

The morphism ϕ has an associated $(n \times m)$ matrix A_ϕ with coefficients in R such that

$$\phi(x) = A_\phi x^t.$$

Definition 1.5. The k-th **Fitting ideal** of M is defined as the ideal generated by

$$\begin{cases} 0 & \text{if } k \leq \max\{0, n-m\} \\ 1 & \text{if } k > n \\ \text{minors of } A_\phi \text{ of order } (n-k+1) & \text{otherwise.} \end{cases}$$

Such an ideal is denoted by $F_k(M)$ or simply F_k if no ambiguity seems likely to arise.

Remarks 1.6.

1. The ideal $F_k(M)$ depends only on M and not on the chosen free finite presentation ϕ. For a proof see for instance [**42**] Theorem IV.21.
2. The Fitting ideals of a module M form an increasing sequence

$$\{0\} = ... = F_{-1} = F_0 \subseteq F_1 \subseteq ... \subseteq F_n \subseteq F_{n+1} = F_{n+2} = ... = R.$$

Before defining characteristic varieties let us recall what the support of a module is.

Definition 1.7. Let M be a module over a commutative ring R. The **support** of M can be defined as

$$\text{Supp}_R(M) := \{\mathfrak{p} \in \text{Spec}(R) \mid M_\mathfrak{p} \neq 0\},$$

where $M_\mathfrak{p}$ is the R-module M localized at the prime ideal $\mathfrak{p}$.

Remarks 1.8.

1. Note that $M_\mathfrak{p} = 0$ if and only if there exists $r \in R \setminus \mathfrak{p}$ such that $rM = 0$. Hence one can give another equivalent definition of $\text{Supp}(M)$ as the set of prime ideals containing the annihilator of M.
2. One can detect the subset of maximal ideals in $\text{Supp}_R(M)$ as follows

$$\text{MaxSupp}_R(M) = \{\mathfrak{m} \in \text{MaxSpec}(R) \mid M/\mathfrak{m}M \neq 0\},$$

where $M/\mathfrak{m}M$ is the quotient module of M by the submodule $\mathfrak{m}M$.

Proof. Suppose $\mathfrak{m} \in \operatorname{Supp}_R(M)$ is a maximal ideal in R. Since $\mathfrak{m} \supset \operatorname{Ann}_R(R/\mathfrak{m})$, one has $\mathfrak{m} \in \operatorname{Supp}_R(R/\mathfrak{m})$. By [**48**], p. 3

$$\mathfrak{m} \in \operatorname{Supp}_R(M) \cap \operatorname{Supp}_R(R/\mathfrak{m}) = \operatorname{Supp}_R(M \otimes R/\mathfrak{m}).$$

Hence $\operatorname{Supp}_R(M/\mathfrak{m}M) \neq \emptyset$ and so $M/\mathfrak{m}M \neq 0$. Conversely, suppose $M/\mathfrak{m}M = M \otimes R/\mathfrak{m} \neq 0$. Therefore

$$(M/\mathfrak{m}M)_{\mathfrak{m}} = M \otimes (R/\mathfrak{m})_{\mathfrak{m}} = M \otimes R/\mathfrak{m} = M/\mathfrak{m}M \neq 0$$

and hence $\mathfrak{m} \in \operatorname{Supp}_R(M/\mathfrak{m}M) = \operatorname{Supp}_R(M) \cap \operatorname{Supp}_R(R/\mathfrak{m})$. In particular $\mathfrak{m} \in \operatorname{Supp}_R(M)$. $\square$

DEFINITION 1.9 ([**34**] p. 159). Under the above conditions the **k-th characteristic variety** of the R-module M can be defined as

$$\operatorname{Char}_k(M) := \operatorname{Supp}_R\left(R/F_k(M)\right).$$

The subindex k is also known as the **depth** of a characteristic variety.

REMARK 1.10. By [**11**] Cor. 1.3 one has

$$\left(\operatorname{Ann}_R\left(\wedge^k M\right)\right)^{\ell} \subseteq \operatorname{Ann}_R\left(R/F_k(M)\right) = F_k(M) \subseteq \operatorname{Ann}_R\left(\wedge^k M\right),$$

then

$$\operatorname{Supp}_R\left(R/F_k(M)\right) = \operatorname{Supp}_R\left(\wedge^k M\right).$$

Hence definitions 0.2 and 1.9 are equivalent.

Next one can extend the definition of characteristic varieties to finitely presented groups via the following construction. Suppose G is a finitely presented group. Then G'/G'' has a structure of G/G'-module given by conjugation

$$\begin{array}{ccc} G'/G'' \times G/G' & \longrightarrow & \underline{G/G'} \\ (\overline{g}_1, \overline{g}) & \mapsto & \overline{g_1 g g_1^{-1}}. \end{array}$$

To check that this action is well defined, suppose $h_2 \in G''$ and $h_1 \in G'$, then

$$(g_1 g^{-1} g_1^{-1})(g_1 h_2 g h_1 h_2^{-1} g_1^{-1}) = (g_1 (g^{-1} h_2 g) h_1 h_2^{-1} g_1^{-1})$$

and since $(g^{-1} h_2 g) = h' \in G'$ one has

$$(g_1 h' h_1 h_2^{-1} g_1^{-1}) \in G'.$$

Let us consider the finite CW-complex X associated with the presentation of G. In this case G'/G'' can be interpreted as $H_1(\tilde{X}_{ab}, \mathbb{Z})$ where $\tilde{X}_{ab}$ is the universal abelian cover of X. The group of deck transformations of the covering is $H_1(X, \mathbb{Z})$. Then $H_1(\tilde{X}_{ab}, \mathbb{Z})$ has a structure of $H_1(X, \mathbb{Z})$-module which coincides with the one described above. Tensoring $H_1(\tilde{X}_{ab}, \mathbb{Z})$ and $\mathbb{Z}[H_1(X, \mathbb{Z})]$ by $\mathbb{C}$ one obtains a module $M_G = H_1(\tilde{X}_{ab}; \mathbb{C})$ over the noetherian ring $R_G = \mathbb{C}[H_1(X, \mathbb{Z})]$.

DEFINITION 1.11. Let X be a topological space with a finitely presented fundamental group $G = \pi_1(X)$. Under the notation above, the **k-th characteristic variety** of X is defined as the k-th characteristic variety of the R_G-module M_G. The k-th characteristic variety of X shall be denoted by either $\operatorname{Char}_k(X)$ or $\operatorname{Char}_k(G)$.

EXAMPLES 1.12.

1. Let $\mathbb{F}_r$ be the free group generated by r elements. The bouquet of r 1-dimensional spheres $X = \bigvee_1^r S^1$ is the standard $K(\pi, 1)$ space associated with $\mathbb{F}_r$. Note that

$$C_i(\tilde{X}_{ab}) = C_i(X) \otimes \mathbb{C}[\mathbb{Z}^r]. \tag{1.6}$$

Let $s_1, ..., s_r$ be the natural set of generators of $H_1(X)$. By (1.6) one has

$$C_0(\tilde{X}_{ab}) = \mathbb{C}[\mathbb{Z}^r] = \Lambda,$$
$$C_1(\tilde{X}_{ab}) = \Lambda s_1 \oplus ... \oplus \Lambda s_r$$

and

$$\partial_1(s_i) = t_i - 1,$$

where t_i is the image of the base point in $\tilde{X}_{ab}$ under the transformation s_i. One has the following exact sequence

$$0 \longrightarrow H_1(\tilde{X}_{ab};\mathbb{C}) \longrightarrow \Lambda^r \xrightarrow{\partial_1} I \longrightarrow 0 \tag{1.7}$$

where the middle term corresponds to $C_1(\tilde{X}_{ab})$ and the right-most term is the augmentation ideal, that is, $\partial_1(C_1(\tilde{X}_{ab}))$. Consider a non-trivial point $p \in \mathbb{C}^{*r}$ ($p \neq \overline{1} = (1, ..., 1)$). Localizing the short exact sequence (1.7) at the maximal ideal $\mathfrak{m}_p$, one obtains

$$0 \longrightarrow H_1(\tilde{X}_{ab};\mathbb{C})_{\mathfrak{m}_p} \longrightarrow \Lambda^r_{\mathfrak{m}_p} \xrightarrow{\partial_1} I_{\mathfrak{m}_p} \longrightarrow 0. \tag{1.8}$$

Since $p \neq \overline{1}$, one can assume $t_1 - 1 \notin \mathfrak{m}_p$. Hence the morphism

$$\begin{array}{rccc} \chi_1 : & I_{\mathfrak{m}_p} & \longrightarrow & \Lambda^r_{\mathfrak{m}_p} \\ & f & \mapsto & \frac{f}{t_1-1}s_1 \end{array}$$

satisfies

$$\partial \circ \chi_1(f) = \partial_1(\frac{f}{t_1 - 1}s_1) = \frac{f}{t_1 - 1}\partial(s_1) = f$$

and therefore (1.8) splits. Note also that the generators $t_i - 1$ of I are invertible in $I_{\mathfrak{m}_p}$, and hence $I_{\mathfrak{m}_p} \cong \Lambda_{\mathfrak{m}_p}$. Therefore

$$H_1(\tilde{X}_{ab};\mathbb{C})_{\mathfrak{m}_p} \cong \Lambda^{r-1}_{\mathfrak{m}_p}$$

and thus $\operatorname{Char}_k(\mathbb{F}_r) = \mathbb{C}^{*r}$ for $k = 1, ..., r-1$. In the case $p = \overline{1}$ the local ring $I_{\mathfrak{m}_{\overline{1}}}$ is no longer free. Using the Koszul resolution of the ideal I, one can check –c.f. [**16**] or [**37**]– that

$$\operatorname{Char}_k(\mathbb{F}_r) = \begin{cases} \overline{1} & \text{for } k = r, ..., \binom{r}{2} \\ \emptyset & \text{if } k > \binom{r}{2}. \end{cases}$$

2. Let $L \in S^3$ be a link in the 3-dimensional sphere S^3. The fundamental group $G = \pi_1(S^3 \setminus L)$ of the complement of the link is a finitely presented group. The fact that any link can be seen as a closed braid allows to give a very special presentation of the fundamental group with deficiency 1 –that is, a presentation with one more generators (n) than relations ($n-1$)). This provides an $(n-1) \times n$ matrix for the Burau representation of the braid (which corresponds to the action of $H_1(X)$ on $H_1(\tilde{X}_{ab})$). Finally, since the Burau representation is reducible to an $(n-1) \times (n-1)$ representation, the (multi-variable) Alexander ideal turns out to be principal, and hence, its zero locus coincides with the first characteristic variety $\operatorname{Char}_1(G)$ –c.f. [**8**] Section 3.3.
3. Let $\mathcal{C}$ be an algebraic curve in $\mathbb{C}^2$ with r irreducible components. Denote by X the complement of $\mathcal{C}$. It is well known that X has the homotopy type of a 2-dimensional CW-complex satisfying $H_1(X) = \mathbb{Z}^r$ –c.f. [**33**]. From a presentation of $\pi_1(X)$ one can use Fox calculus –c.f. [**22**], [**8**] Ch.3 or [**27**]– to calculate the $\mathbb{C}[\mathbb{Z}^r]$-module $H_1(\tilde{X}_{ab}, \widetilde{pts};\mathbb{C})$. This module is defined as

the first homology of $\tilde{X}_{ab}$ relative to the inverse image of the base point in X and it fits in the following exact sequence

$$0 \to H_1(\tilde{X}_{ab};\mathbb{C}) \to H_1(\tilde{X}_{ab},\widetilde{pts};\mathbb{C}) \xrightarrow{\partial} H_0(\widetilde{pts};\mathbb{C}). \tag{1.9}$$

Since X is connected it can be assumed to have only one 0-cell. Hence

$$H_0(\widetilde{pts};\mathbb{C}) = \Lambda = \mathbb{C}[\mathbb{Z}^r].$$

Also note that $\partial(H_1(\tilde{X}_{ab},\widetilde{pts};\mathbb{C})) = I \subset \Lambda$. After localizing (1.9) at the maximal $\mathfrak{m}_p$ of any point $p \in \mathbb{C}^{*r} \setminus \overline{1}$, it becomes

$$0 \quad \to \quad H_1(\tilde{X}_{ab};\mathbb{C})_{\mathfrak{m}_p} \quad \to \quad H_1(\tilde{X}_{ab},\widetilde{pts};\mathbb{C})_{\mathfrak{m}_p} \quad \xrightarrow{\partial} \quad \Lambda_{\mathfrak{m}_p} \quad \to \quad 0.$$

As in example 1 this exact sequence splits, resulting in

$$H_1(\tilde{X}_{ab},\widetilde{pts};\mathbb{C})_{\mathfrak{m}_p} = H_1(\tilde{X}_{ab};\mathbb{C})_{\mathfrak{m}_p} \oplus \Lambda_{\mathfrak{m}_p}.$$

Suppose one has a finite free presentation for $H_1(\tilde{X}_{ab};\mathbb{C})$

$$\Lambda^m \quad \xrightarrow{\phi} \quad \Lambda^n \quad \longrightarrow \quad H_1(\tilde{X}_{ab};\mathbb{C}) \quad \longrightarrow \quad 0 \ .$$

Therefore

$$\Lambda^m_{\mathfrak{m}_p} \quad \xrightarrow{\tilde{\phi}_p} \quad \Lambda^n_{\mathfrak{m}_p} \oplus \Lambda_{\mathfrak{m}_p} \quad \longrightarrow \quad H_1(\tilde{X}_{ab},\widetilde{pts};\mathbb{C})_{\mathfrak{m}_p} \quad \longrightarrow \quad 0$$

is a finite free presentation of $H_1(\tilde{X}_{ab},\widetilde{pts};\mathbb{C})_{\mathfrak{m}_p}$. The matrix $A_{\tilde{\phi}_p}$ corresponding to $\tilde{\phi}_p$ can be written as

$$\begin{pmatrix} & \phi_p & \\ & & \\ 0 & \dots & 0 \end{pmatrix}$$

where the minors of order $(n-k+1)$ of A_{ϕ_p} become the minors of order $((n+1)-(k+1)+1)$ of $A_{\tilde{\phi}_p}$. Therefore, outside $\overline{1}$ one has

$$\mathrm{Char}_k(X) = \mathrm{Char}_{k+1}(H_1(\tilde{X}_{ab},\widetilde{pts};\mathbb{C})). \tag{1.10}$$

To finish this section we want to establish a connection between characteristic varieties of topological spaces and homology with coefficients in rank one local systems –c.f. [**29**] and [**37**]. Let X be a topological space such that $\pi_1(X) = G$ is finitely presented and $H_1(X) = \mathbb{Z}^r$. Note that

$$\mathrm{Hom}(H_1(X),\mathbb{C}^*) \cong \mathbb{C}^{*r}$$

parametrizes the space of rank one representations of G. One can stratify this space as follows –c.f. [**1**]–

$$\Sigma^k(X) := \{\rho = (\rho_1,\dots,\rho_r) \in \mathrm{Hom}(H_1(X),\mathbb{C}^*) \mid \dim H^1(X,\rho) \geq k\}.$$

Note that $\mathbb{C}_\rho$ is isomorphic –as a $\mathbb{C}[H_1(X)]$-module– to $\mathbb{C}[t_1^{\pm 1},\dots,t_r^{\pm 1}]/\mathfrak{m}_\rho$ where $\mathfrak{m}_\rho$ denotes the maximal ideal at the point ρ –generated by $(t_1-\rho_1,\dots,t_r-\rho_r)$. On the other hand, $\mathrm{Char}_k(X)$ defines a variety in $\mathbb{C}^{*r}$ whose closed points are in one-to-one correspondence with the maximal ideals in $\mathrm{Char}_k(X)$. We will denote this set of closed points by $\mathrm{Char}^{\mathfrak{m}}_k(X)$. By (1.3), Proposition 1.4 and Remark 1.8.2, one has the following

Proposition 1.13. *Let X be as above, then*

$$\mathrm{Char}_k^{\mathfrak{m}}(X) \setminus \bar{1} \cong \Sigma^k(X) \setminus \bar{1}$$

are isomorphic as subspaces of $\mathbb{C}^{*r}$. Moreover, the image of the maximal ideal $\mathfrak{m}_\rho = (t_1 - \rho_1, ..., t_r - \rho_r) \in \mathrm{Char}_k^{\mathfrak{m}}(X)$ by this isomorphism is the character $\rho^* = (\rho_1^{-1}, ..., \rho_r^{-1})$.

Proof. By definition, if the maximal ideal $\mathfrak{m}_\rho$ corresponding to the point $\rho \in \mathbb{C}^{*r} \setminus \bar{1}$ is in $\mathrm{Char}_k^{\mathfrak{m}}(X)$, then $\mathfrak{m}_\rho \in \mathrm{Supp}_{\mathbb{C}[H_1(X)]}(\wedge^k H_1(\tilde{X}_{ab}; \mathbb{C}))$. Therefore, by Remark 1.8.2,

$$\wedge^k H_1(\tilde{X}_{ab}; \mathbb{C}) \otimes \mathbb{C}[H_1(X)]/\mathfrak{m}_\rho = \left(\wedge^k H_1(\tilde{X}_{ab}; \mathbb{C})\right) \otimes \mathbb{C}_\rho = \\ = \wedge^k \left(H_1(\tilde{X}_{ab}; \mathbb{C}) \otimes \mathbb{C}_\rho\right) \neq 0$$

and hence

$$\dim H_1(\tilde{X}_{ab}; \mathbb{C}) \otimes \mathbb{C}_\rho \geq k. \tag{1.11}$$

Since ρ is non-trivial one can apply Proposition 1.3 to equation (1.11). Hence

$$\dim H_1(X, \rho) \geq k.$$

By duality –c.f. 1.4), one has

$$\dim H^1(X, \rho^*) \geq k$$

and hence $\rho^* \in \Sigma^k(X)$.

The converse follows by simply retracing the previous steps. $\square$

Remark 1.14. The reason why Proposition 1.13 does not apply to the trivial element $\bar{1}$ is that Proposition 1.3 does not hold for the trivial representation. Instead, one has the following result –c.f. [**34**] Prop. 1.1–

$$\min\{k \in \mathbb{N} \mid \bar{1} \in \mathrm{Char}_k(X)\} = \dim(\ker \cup) - 1$$

where $\cup$ denotes the cup product,

$$\cup : \quad \wedge^2 H^1(X; \mathbb{C}) \quad \longrightarrow \quad H^2(X; \mathbb{C}).$$

1.3. C^∞ log complex of quasi-projective algebraic varieties

The aim of this section is to present an appropriate setting for the study of the cohomology ring of the complement to rational arrangements. This includes definitions and basic properties of logarithmic sheaves and the definition of a very useful operator on these sheaves: the Poincaré residue operator. Most definitions and results in this section follow from [**25**]. We refer to Chapter 5 of this classical paper for detailed proofs.

Let X be a smooth, quasi-projective algebraic variety over $\mathbb{C}$ and $\overline{X}$ be a smooth compactification of X. We will assume $\overline{X}$ to be a smooth projective variety such that $X = \overline{X} \setminus \overline{\mathcal{D}}$, where $\overline{\mathcal{D}}$ is a union $\overline{\mathcal{D}}_1 \cup ... \cup \overline{\mathcal{D}}_N$ of smooth divisors on $\overline{X}$ with normal crossings. The condition of normal crossing on $\overline{\mathcal{D}}$ means that locally at $P \in \overline{X}$, the divisor $\overline{\mathcal{D}}$ is given by

$$\{(z_1, ..., z_n) \mid z_{i_1} \cdot ... \cdot z_{i_m} = 0\} = \{(z_1, ..., z_n) \mid z_{I_P} = 0\},$$

where $I_P = (i_1, ..., i_m)$ is a string whose coordinates belong to $\{1, ..., n\}$. The set of coordinates of I_P will be denoted –by abuse of notation– as $\{I_P\}$. This multi-index notation will be used throughout the section. Note that the variables $z_{i_1}, ..., z_{i_m}$ are local coordinates of certain components, say $\overline{\mathcal{D}}_{j_1}, ..., \overline{\mathcal{D}}_{j_m}$ respectively. We will denote by J_P the corresponding string $(j_1, ..., j_m)$ with $\{J_P\} \subset \{1, ..., N\}$.

DEFINITION 1.15.

1. Let $\mathcal{A}_{\overline{X}}$ be the sheaf of C^∞ forms on $\overline{X}$. Denote by $\mathcal{A}^0_{\overline{X}}$ the sheaf of C^∞ functions on $\overline{X}$. Note that $\mathcal{A}_{\overline{X}}$ is a sheaf of graded algebras over $\mathcal{A}^0_{\overline{X}}$. The **sheaf of C^∞ log forms** $\mathcal{A}_{\overline{X}}(\log\langle\overline{\mathcal{D}}\rangle)$ can be defined locally at a point $P \in \overline{X}$ as the graded algebra over $(\mathcal{A}^0_{\overline{X}})_P$ of C^∞ forms $\varphi \in (\mathcal{A}_X)_P$ –c.f. Remark 1.16– such that
$$z_{I_P}\varphi \quad \text{and} \quad z_{I_P}d\varphi$$
are C^∞ forms in $(\mathcal{A}_{\overline{X}})_P$.
2. A form φ shall be called **logarithmic (with respect to $\overline{\mathcal{D}}$)** if $\varphi \in \mathcal{A}_{\overline{X}}(\log\langle\overline{\mathcal{D}}\rangle)$

REMARK 1.16. In Definition 1.15, we wrote $(\mathcal{A}_X)_P$ for the stalk at P of the direct image $(i_*\mathcal{A}_X)$ given by the inclusion $i : X \hookrightarrow \overline{X}$. Recall that any sheaf $\mathcal{F}$ over X can be extended to a sheaf over $\overline{X}$ where the stalk at a point $P \in \overline{X} \setminus X$ is the direct limit of $\mathcal{F}(U_P \cap X)$. By abuse of notation, and if no ambiguity seems likely to arise, this will be simply denoted by $\mathcal{F}_P$.

The sheaf $\mathcal{A}_{\overline{X}}(\log\langle\overline{\mathcal{D}}\rangle)$ is a locally free and finitely generated $\mathcal{A}^0_{\overline{X}}$-algebra, as follows from

LEMMA 1.17 ([**25**] 5.7). $\mathcal{A}_{\overline{X}}(U_P)(\log\langle\overline{\mathcal{D}}\rangle) \cong \mathcal{A}_{\overline{X}}(U_P)\{\frac{dz_i}{z_i}\}_{i\in I_P}$

By definition, $\mathcal{A}_{\overline{X}}(\log\langle\overline{\mathcal{D}}\rangle)$ is closed under the exterior derivation d. This Lemma shows that it is in fact closed under the exterior product and generated by $\mathcal{A}^1_{\overline{X}}(\log\langle\overline{\mathcal{D}}\rangle)$.

In what follows, a weight filtration is defined in this sheaf of graded algebras that is compatible with the differential d.

DEFINITION 1.18. If $\ell \geq 0$ we shall define the **sheaf of C^∞ log forms of weight ℓ** as the subsheaf of $\mathcal{A}_{\overline{X}}(\log\langle\overline{\mathcal{D}}\rangle)$ given locally as the $(\mathcal{A}_{\overline{X}})^0_P$-submodule of $\mathcal{A}_{\overline{X}}(\log\langle\overline{\mathcal{D}}\rangle)_P$ of those forms φ such that
$$\varphi \in \sum_{\substack{I\subset I_P,\\ |I|\leq\ell}} \mathcal{A}_{\overline{X}}\left\{\frac{dz_i}{z_i}\right\}_{i\in I}.$$

Such a sheaf will be denoted by $\mathcal{W}_\ell := \mathcal{W}_\ell\left(\mathcal{A}_{\overline{X}}(\log\langle\overline{\mathcal{D}}\rangle)\right)$. Otherwise, that is, if $\ell < 0$, we will assume $\mathcal{W}_\ell := \{0\}$.

REMARK 1.19. Note that
$$\mathcal{W}_\ell \subset \mathcal{W}_{\ell+1},$$
$$d\mathcal{W}_\ell \subset \mathcal{W}_\ell \quad \text{and}$$
$$\mathcal{W}_\ell \wedge \mathcal{W}_{\ell'} \subset \mathcal{W}_{\ell+\ell'}$$
are obvious consequences of Definition 1.18.

NOTATIONS 1.20.

1. Let us denote by $\overline{\mathcal{D}}^{[k]}$ the disjoint union of codimension k intersections of components of $\overline{\mathcal{D}}$, that is,
$$\overline{\mathcal{D}}^{[k]} := \bigsqcup_{|J|=k} \overline{\mathcal{D}}_J$$
where $\overline{\mathcal{D}}_J = \cap_{i\in J}\overline{\mathcal{D}}_i$.
2. There is an injection $\overline{\mathcal{D}}_J \overset{i_J}{\hookrightarrow} \overline{X}$ for each $\overline{\mathcal{D}}_J \in \overline{\mathcal{D}}^{[k]}$. Denoting by i_k the corresponding map on $\overline{\mathcal{D}}^{[k]}$, one has the following sheaf on $\overline{X}$
$$\mathcal{A}^*_{\overline{\mathcal{D}}^{[k]}} = (i_k)_* \bigoplus_{|J|=k} \mathcal{A}^*_{\overline{\mathcal{D}}_J}$$

DEFINITION 1.21. Under the notations above, the **Poincaré residue operator**
$$R^{[k]} : \mathcal{W}_k\left(\mathcal{A}^*_{\overline{X}}(\log\langle\overline{\mathcal{D}}\rangle)\right) \longrightarrow \mathcal{A}^{*-k}_{\overline{\mathcal{D}}^{[k]}}$$
can be defined locally by
$$R^{[k]}\left(\alpha_P \wedge \frac{dz_I}{z_I}\right) = (-1)^{\sigma(\overline{J})}\alpha_P \mid_{\overline{\mathcal{D}}_J},$$
where:

i) $\frac{dz_I}{z_I}$ denotes $\frac{dz_{i_1}}{z_{i_1}} \wedge ... \wedge \frac{dz_{i_k}}{z_{i_k}}$,
ii) $z_{i_1}, ..., z_{i_k}$ are local coordinates of the components $\overline{\mathcal{D}}_{j_1}, ..., \overline{\mathcal{D}}_{j_k}$ resp. and
iii) $\sigma(\overline{J}) := sign(j_{k+1}, ..., j_N, j_1, ..., j_k)$, where $j_{k+1} < ... < j_N$ are the elements of $\{1, ..., N\} \setminus J = \{1, ..., N\} \setminus \{j_1, ..., j_k\}$.

REMARK 1.22. Note that for any $\overline{\mathcal{D}}_{J'}$ and $\overline{\mathcal{D}}_J$ with $|J'| = k+1$ and $|J| = k$ one can define a smooth divisor on $\overline{\mathcal{D}}_J$ as follows:
$$\overline{\mathcal{D}}_J \mid_{\overline{\mathcal{D}}_{J'}} := \begin{cases} \overline{\mathcal{D}}_{J'} & \text{if } J \subset J' \\ \emptyset & \text{otherwise} \end{cases}$$
Moreover, the union
$$\overline{\mathcal{D}}_J \mid_{\overline{\mathcal{D}}^{[k+1]}} := \sum_{|J'|=k+1} \overline{\mathcal{D}}_J \mid_{\overline{\mathcal{D}}_{J'}}$$
provides a divisor with normal crossings in $\overline{\mathcal{D}}_J \subset \overline{\mathcal{D}}^{[k]}$. Hence $\overline{\mathcal{D}}^{[k]}$ can be regarded as a disjoint union of smooth compact algebraic varieties, each component containing a divisor with normal crossings. Therefore one can consider the sheaf of C^∞ log forms on each smooth algebraic variety $\overline{\mathcal{D}}_J$ with respect to $\overline{\mathcal{D}}_J \mid_{\overline{\mathcal{D}}^{[k+1]}}$, denoted by $\mathcal{A}_{\overline{\mathcal{D}}_J}(\log\langle\overline{\mathcal{D}}^{[k+1]}\rangle)$.

DEFINITION 1.23. By means of the inclusions
$$\overline{\mathcal{D}}_J \overset{i_k}{\hookrightarrow} \overline{X}$$
as in Notation 1.20.2 one can also define **log sheaves on $\overline{\mathcal{D}}^{[k]}$ relative to $\overline{\mathcal{D}}^{[k+1]}$** as subsheaves of the direct sum of log sheaves for each component satisfying certain compatibility relations. That is,
$$\mathcal{A}_{\overline{\mathcal{D}}^{[k]}}(\log\langle\overline{\mathcal{D}}^{[k+1]}\rangle) \subset \bigoplus_{|J|=k} \mathcal{A}_{\overline{\mathcal{D}}_J}(\log\langle\overline{\mathcal{D}}^{[k+1]}\rangle),$$

defined by the following local condition. For any strings I_1, I_2, I'_1, I'_2 such that $|I_i| = |I'_i| = k_i$, $i = 1, 2$ and

$$\{I_1 + I_2\} = \{I'_1 + I'_2\},$$

and for any pair of forms

$$\alpha_P \frac{dz_{I_2}}{z_{I_2}} \in \left(\mathcal{A}^*_{\overline{\mathcal{D}}_{J_1}}(\log\langle\overline{\mathcal{D}}^{[k_1+1]}\rangle)\right)_P \quad ,$$

$$\beta_P \frac{dz_{I'_2}}{z_{I'_2}} \in \left(\mathcal{A}^*_{\overline{\mathcal{D}}_{J'_1}}(\log\langle\overline{\mathcal{D}}^{[k_1+1]}\rangle)\right)_P \quad ,$$

then

$$(-1)^{\sigma(\overline{J_1})}(-1)^{\sigma(\overline{J_2+J_1})}\alpha_P \mid_{\overline{\mathcal{D}}_{J_1}} = (-1)^{\sigma(\overline{J'_1})}(-1)^{\sigma(\overline{J'_2+J'_1})}\beta_P \mid_{\overline{\mathcal{D}}_{J'_1}}, \tag{1.12}$$

where $\overline{J_i}$ and $\overline{J'_i}$ are defined as in 1.21 and $I + I'$ denotes juxtaposition of strings. For simplification this sheaf will be denoted by

$$\mathcal{A}^*_k(\log\langle\overline{\mathcal{D}}\rangle)$$

and its projection on $\overline{\mathcal{D}}_J$ (for $|J| = k$) by

$$\mathcal{A}^*_{k,J}(\log\langle\overline{\mathcal{D}}\rangle).$$

There also exists an obvious weight filtration on $\mathcal{A}_k(\log\langle\overline{\mathcal{D}}\rangle)$, denoted by $\mathcal{W}^k_\ell$. Note that

$$\mathcal{W}^0_\ell = \mathcal{W}_\ell(\mathcal{A}_{\overline{X}}(\log\langle\overline{\mathcal{D}}\rangle))$$

and

$$\mathcal{W}^k_0 = \mathcal{A}_{\overline{\mathcal{D}}^{[k]}}.$$

The compatibility relations given in (1.12) allow for a generalization of the Poincaré residue operator to be defined as follows

$${}^\ell R^{[k]} : \mathcal{W}^0_\ell \longrightarrow \mathcal{W}^k_{\ell-k},$$

where locally,

$${}^\ell R^{[k]}\left(\alpha_P \wedge \frac{dz_I}{z_I}\right) = (-1)^{\sigma(\overline{J})}\alpha_P \mid_{\overline{\mathcal{D}}_J} .$$

Note that

$${}^\ell R^{[k]}(\mathcal{W}^0_\ell) \subset \mathcal{W}^k_{\ell-k}. \tag{1.13}$$

Analogously, one can generalize the Poincaré residue operator to all the log sheaves relative to $\overline{\mathcal{D}}$, namely

$${}^\ell R^{[k]}_m : \mathcal{W}^m_\ell \longrightarrow \mathcal{W}^{m+k}_{\ell-k}.$$

In order to give a local description of ${}^\ell R^{[k]}_m$ let us consider a point $P \in \overline{X}$ and a form

$$\varphi \in \left(\mathcal{A}^*_k(\log\langle\overline{\mathcal{D}}\rangle)\right)_P \quad .$$

Let us denote by

$$\left({}^\ell R^{[k]}_m \varphi\right)_J$$

the coordinate of ${}^{\ell}R_m^{[k]}\varphi$ on $\mathcal{A}^{*-k}_{k+m,J}(\log\langle\overline{\mathcal{D}}\rangle)_P$ –where $|J| = m + k$. In order to calculate this coordinate take two disjoint strings I_1 and I_2 such that $|I_1| = m$, and $\overline{\mathcal{D}}_J = \{z_{I_1+I_2} = 0\}$ –and hence $|I_2| = k$. The form φ can be written as

$$\alpha \frac{dz_{I_2}}{z_{I_2}} \in \left(\mathcal{A}^*_{k,I_1}(\log\langle\overline{\mathcal{D}}\rangle)\right)_P .$$

Thus one can define

$$\left({}^{\ell}R_m^{[k]}\varphi\right)_J := (-1)^{\sigma(\overline{J_1})}(-1)^{\sigma(\overline{J_2+J_1})}\alpha \mid_{\overline{\mathcal{D}}_J}$$

Again by (1.12) the definition of $\left({}^{\ell}R_m^{[k]}\varphi\right)_J$ does not depend on the choice of I_1 and I_2.

REMARKS 1.24.

1. It is not difficult to check that ${}^{\ell}R_m^{[k]}$ commutes with the differential d, and hence that ${}^{\ell}R_m^{[k]}$ is a homomorphism of complexes.
2. Note also that ${}^{\ell}R_0^{[k]} = {}^{\ell}R^{[k]}$ and ${}^{k}R_0^{[k]} = R^{[k]}$.

One has the following result of commutativity of operators

LEMMA 1.25. ${}^{(\ell-k_1)}R^{[k_2]}_{m+k_1} \circ {}^{\ell}R_m^{[k_1]} = {}^{\ell}R_m^{[k_1+k_2]}$.

Proof. It is enough to check the result over an intersection $\overline{\mathcal{D}}_{J_2+J_1+J}$ where $|J| = m$ and $|J_i| = k_i$ for $i = 1, 2$. The lemma is now a consequence of the following fact

$$(-1)^{\sigma(\overline{J})}(-1)^{\sigma(\overline{J_2+J_1+J})} = (-1)^{\sigma(\overline{J})}(-1)^{\sigma(\overline{J_1+J})}(-1)^{\sigma(\overline{J_1+J})}(-1)^{\sigma(\overline{J_2+J_1+J})}. \quad \square$$

EXAMPLE 1.26. Consider the divisor $\mathcal{D} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3$ in $\mathbb{C}^2$ where

$$\mathcal{C}_i := \{z_i = 0\}$$

and $z_1 = x$, $z_2 = y$, $z_3 = x + y - 1$.

The 2-form

$$\alpha = \frac{dx \wedge dy}{z_1 z_2 z_3} = \frac{dx \wedge dy}{xy(x+y-1)}$$

belongs to $\mathcal{W}_2\left(\mathcal{A}^2_{\mathbb{C}^2}(\log\langle\mathcal{D}\rangle)\right)$. Note that

$$\mathcal{A}^1_1(\log\langle\mathcal{D}\rangle) \subset \sum_{\substack{i=1\\ \{i,j,k\}=\{1,2,3\}}}^{3} \mathcal{A}^0_{\mathcal{C}_i}\left\{\frac{dz^i_j}{z^i_j}\right\} \oplus \mathcal{A}^0_{\mathcal{C}_i}\left\{\frac{dz^i_k}{z^i_k}\right\}$$

where $z^i_j := z_j \mid_{\mathcal{C}_i}$, is the sheaf of 3-tuples of 1-forms

$$\left(\varphi_1(y)\frac{dy}{y(y-1)}, \varphi_2(x)\frac{dx}{x(x-1)}, \varphi_3(x)\frac{dx}{x(x-1)}\right)$$

satisfying

$$(-1)^{\sigma(2,3,1)+\sigma(3,2,1)}\varphi_1(0) = (-1)^{\sigma(1,3,2)+\sigma(3,1,2)}\varphi_2(0) \quad i.e.\ \ \varphi_1(0) = \varphi_2(0),$$

$$(-1)^{\sigma(1,3,2)+\sigma(1,3,2)}\varphi_2(1) = (-1)^{\sigma(1,2,3)+\sigma(1,2,3)}\varphi_3(1) \quad i.e.\ \ \varphi_2(1) = \varphi_3(1),$$

$$(-1)^{\sigma(1,2,3)+\sigma(2,1,3)}\varphi_1(1) = (-1)^{\sigma(2,3,1)+\sigma(2,3,1)}\varphi_3(0) \quad i.e.\ \ -\varphi_1(1) = \varphi_3(0).$$

Also note that

$$\mathcal{A}_2(\log\langle\overline{\mathcal{D}}\rangle) = \mathbb{C}_{(0,0)} \oplus \mathbb{C}_{(1,0)} \oplus \mathbb{C}_{(0,1)}.$$

Applying the definitions, one has

$$^2R_0^{[1]}\alpha = \left(-\frac{dy}{y(y-1)}, -\frac{dx}{x(x-1)}, \frac{dx}{x(1-x)} = -\frac{dy}{(1-y)y}\right)$$

and

$$^2R_0^{[2]}\alpha = (-1,-1,-1).$$

One can calculate, for instance, the coordinate of

$$^1R_1^{[1]} \circ {}^2R_0^{[1]}\alpha$$

corresponding to the intersection $(0,0)$. This can be computed as

$$\left({}^1R_1^{[1]}\,|_{\mathcal{C}_i} \circ \left({}^2R_0^{[1]}(\alpha)\right)_{\mathcal{C}_i}\right)_{(0,0)}$$

for $i=1,2$.

For $i=1$, one has

$$\left({}^1R_1^{[1]}\,|_{\mathcal{C}_1} -\frac{dy}{y(y-1)}\right)_{(0,0)} = -(-1)^{\sigma(2,3,1)+\sigma(3,2,1)}\frac{1}{0-1} = -1.$$

For $i=2$,

$$\left({}^1R_1^{[1]}\,|_{\mathcal{C}_2} -\frac{dx}{x(x-1)}\right)_{(0,0)} = -(-1)^{\sigma(1,3,2)+\sigma(3,1,2)}\frac{1}{0-1} = -1.$$

This agrees with the direct calculation

$$\left({}^2R_0^{[2]}\alpha\right)_{(0,0)} = (-1)^{\sigma(3,1,2)}\frac{1}{0+0-1} = -1.$$

Analogously, one can readily check the remaining cases.

This exemplifies (1.12).

The classical application of the Poincaré residue operator is to describe the Mixed Hodge structure on a divisor with normal crossings via the given weight filtration. In order to do so, one has to apply the residue operator to the successive quotients of the weight filtration and prove that they provide isomorphisms with the cohomology of the spaces $\overline{\mathcal{D}}^{[k]}$. Something analogous can be done for these newly defined residue operators.

Note that, analogously to (1.13) it can be verified that

$$^\ell R_m^{[k]}\left(\mathcal{W}_\ell^m\right) \subset \mathcal{W}_{\ell-k}^{m+k}.$$

Let us denote $\mathcal{W}_\ell^m\left(\mathcal{A}_m^*(\log\langle\overline{\mathcal{D}}\rangle)\right)$ by $\mathcal{W}_\ell^{m,*}$. It holds that

$$d\mathcal{W}_\ell^{m,*} \subset \mathcal{W}_\ell^{m,*+1}.$$

Hence the complex

$$\left(\mathcal{W}_\ell^{m,\bullet}/\mathcal{W}_{\ell-1}^{m,\bullet}, d\right)$$

is well defined. The Poincaré residue operator can also be defined for this quotient complex

$$^\ell\tilde{R}_m^{[k]} : \mathcal{W}_\ell^{m,*}/\mathcal{W}_{\ell-1}^{m,*} \longrightarrow \mathcal{W}_{\ell-k}^{m,*-k}/\mathcal{W}_{\ell-k-1}^{m,*-k}.$$

One has the following result

LEMMA 1.27. *The induced mapping on cohomology sheaves*

$$ {}^{\ell}\tilde{R}_0^{[k]} : \mathcal{H}^{*}\left(\mathcal{W}_{\ell}^{0}/\mathcal{W}_{\ell-1}^{0}, d\right) \longrightarrow \mathcal{H}^{*-k}\left(\mathcal{W}_{\ell-k}^{k}/\mathcal{W}_{\ell-k-1}^{k}, d\right) $$

is an isomorphism of sheaves.

Proof. The proof given in [**25**] 5.13 for the case $\ell = k$ is also valid for any ℓ. □

Let us denote by W_{ℓ}^{k} the ring of global sections of the sheaf $\mathcal{W}_{\ell}^{k}$. As a consequence of the acyclicity of the C^{∞} complexes provided by the existence of partitions of unity –c.f. [**24**] p. 42– one has the following

THEOREM 1.28. *The residue mapping* ${}^{\ell}\tilde{R}_0^{[k]}$ *defined on the complex of global sections*

$$ {}^{\ell}\tilde{R}_0^{[k]} : (W_{\ell}^{0,*}/W_{\ell-1}^{0,*}) \longrightarrow (W_{\ell-k}^{0,*-k}/W_{\ell-k-1}^{0,*-k}) $$

induces isomorphism on d-cohomology.

This theorem generalizes [**25**] 5.15.

Consequently, one has the following

THEOREM 1.29. *Any residue mapping* ${}^{\ell}\tilde{R}_m^{[k]}$ *on the complex of global sections induces isomorphism on d-cohomology.*

Proof. By Theorem 1.28, the operator ${}^{(\ell+m)}\tilde{R}_0^{[i]}$ is an isomorphism for any i. Applying Lemma 1.25 to the quotient, one has

$$ {}^{\ell}\tilde{R}_m^{[k]} \circ {}^{(\ell+m)}\tilde{R}_0^{[m]} = {}^{(\ell+m)}\tilde{R}_0^{[m+k]}. $$

Hence the Poincaré residue operator ${}^{\ell}\tilde{R}_m^{[k]}$ is an isomorphism. □

CHAPTER 2

Ring structure of $H^\bullet(\mathbb{P}^2 \setminus \mathcal{R}; \mathbb{C})$

Let us start with some basic definitions.

DEFINITIONS 2.1.

1. A reduced divisor in $\mathbb{P}^2$ will be called an **arrangement**. If all irreducible components $\mathcal{C}_0, ..., \mathcal{C}_n$ in an arrangement $\mathcal{D}$ are rational curves (possibly singular), we shall say $\mathcal{D}$ is a **rational arrangement**. In the special case where all components are lines, $\mathcal{D}$ will be referred to as a **line arrangement**.
2. Let us fix $\pi : \overline{X}_{\mathcal{D}} \longrightarrow \mathbb{P}^2$ a resolution of the singularities of an arrangement $\mathcal{D}$ so that the reduced divisor $\overline{\mathcal{D}} = (\pi^*(\mathcal{D}))_{red}$ is a union of smooth divisors on $\overline{X}_{\mathcal{D}}$ with normal crossings as described in section 1.3.

From now on, $\mathcal{D}$ will denote an arrangement in $\mathbb{P}^2$, $\mathcal{R}$ a rational arrangement and $\mathcal{L}$ a line arrangement. The complement of $\mathcal{D}$ (resp. $\mathcal{R}$, or $\mathcal{L}$) will be denoted as $X_{\mathcal{D}}$ (resp. $X_{\mathcal{R}}$, or $X_{\mathcal{L}}$.

A first approach to study $H^*(X_{\mathcal{R}}; \mathbb{C})$ is given by the de Rham Theorem [**26**]. This theorem implies that the spaces $H^p(X_{\mathcal{D}}; \mathbb{C})$ (for a general divisor $\mathcal{D}$) are generated by differential global p-forms on $X_{\mathcal{D}}$. However, the spaces of differential forms are in general hard to handle. Another possible strategy to study $H^p(X_{\mathcal{D}}; \mathbb{C})$ is to impose restrictions on the type of forms one wants to work with. This can be achieved, in a more general context, using the holomorphic logarithmic complex as follows. Let $\overline{X}$ be a smooth complex variety and $\overline{\mathcal{D}}$ be a union of smooth divisors with normal crossings in $\overline{X}$ such that $X = \overline{X} \setminus \overline{\mathcal{D}}$. The hypercohomology of the complex of sheaves of holomorphic logarithmic forms on $\overline{X}$ with the exterior differential is isomorphic to the cohomology of X. The logarithmic de Rham Theorem –[**25**] Prop. 5.14– exhibits a combination of the attributes of these two theorems. It states that the global differential logarithmic forms on $\overline{X}$ with poles along $\overline{\mathcal{D}}$ are enough to generate the cohomology of X. In some particular cases, such as complex line arrangements –and even general hyperplane arrangements– the cohomology of the complement is in fact generated by holomorphic logarithmic forms [**9**]. Moreover, there is a ring isomorphism

$$\wedge^* H^0(\overline{X_{\mathcal{L}}}, \Omega^1(\log\langle\mathcal{L}\rangle)) \cong H^*(X_{\mathcal{L}}; \mathbb{C}). \tag{2.1}$$

Our aim in this Chapter is to prove an analogue of (2.1) for rational arrangements.

2.1. The spaces $H^k(\mathbb{P}^2 \setminus \mathcal{D}; \mathbb{C})$ and the residue maps

In this first section some general formulæ are given about the dimensions of the vector spaces $H^*(X_{\mathcal{D}}; \mathbb{C})$. In the sequel, residue maps are defined from the cohomology of $X_{\mathcal{D}}$ to the cohomology of the subspaces $\overline{\mathcal{D}}^{[*]}$. These maps can be described by means of the Poincaré residue operator – c.f. section 1.3. Finally, it is proved that, for rational arrangements, these maps are injective. These residues provide a very useful tool for comparing cohomology classes.

PROPOSITION 2.2. *Let $\mathcal{D}$ be an arrangement, then*

$$\begin{aligned} H^2(X;\mathbb{C}) &\cong H_1(\mathcal{D};\mathbb{C}), \text{ and} \\ H^1(X_\mathcal{D};\mathbb{C}) &\cong H_2(\mathcal{D};\mathbb{C})/\mathbb{C}. \end{aligned}$$

Proof. Consider the following exact sequence of relative pairs

$$\begin{aligned} 0 \overset{i^1}{\to} \ & H^1(X_\mathcal{D};\mathbb{C}) \overset{\delta^2}{\to} H^2(\mathbb{P}^2, X_\mathcal{D};\mathbb{C}) \overset{j^2}{\to} H^2(\mathbb{P}^2;\mathbb{C}) \cong \mathbb{C} \overset{i^2}{\to} \\ \overset{i^2}{\to} \ & H^2(X_\mathcal{D};\mathbb{C}) \overset{\delta^3}{\to} H^3(\mathbb{P}^2, X_\mathcal{D};\mathbb{C}) \overset{j^3}{\to} H^3(\mathbb{P}^2;\mathbb{C}) \cong 0 \end{aligned} \tag{2.2}$$

Let $\mathcal{TD}$ be a regular neighborhood of $\mathcal{D}$. Note that $X_\mathcal{D}$ has the homotopy type of $\mathbb{P}^2 \setminus \mathcal{TD}$. Hence, using excision and Lefschetz duality, one obtains

$$H^n(\mathbb{P}^2, X_\mathcal{D};\mathbb{C}) \overset{exc.}{\cong} H^n(\mathcal{TD}, \partial\mathcal{TD};\mathbb{C}) \overset{L}{\cong} H_{4-n}(\mathcal{TD};\mathbb{C}) \cong H_{4-n}(\mathcal{D};\mathbb{C}).$$

Thus, the sequence (2.2) becomes

$$0 \overset{i^1}{\to} H^1(X_\mathcal{D};\mathbb{C}) \overset{\delta^2}{\to} H_2(\mathcal{D};\mathbb{C}) \overset{j^2}{\to} \mathbb{C} \overset{i^2}{\to} H^2(X_\mathcal{D};\mathbb{C}) \overset{\delta^3}{\to} H_1(\mathcal{D};\mathbb{C}) \overset{j^3}{\to} 0.$$

Since $H_2(\mathcal{D};\mathbb{C})$ is non-trivial and j^2 is non-zero, j^2 is in fact surjective. Therefore

$$H^2(X_\mathcal{D};\mathbb{C}) \cong H_1(\mathcal{D};\mathbb{C}), \text{ and } H^1(X_\mathcal{D};\mathbb{C}) \cong H_2(\mathcal{D};\mathbb{C})/\mathbb{C}. \quad \square$$

NOTATION 2.3. Let Y be a topological space. In what follows we will denote by $h_i(Y)$ (resp. $h^i(Y)$) the dimension of the vector space $H_i(Y;\mathbb{C})$ (resp. $H^i(Y;\mathbb{C})$). Note that, by the Universal Coefficient Theorem, $h_i(Y) = h^i(Y)$.

PROPOSITION 2.4. *Let $\mathcal{D}$ be an arrangement with $n+1$ irreducible components. Then*

$$\dim H^2(X_\mathcal{D};\mathbb{C}) = 1 + \sum_{P \in \operatorname{Sing}\mathcal{D}} (r_P - 1) - \sum_{i=0}^{n} (\chi(\hat{\mathcal{C}}_i) - 1),$$

where r_P is the number of branches of $\mathcal{D}$ passing through P and $\hat{\mathcal{C}}_i$ is the normalization of the irreducible component $\mathcal{C}_i$.

Proof. By Proposition 2.2, $h^2(X_\mathcal{D}) = h_1(\mathcal{D})$. On the other hand

$$h_2(\mathcal{D}) = n + 1$$

and

$$\chi(\mathcal{D}) = \sum_{i=0}^{n} \chi(\hat{\mathcal{C}}_i) - \sum_{P \in \operatorname{Sing}\mathcal{D}} (r_P - 1).$$

Therefore

$$\begin{aligned} h_1(\mathcal{D}) &= 1 + h_2(\mathcal{D}) - \chi(\mathcal{D}) = \\ &= 1 + (n+1) - \textstyle\sum_{i=0}^{n} \chi(\hat{\mathcal{C}}_i) + \sum_{P \in \operatorname{Sing}\mathcal{D}} (r_P - 1) = \\ &= 1 + \textstyle\sum_{P \in \operatorname{Sing}\mathcal{D}} (r_P - 1) - \sum_{i=0}^{n} (\chi(\hat{\mathcal{C}}_i) - 1). \quad \square \end{aligned}$$

In the following proposition the residue maps are defined and applied to the case of rational arrangements.

PROPOSITION 2.5. *Let $\mathcal{D}$ be an arrangement. Then there is an injection*

$$H^1(X_\mathcal{D};\mathbb{C}) \overset{\mathrm{Res}^{[1]}}{\hookrightarrow} H^0(\overline{\mathcal{D}}^{[1]};\mathbb{C})$$

and a map

$$H^2(X_\mathcal{D};\mathbb{C}) \overset{\mathrm{Res}^{[2]}}{\to} H^0(\overline{\mathcal{D}}^{[2]};\mathbb{C})$$

via the Poincaré residue operator – c.f. Definition 1.21.
Moreover, $\mathrm{Res}^{[2]}$ *is injective if and only if $\mathcal{D}$ is a rational arrangement.*
Such morphisms $\mathrm{Res}^{[*]}$ *will be called* **residue maps**.

Proof. By Theorem 1.28, the Poincaré residue operator

$$R^{[i]}: W_i^{0,*}/W_{i-1}^{0,*} \longrightarrow A^{*-i}_{\overline{\mathcal{D}}^{[i]}}$$

induces an isomorphism on d-cohomology. Therefore

$$H^k(W_i^{0,*}/W_{i-1}^{0,*},d) \cong H^k(A^{*-i}_{\overline{\mathcal{D}}^{[i]}},d) \cong H^{k-i}(\overline{\mathcal{D}}^{[i]};\mathbb{C}). \tag{2.3}$$

Consider the following short exact sequence of complexes

$$0 \to W_{i-1} \to W_i \to W_i/W_{i-1} \to 0$$

where W_i denotes the complex

$$\left(W_i^{0,*},d\right).$$

Its corresponding long exact sequence of d-cohomology is

$$\cdots \to H^i(W_{i-1},d) \to H^i(W_i,d) \to H^i(W_i/W_{i-1},d) \to$$
$$\to H^{i+1}(W_{i-1},d) \to H^{i+1}(W_i,d) \to H^{i+1}(W_i/W_{i-1},d) \to \dots \tag{2.4}$$

By the de Rham Theorem $H^i(W_i,d)\cong H^i(X_\mathcal{D};\mathbb{C})$. Hence, there exist maps

$$H^i(X_\mathcal{D};\mathbb{C}) \overset{\mathrm{Res}^{[i]}}{\to} H^0(\overline{\mathcal{D}}^{[i]};\mathbb{C}).$$

The injectivity of $\mathrm{Res}^{[1]}$ follows immediately from (2.4) (for $i=1$) and $H^1(W_0,d)=0$. In order to prove the injectivity of $\mathrm{Res}^{[2]}$ note that, from (2.4) (for $i=2$), it is enough to prove that $H^2(W_1,d)\cong 0$ provided $\mathcal{D}$ is a rational arrangement. Assuming now that $\mathcal{D}$ is a rational arrangement one has

$$H^1(W_0,d) \cong H^1(\overline{X_\mathcal{D}};\mathbb{C}) = 0$$
$$H^1(W_1,d) \cong H^1(X_\mathcal{D};\mathbb{C}) \cong H_2(\mathcal{R};\mathbb{C})/\mathbb{C} \cong \mathbb{C}^n \text{ by Proposition 2.2,}$$
$$H^1(W_1/W_0,d) \cong H^0(\overline{\mathcal{D}}^{[1]};\mathbb{C}),$$
$$H^2(W_0,d) \cong H^2(\overline{X}_\mathcal{D};\mathbb{C}) \text{ and}$$
$$H^2(W_1/W_0,d) \cong H^1(\overline{\mathcal{D}}^{[1]};\mathbb{C}) \cong \mathbb{C}^g,$$

where g is the sum of the genera of the irreducible components of $\mathcal{D}$. Hence (2.4) (for $i=1$) becomes

$$0 \to \mathbb{C}^n \to \mathbb{C}^{n+1+e} \to H^2(\overline{X}_\mathcal{D};\mathbb{C}) \to H^2(W_1,d) \to \mathbb{C}^g \to 0,$$

where e is the number of exceptional components in the resolution of $\mathcal{D}$. The euler characteristic of a long exact sequence is zero, thus

$$n-(n+1+e)+h^2(\overline{X}_\mathcal{D})-h^2(W_1)+g=0. \tag{2.5}$$

The euler characteristic of $\overline{X}_\mathcal{D}$ can be computed as

$$\chi(\overline{X}_\mathcal{D}) = h^0(\overline{X}_\mathcal{D}) - h^1(\overline{X}_\mathcal{D}) + h^2(\overline{X}_\mathcal{D}) = 1-0+h^2(\overline{X}_\mathcal{D})$$

or

$$\chi(\overline{X}_{\mathcal{D}}) = \chi(\mathbb{P}^2) + e = 2 + e.$$

Hence

$$h^2(\overline{X}_{\mathcal{D}}) = (e+2) - 1 = e + 1.$$

Substituting this in (2.5), one has

$$n - (n+1+e) + (e+1) + g = h^2(W_1)$$

which yields $H^2(W_1, d) \cong \mathbb{C}^g$. □

2.2. Logarithmic forms on non-normal crossing divisors

In this section, the notion of logarithmic forms on a non-normal crossing divisor $\mathcal{D}$ relative to a resolution is presented. In general, the sheaf of such forms does not coincide with the classical sheaf of logarithmic forms – c.f. [**30**]. Logarithmic forms on non-normal crossing divisors can be connected to the usual logarithmic forms via a resolution of singularities. The purpose of the forthcoming sections is to construct global logarithmic forms on $\mathcal{D}$ that provide a basis for the spaces $H^k(X_{\mathcal{D}}; \mathbb{C})$.

NOTATION 2.6. In what follows we shall consider $\mathcal{D} = \mathcal{C}_0 \cup \mathcal{C}_1 \cup \ldots \cup \mathcal{C}_n$ an arrangement of curves containing at least one transversal line $\mathcal{C}_0$ which shall be referred to as the line at infinity. This technicality does not represent a serious restriction, as explained in 5.1. Consider a system of coordinates $[X : Y : Z]$ in $\mathbb{P}^2$ such that $\mathcal{C}_0 := \{Z = 0\}$. If one writes $\mathcal{D} := \{D = 0\}$, D can be expressed as a product $C_0 \cdot C_1 \cdot \cdots \cdot C_n$ where $\mathcal{C}_i := \{C_i = 0\}$, C_i are irreducible components of D and $d_i := deg(C_i)$. The affine arrangement in $\mathbb{C}^2 = \mathbb{P}^2 \setminus \mathcal{C}_0$ – with coordinates (x, y) – will be denoted by $\mathcal{D}^* = \mathcal{C}_1^* \cup \cdots \cup \mathcal{C}_n^*$, where C_i^* is the equation of $\mathcal{C}_i^*$. Note that C_i is the homogenization of C_i^* with respect to Z, that is $C_i = Z^{d_i} C_i^*(\frac{X}{Z}, \frac{Y}{Z})$.

DEFINITIONS 2.7.

1. A C^∞ form φ on $X_{\mathcal{D}}$ shall be called **logarithmic (with respect to a divisor $\mathcal{D}$ and a resolution π)** if $\pi^*\varphi$ is logarithmic on $\overline{X}_{\mathcal{D}}$ with respect to the normal crossing divisor $\overline{\mathcal{D}} = (\pi^*(\mathcal{D}))_{red}$ – c.f. Definition 1.15.2. Therefore, one has the corresponding sheaf

$$\pi_* \mathcal{A}_{\overline{X}_{\mathcal{D}}}(\log\langle \overline{\mathcal{D}} \rangle).$$

In the sequel, a logarithmic form with respect to a divisor $\mathcal{D}$ and a resolution π will be referred to as simply a logarithmic form if $\mathcal{D}$ and π are known and no ambiguity is likely to arise.

2. Once $\mathcal{D}$ and π are fixed one can define the **residue map** $\mathrm{Res}_\pi^{[*]}(\varphi)$ **of a logarithmic form** φ as follows

$$\begin{array}{ccc} \pi_* \mathcal{A}^k_{\overline{X}_{\mathcal{D}}}(\log\langle \overline{\mathcal{D}} \rangle) & \xrightarrow{\mathrm{Res}_\pi^{[k]}} & H^0(\overline{\mathcal{D}}^{[k]}; \mathbb{C}) \\ \varphi & \mapsto & \mathrm{Res}^{[k]}(\pi^* \varphi). \end{array}$$

2.3. A basis for $H^1(\mathbb{P}^2 \setminus \mathcal{D};\mathbb{C})$

For line arrangements, the first cohomology space of $X_{\mathcal{L}}$ is generated by holomorphic 1-forms, each one having poles along a line. This result is also valid for general arrangements as shown in this section.

According to 2.6 one can consider the following differential forms on $\mathbb{C}^3 \setminus \{C_i = 0\}$

$$\frac{d\mathcal{C}_i}{\mathcal{C}_i} := d\left(\log \frac{C_i}{C_0^{d_i}}\right) = d(\log C_i) - d_i\ d(\log C_0),$$

where $i = 1, ..., n$. Note that, since any two determinations of $\log \frac{C_i}{C_0^{d_i}}$ differ by a constant, their differential is well defined.

Moreover, such forms are well defined on $\mathbb{P}^2 \setminus \mathcal{C}_i$. This is a consequence of the following two facts:

1. Each $\frac{d\mathcal{C}_i}{\mathcal{C}_i}$ is invariant under the $\mathbb{C}^*$-action
$$\lambda \cdot (X, Y, Z) = (\lambda X, \lambda Y, \lambda Z).$$
2. They vanish on the space tangent to the fibers of the natural projection
$$\begin{array}{ccc} \mathbb{C}^3 \setminus \{C_i = 0\} & \xrightarrow{j} & \mathbb{P}^2 \setminus \mathcal{C}_i \\ (X, Y, Z) & \mapsto & [X : Y : Z] \end{array}$$
at any point.
Proof. The vector
$$E(X, Y, Z) = X\frac{\partial}{\partial X} + Y\frac{\partial}{\partial Y} + Z\frac{\partial}{\partial Z}$$
generates the space tangent to the fibers of j at (X, Y, Z). Hence one has to check
$$\frac{d\mathcal{C}_i}{\mathcal{C}_i}(E(X, Y, Z)) = 0.$$
Note that
$$\frac{d\mathcal{C}_i}{\mathcal{C}_i}(E(X, Y, Z)) = \left(X\frac{C_{i,X}}{C_i} + Y\frac{C_{i,Y}}{C_i} + Z\frac{C_{i,Z}}{C_i} - d_i Z\frac{dZ}{Z}\right)$$
where $C_{i,X}$, $C_{i,Y}$ and $C_{i,Z}$ are the derivatives of C_i with respect to X, Y and Z respectively. Finally, by the Euler identity
$$XC_{i,X} + YC_{i,Y} + ZC_{i,Z} = d_i C_i$$
and therefore
$$\left(X\frac{C_{i,X}}{C_i} + Y\frac{C_{i,Y}}{C_i} + Z\frac{C_{i,Z}}{C_i} - d_i Z\frac{dZ}{Z}\right) = \frac{d_i C_i}{C_i} - d_i = 0. \quad \square$$

DEFINITION 2.8. From the discussion above, $\frac{d\mathcal{C}_i}{\mathcal{C}_i}$ defines a global differential 1-form on $X_{\mathcal{D}}$ for any $i = 1, ..., n$.

REMARKS 2.9.

1. If π' dominates π, that is, if one has the following commutative diagram of birational morphisms
$$\begin{array}{ccc} \overline{X}'_{\mathcal{D}} & \xrightarrow{\rho} & \overline{X}_{\mathcal{D}} \\ \pi' & \searrow & \downarrow \pi \\ & & \mathbb{P}^2 \end{array}$$

then

$$\pi'_* \mathcal{A}_{\overline{X}'_{\mathcal{D}}}(\log\langle \overline{D}' \rangle) \subset \pi_* \mathcal{A}_{\overline{X}_{\mathcal{D}}}(\log\langle \overline{D} \rangle).$$

Proof. The result can be checked locally. We will do it in detail for 2-forms. For 1-forms or functions the proof is essentially the same. Let ψ be a logarithmic 2-form at $P \in \mathbb{P}^2$. By Lemma 1.17, ψ can be written locally as

$$\varphi_Q \, \frac{dx \wedge dy}{xy}$$

at any point $Q \in \pi^{-1}(P)$. If the map ρ doesn't blow-up Q, then its inverse image is automatically logarithmic on $\overline{X}'_{\mathcal{D}}$. Otherwise, Q is the center of a blow-up ε. Then, on one chart, its inverse image becomes

$$\varepsilon^* \varphi_Q(x,y) \, \frac{dx \wedge dy}{xy} = \varphi_Q(u,uv) \, \frac{du \wedge d(uv)}{u^2 v} = \varphi_Q(u,uv) \, \frac{du \wedge dv}{uv}$$

which is also logarithmic along $\{u = 0\}$. Therefore, since ρ is a finite sequence of blow-ups the result follows. □

2. A simple consequence of the previous proof. Let ψ be a logarithmic 2-form locally at a point P, and assume $\mathcal{D}$ is a normal crossing divisor at P. If one blows up the point P, then the pull-back of ψ is a logarithmic 2-form on the pull-back of $\mathcal{D}$ and has poles along the strict transform of $\mathcal{D}$ and the exceptional divisor.

PROPOSITION 2.10. *The pull-back $\pi^* \frac{d\mathcal{C}_i}{\mathcal{C}_i}$ defines a logarithmic 1-form on $\overline{X}_{\mathcal{D}}$.*

Proof. Since the 1-forms $\frac{d\mathcal{C}_i}{\mathcal{C}_i}$ are C^∞ on $X_{\mathcal{D}}$ the statement is trivial on $X_{\mathcal{D}}$. Hence, it is enough to check the statement locally at the points of $\overline{X}_{\mathcal{D}} \setminus X_{\mathcal{D}}$. Note that, on the affine chart $\mathbb{C}^2$, the form $\frac{d\mathcal{C}_i}{\mathcal{C}_i}$ can be written as $\frac{d\mathcal{C}_i^*}{\mathcal{C}_i^*}$. Let $P \in \overline{\mathcal{D}}$ be a point on the inverse image of $\mathcal{D}$. Hence, the pull-back of $\frac{d\mathcal{C}_i^*}{\mathcal{C}_i^*}$ by π can be written locally at P as

$$\frac{d(x^n y^m)}{x^n y^m} = n \frac{dx}{x} + m \frac{dy}{y}$$

which is logarithmic at P. Finally, if P is a point on the line at infinity, then $P \in \operatorname{Reg}(\mathcal{C}_i)$ and thus our form can be written locally as

$$\frac{du}{u} - d_i \frac{dz}{z}$$

where u is a local equation of $\mathcal{C}_i$ at P. □

Finally, the injectivity of the residue map $\operatorname{Res}^{[1]}$ will prove that the forms of the previous proposition define a basis for $H^1(X_{\mathcal{D}}; \mathbb{C})$.

THEOREM 2.11. *The cohomology classes of*

$$v_1(\mathcal{D}) := \left\{ \pi^* \frac{d\mathcal{C}_1}{\mathcal{C}_1}, \ldots, \pi^* \frac{d\mathcal{C}_n}{\mathcal{C}_n} \right\}$$

constitute a basis for $H^1(X_{\mathcal{D}}; \mathbb{C})$.

Proof. By the de Rham theorem, the classes $\left[\frac{d\mathcal{C}_i}{\mathcal{C}_i} \right]$ with respect to d-cohomology are elements of $H^1(X_{\mathcal{D}}; \mathbb{C})$. Moreover, their residues can be obtained as follows

–c.f. 2.5–

$$\left(\operatorname{Res}^{[1]}_{\pi}\frac{d\mathcal{C}_i}{\mathcal{C}_i}\right)_{\tilde{\mathcal{C}}_j} = \left(\operatorname{Res}^{[1]}\pi^*\frac{d\mathcal{C}_i}{\mathcal{C}_i}\right)_{\tilde{\mathcal{C}}_j} = \begin{cases} (-1)^{n-i} & \text{if } i=j\neq 0\\ 0 & \text{if } i\neq j\neq 0\\ (-1)^{n+1}d_i & \text{if } j=0\end{cases}$$

where $\tilde{\mathcal{C}}_j \in \overline{\mathcal{D}}^{[1]}$ is the strict transform of $\mathcal{C}_j$. Since the images are linearly independent in $H^0(\overline{\mathcal{D}}^{[1]};\mathbb{C})$, the forms $\frac{d\mathcal{C}_i}{\mathcal{C}_i}$ are also linearly independent in $H^1(X_{\mathcal{D}};\mathbb{C})$. Finally, by Proposition 2.2, $v_1(\mathcal{D})$ has maximal cardinality. □

2.4. The problem with the 2-forms

One can easily check that Brieskorn's theorem –c.f. (2.1)– is no longer true for rational arrangements. For instance, let $\mathcal{R} = \mathcal{C}_0 \cup \mathcal{C}_1 \cup \mathcal{C}_2$ be a plane quartic, where $\mathcal{C}_1$ is a conic, $\mathcal{C}_2$ is a line tangent to $\mathcal{C}_1$, and $\mathcal{C}_0$ is the line at infinity. By propositions 2.4 and 2.2, one has $h^1(X_{\mathcal{R}}) = h^2(X_{\mathcal{R}}) = 2$ and hence

$$\wedge^2 H^1(X_{\mathcal{R}};\mathbb{C}) \ncong H^2(X_{\mathcal{R}};\mathbb{C}). \tag{2.6}$$

In general, what happens is that the left-hand side in (2.6) is strictly contained in the right-hand side. The first problem one should deal with in order to construct a basis for $H^2(X_{\mathcal{R}};\mathbb{C})$ is finding enough 2-forms.

Going back to the 1-forms defined in 2.8, it is straightforward to check that

$$\frac{d\mathcal{C}_i}{\mathcal{C}_i}\wedge\frac{d\mathcal{C}_j}{\mathcal{C}_j} = d\left(\log\frac{C_i}{C_0^{d_i}}\right)\wedge d\left(\log\frac{C_j}{C_0^{d_j}}\right) =$$

$$\left(\frac{dC_i}{C_i} - d_i\frac{dZ}{Z}\right)\wedge\left(\frac{dC_j}{C_j} - d_j\frac{dZ}{Z}\right) = \frac{C_{i,X}C_{j,Y} - C_{i,Y}C_{j,X}}{ZC_iC_j}\,\omega \tag{2.7}$$

where $\omega := (ZdX\wedge dY - YdX\wedge dZ + XdY\wedge dZ)$. That is, the product of two 1-forms in $v_1(\mathcal{D})$ is a 2-form vanishing along

$$J_{\mathcal{C}_i,\mathcal{C}_j} := \{J_{C_i,C_j} = C_{i,X}C_{j,Y} - C_{i,Y}C_{j,X} = 0\}.$$

This set of zeroes is non-vacuous unless both $\mathcal{C}_i$ and $\mathcal{C}_j$ are lines.

DEFINITION 2.12. We shall refer to J_{C_i,C_j} as the **Jacobian of C_i and C_j with respect to** $Z(=C_0)$ (or simply as **Jacobian** if no ambiguity seems likely to arise). In order to emphasize the distinction between projective and affine we shall call **affine Jacobian** $J_{C_i^*,C_j^*}$ the dehomogenization of J_{C_i,C_j}.

REMARK 2.13. The affine Jacobian is invariant under rotation in $\mathbb{C}^2 = \mathbb{P}^2 \setminus \{Z=0\}$.

Proof. Let us consider (x,y) an affine system of coordinates in $\mathbb{C}^2$. Let $A \in SL(2,\mathbb{C})$ be a rotation, so that $\binom{x}{y} = A\binom{u}{v}$. Let $f,g \in \mathbb{C}[x,y]$ be two polynomials. Writing $\tilde{f}$ and $\tilde{g}$ for f and g as polynomials of u and v, one has

$$\begin{pmatrix} f_x & f_y \\ g_x & g_y\end{pmatrix} = \begin{pmatrix}\tilde{f}_u & \tilde{f}_v \\ \tilde{g}_u & \tilde{g}_v\end{pmatrix} A.$$

Therefore,

$$J_{f,g}(x,y) = J_{\tilde{f},\tilde{g}}(u,v). \quad \square$$

EXAMPLES 2.14.

1. If $C_i^* = x^2 - y^3$ and $C_j^* = x - y^3$ then
$$J_{C_i,C_j} = 3Y^2 Z(Z - 2X).$$
2. The Jacobian of C_i and C_j can be zero even for non-constant polynomials. For example, that is the case when $k = d_i = d_j$, $C_i = C$ and $C_j = C + Z^k$. Then
$$\begin{aligned}\frac{d\mathcal{C}_i}{\mathcal{C}_i} \wedge \frac{d\mathcal{C}_j}{\mathcal{C}_j} &= \left(\frac{dC_i}{C_i} - k\frac{dZ}{Z}\right) \wedge \left(\frac{dC_j}{C_j} - k\frac{dZ}{Z}\right) = \\ &= \left(\frac{dC}{C} - k\frac{dZ}{Z}\right) \wedge \left(\frac{d(C+Z^k)}{C+Z^k} - k\frac{dZ}{Z}\right) = \\ &= kZ^{k-1}\frac{dC \wedge dZ}{C(C+Z^k)} - k\frac{dC \wedge dZ}{CZ} - k\frac{dZ \wedge dC}{Z(C+Z^k)} = 0.\end{aligned}$$

Fortunately, this "unwanted" situation does not appear in arrangements with a transversal line at infinity.

PROPOSITION 2.15. *If the line at infinity $\mathcal{C}_0$ is transversal to $\mathcal{C}_i \cup \mathcal{C}_j$, then the Jacobian J_{C_i,C_j} of C_i and C_j with respect to C_0 has degree $d_i + d_j - 2$ and is not divisible by C_0.*

Proof. It is enough to prove that the affine Jacobian has degree $d_i + d_j - 2$. Let's decompose C_i^* (resp. C_j^*) into a sum of homogeneous terms $h_{i,d}$ –resp. $h_{j,d}$– of degree d. By transversality of the line at infinity, $h_{i,d_i} = f$ and $h_{j,d_j} = g$ are non-zero. Therefore all one needs to prove is that
$$f_x g_y - f_y g_x \neq 0. \tag{2.8}$$
Since f and g are homogeneous on x and y, one has $xf_x + yf_y = d_i f$ and $xg_x + yg_y = d_j g$ by Euler's identity. Hence, multiplying the left-hand side of (2.8) by xy one has
$$\begin{aligned}xf_x y g_y - xyf_y g_x &= (d_i f - yf_y)(d_j g - xg_x) - xyf_y g_x = \\ &= d_i d_j fg - d_i xfg_x - d_j yf_y g. \end{aligned}\tag{2.9}$$
Let (x_0, y_0) be a point such that $[x_0 : y_0 : 0] \in C_0 \cap C_i$. Rotating the affine system of coordinates (x, y) if necessary –c.f. 2.13– it can be assumed that $y_0 f_y(x_0, y_0) \neq 0$. Note that, since $C_i(x_0, y_0, 0) = f(x_0, y_0) = 0$ and $C_j(x_0, y_0, 0) = g(x_0, y_0) \neq 0$, (2.9) becomes
$$-d_j y_0 f_y(x_0, y_0) g(x_0, y_0) \neq 0. \quad \square$$
From the discussion above one can give the following

DEFINITION 2.16. Let C_i and C_j be equations of the components $\mathcal{C}_i$ and $\mathcal{C}_j$ of $\mathcal{D}$ as in 2.6. We shall define the following 2-form
$$s_{i,j} := \frac{Z^{d_{ij}-3}}{C_{ij}}\,\omega,$$
where $d_{ij} := \deg(C_{ij})$ and
$$C_{ij} := \begin{cases} C_i C_j & if\ i \neq j \\ C_i & if\ i = j \end{cases}.$$

REMARKS 2.17.

1. By (2.7), it is readily seen that
$$s_{i,j} := \begin{cases} \frac{Z^{d_{ij}-2}}{J_{C_i,C_j}}\left(\frac{d\mathcal{C}_i}{\mathcal{C}_i}\wedge\frac{d\mathcal{C}_j}{\mathcal{C}_j}\right) & if\ i \neq j \\ \\ \frac{Z^{d_{ij}-2}C_k}{J_{C_i,C_k}}\left(\frac{d\mathcal{C}_i}{\mathcal{C}_i}\wedge\frac{d\mathcal{C}_k}{\mathcal{C}_k}\right) & if\ i = j\ for\ any\ \mathcal{C}_k \neq \mathcal{C}_i, \end{cases}$$
where J_{C_i,C_j} denotes the Jacobian of C_i and C_j with respect to C_0 –c.f. Definition 2.12.
2. The restriction $s^*_{i,j}$ of $s_{i,j}$ to $\mathbb{C}^2$ is
$$s^*_{i,j} = \frac{dx \wedge dy}{C^*_{ij}}.$$
Let us consider the map
$$H^2(X_{\mathcal{D}};\mathbb{C}) \xrightarrow{\mathrm{Res}^{[2]}} H^0(\overline{\mathcal{D}}^{[2]};\mathbb{C}) \xrightarrow{j} H^0(\overline{\mathcal{D}}^{*[2]};\mathbb{C}) \quad ,$$
where $\mathrm{Res}^{[2]}$ is the residue map described in Proposition 2.5 and j is the restriction map. Suppose one has a projective 2-form s which is logarithmic at a point $P \in \overline{\mathcal{D}}^{*[2]}$. For convenience, we shall often write $\mathrm{Res}^{[2]}(s^*)$ for
$$(j \circ \mathrm{Res}^{[2]})(s).$$
3. Note that $s_{i,j}$ is not necessarily logarithmic. For example, if $\mathcal{C}_i$ has a cusp at P, even if $\mathcal{C}_j$ does not pass through P, one has locally at $P = 0$ –and up to units– the following
$$\frac{dC_i \wedge dC_j}{J_{C_i,C_j}C_iC_j} = \frac{dx \wedge dy}{x^2 - y^3} \overset{\substack{x=u_1v_1\\ y=v_1}}{\longleftarrow} \frac{du_1 \wedge dv_1}{v_1(u_1^2 - v_1)}$$
$$\overset{\substack{u_1=u_2\\ v_1=u_2v_2}}{\longleftarrow} \frac{du_2 \wedge dv_2}{u_2v_2(u_2 - v_2)} \overset{\substack{u_2=u_3\\ v_2=u_3v_3}}{\longleftarrow} \frac{du_3 \wedge dv_3}{u_3^2 v_3}$$
which is not logarithmic.
4. If $\mathcal{C}_0$ is transversal to $\mathcal{C}_{ij}$, then $s_{i,j}$ is clearly logarithmic at infinity.
5. As mentioned at the beginning of this section (2.1), for line arrangements, the 2-forms $\frac{d\mathcal{C}_i}{\mathcal{C}_i}\wedge\frac{d\mathcal{C}_j}{\mathcal{C}_j}$ generate $H^2(X_{\mathcal{L}};\mathbb{C})$. Note that in that case J_{C_i,C_j} are constant and hence the 2-forms $s_{i,j}$ also generate $H^2(X_{\mathcal{L}};\mathbb{C})$ –since they are multiples of $\frac{d\mathcal{C}_i}{\mathcal{C}_i}\wedge\frac{d\mathcal{C}_j}{\mathcal{C}_j}$.

2.5. Logarithmic ideals

In this section, logarithmic ideals and sheaves associated with an arrangement $\mathcal{D}$ and a resolution π are defined. Their global sections allow for a construction of logarithmic 2-forms on $\mathcal{D}$ with respect to π. A useful description of logarithmic ideals results from the use of multiplicity trees.

DEFINITIONS 2.18.

1. Let $f \in \mathbb{C}\{x,y\}$ be a holomorphic germ at P and π a resolution of V_f. An ideal $I \subset \mathbb{C}\{x,y\}$ is called a **logarithmic ideal for f at P** if for any germ $h \in I$ the 2-form
$$h\,\frac{dx \wedge dy}{f} \tag{2.10}$$
is logarithmic at P –with respect to V_f and the resolution π.
2. Let $\mathcal{D} = (D)$ be an arrangement and let us fix a resolution π of $\mathcal{D}$. An ideal sheaf $\mathcal{I}$ on $\mathbb{P}^2$ is called a **logarithmic ideal sheaf for $\mathcal{D}$** if its stalks $\mathcal{I}_P$ are logarithmic ideals for the germs D_P of D at any $P \in \mathbb{P}^2$ (with respect to V_{D_P} and the resolution π restricted to a neighborhood of P).
3. If $\mathcal{I}$ is an ideal sheaf of $\mathcal{O}_{\mathbb{P}^2}$, we will denote by $\mathcal{I}(d)$ the twisted ideal sheaf $\mathcal{I} \otimes \mathcal{O}_{\mathbb{P}^2}(d)$ $d \in \mathbb{Z}$, which is an ideal sheaf of $\mathcal{O}_{\mathbb{P}^2}(d)$.

REMARK 2.19. Note that if D either does not vanish, is smooth, or has a double point at P, then any 2-form of type (2.10) is logarithmic at P. In other words, in order to decide whether an ideal sheaf $\mathcal{I}$ is logarithmic or not, one only needs to verify the local conditions on $\mathcal{I}_P$ at a finite number of points P, in fact at a subset of $\operatorname{Sing}(\mathcal{D})$.

In particular, if $\mathcal{D}$ is an arrangement as in 2.6 and $\mathcal{I}$ is an ideal sheaf on $\mathbb{P}^2$ then, by Remark 2.17.4, $\mathcal{I}$ is a logarithmic sheaf for $\mathcal{D}$ if and only if $\mathcal{I}_P$ is a logarithmic ideal for the germ D_P at any singular point P of D in $\mathbb{C}^2$.

In order to better describe logarithmic ideals, some notation has to be presented. Most of the definitions that follow can be found in [**10**] p. 503, in the construction of multiplicity sequences. Many of the ideas in this section were originally used by the Italian school.

DEFINITIONS 2.20.

1. Let $f \in \mathbb{C}\{x,y\}$ be a germ of holomorphic function at P whose set of zeroes is a germ of curve $V_f \subset X_0$ with an isolated singularity at the point P. Consider the sequence of blow-ups
$$X_0 \xleftarrow{\varepsilon_1} X_1 \xleftarrow{\varepsilon_2} X_2 \xleftarrow{\varepsilon_3} \dots \xleftarrow{\varepsilon_m} X_m = \overline{X}$$
in the resolution of X_0 at P, and denote by π_k the composition of the first k blow-ups $\pi_k = \varepsilon_k \circ \dots \circ \varepsilon_1$. The germ of curve $\tilde{V}_{f,k} = \overline{\pi_k^{-1}(V_f \setminus \{P\})}$ shall be called the **strict transform of X_0 in X_k** and its equation denoted by $\tilde{f}_k$. The reduced divisor $\pi_k^*(V_f)_{red}$ shall be denoted by $\overline{V}_{f,k}$ and called the **total transform of X_0 in X_k**. For simplicity let us write $\tilde{V}_f := \tilde{V}_{f,m}$ and $\overline{V}_f := \overline{V}_{f,m}$. The exceptional divisor in X_k resulting from the blow-up of a point in X_{k-1} shall be denoted by E_k and the points $P_k^1, \dots, P_k^{N_k}$ in $E_k \cap \tilde{V}_{f,k}$ called the **infinitely near points to P in E_k**. For convenience, the point P is also considered to be infinitely near to itself. Finally, the multiplicity of $\tilde{V}_{f,k} \subset X_\ell$ at the point P_k^ℓ shall be denoted by $\nu_{P_k^\ell}(f)$, i.e.
$$\nu_{P_k^\ell}(f) := \operatorname{mult}_{P_k^\ell}(\tilde{V}_{f,k}).$$
2. To each resolution of singularities π one can assign the **multiplicity tree of π at P** –denoted by $\mathcal{T}_P(\pi, f)$, or simply by $\mathcal{T}_P(f)$ if the resolution π of X_0 is fixed. $\mathcal{T}_P(f)$ is a tree with weights at each vertex and is defined as follows.

a. The vertices of $\mathcal{T}_P(f)$ are in bijection with the infinitely near points to P (for simplification we shall denote the vertices of the tree as their corresponding infinitely near points).
b. Two vertices of $\mathcal{T}_P(f)$, say Q and Q', are joined by an edge if and only if one of the points, say Q', belongs to X_k; the other point Q belongs to X_{k-1} and $Q' \in \varepsilon_k^{-1}(Q)$.
c. For convenience, this tree is considered to simply be a vertex if P is not a singular point of f. If $f(P) \neq 0$, then $\mathcal{T}_P(f) := \emptyset$.
d. The weight $w(\mathcal{T}_P(f), Q)$ of a vertex Q is $\nu_Q(f)$.

Another tree related to the multiplicity tree will be considered in this chapter, the **extended multiplicity tree at** P, denoted by $\tilde{\mathcal{T}}_P(f)$. It contains the multiplicity tree $\mathcal{T}_P(f)$ as a subtree and it can be constructed as follows.

a'. The vertices of $\tilde{\mathcal{T}}_P(f)$ correspond to the points of $\mathrm{Sing}(\overline{X})$. Note that each extra vertex corresponds to a non-infinitely near point in $\mathrm{Sing}(\overline{X})$, that is, an intersection of two exceptional divisors, say E_i and E_j. They shall be denoted by e_{ij} with the convention $i < j$.
b'. A vertex e_{ij} is joined to another vertex Q if Q belongs to X_k for a certain k and $e_{ij} \in \varepsilon_k^{-1}(Q)$. It is not difficult to see that $k = j$. The remaining edges come from the fact that $\mathcal{T}_P(f)$ is a subtree of $\tilde{\mathcal{T}}_P(f)$.
c'. The weight at each e_{ij} shall be defined to be 0. The weight at the remaining vertices coincide with their weight as vertices of $\mathcal{T}_P(f)$.

3. The set of vertices $|\tilde{\mathcal{T}}_P(f)|$ of an extended multiplicity tree $\tilde{\mathcal{T}}_P(f)$ is endowed with a partial order as follows. Consider the preferred point –or **root**– P and direct the edges of the tree towards P. In this **directed tree**, a point Q is said to be greater than Q' –denoted $Q \geq Q'$– if there is a directed path from Q to Q' –directed path means that all arrows are pointing in the same direction. In graph theory this situation is commonly described by calling Q an **ancestor** of Q', or Q' a **descendant** of Q. Given a set of points $\{P_1, ..., P_n\} \subset \tilde{\mathcal{T}}_P(f)$ one can define
$$\mathrm{Asc}(P_1, ..., P_n) = \{Q \in \mathcal{T}_P(f) \mid Q \geq P_i \;\; i = 1, ..., n\},$$
and
$$\mathrm{Desc}(P_1, ..., P_n) = \{Q \in \mathcal{T}_P(f) \mid Q \leq P_i \;\; i = 1, ..., n\}.$$
Multiplicity trees are **quasi-strongly connected trees**, which means that the set of common descendants $\mathrm{Desc}(P_1, ..., P_n)$ is non vacuous and inherits a linear order from $\mathcal{T}_P(f)$. The maximal element in $\mathrm{Desc}(P_1, ..., P_n)$ is called the **greatest common descendant** and denoted $g.c.d.(P_1, ..., P_n)$ –for a reference see [**7**].
4. The **degree** of a weighted tree $\mathcal{T}$ shall be defined as
$$\deg(\mathcal{T}) := \sum_{Q \in |\mathcal{T}|} \binom{w(\mathcal{T}, Q) + 1}{2}, \tag{2.11}$$
where $w(\mathcal{T}, Q)$ denotes the weight of $\mathcal{T}$ at Q.
Note that if $\mathcal{T} = \mathcal{T}_P(f)$, then $\deg(\mathcal{T})$ is the δ-invariant of the singularity of f at P.
5. Let $g \in \mathbb{C}\{x, y\}$ be another germ at P. Then one can consider the **restriction of** g **to** $\mathcal{T}_P(f)$ –or $\mathcal{T}_P(f)|_g$– as another weighted graph isomorphic to

$\mathcal{T}_P(f)$, but whose weights are now $\nu_Q(g)$. Note that, $\dim_{\mathbb{C}}\left(\mathbb{C}\{x,y\}/\mathfrak{m}^k\right) = \binom{k+1}{2}$. Hence

$$\deg(\mathcal{T}) = \dim_{\mathbb{C}}\left(\mathbb{C}\{x,y\}/\{g \in \mathbb{C}\{x,y\} \mid \mathcal{T}|_g \geq \mathcal{T}\}\right)$$

6. In order to simplify, we shall write $\mathcal{T} \cong \mathcal{T}'$ for two weighted trees that are isomorphic as trees, and $\mathcal{T} = \mathcal{T}'$ (resp. $\geq$, $\leq$, $<$ or $>$) if $\mathcal{T} \cong \mathcal{T}'$ and $w(\mathcal{T}, Q) = w(\mathcal{T}', Q)$ (resp. $\geq$, $\leq$, $<$ or $>$) for any $Q \in |\mathcal{T}| = |\mathcal{T}'|$ (note that we are using the isomorphism of trees to identify the vertices). In this context, $\mathcal{T} - k$ shall be a tree $\mathcal{T}' \cong \mathcal{T}$ so that $w(\mathcal{T}', Q) = max\{w(\mathcal{T}, Q) - k, 0\}$ for any $Q \in |\mathcal{T}|$. Particularly useful will be the tree
$$\mathcal{T}_P^{nul}(f) := \mathcal{T}_P(f) - 1. \tag{2.12}$$
Sometimes it will be necessary to compare empty trees. In this case, the conditions $=, \leq, \geq$ are vacuous and hence always satisfied.
7. Let $\mathcal{D}$ be an arrangement, note that its (extended) multiplicity trees do not depend on the equation of $\mathcal{D}$. Hence they can be denoted as $\mathcal{T}_P(\mathcal{D})$ (resp. $\tilde{\mathcal{T}}_P(\mathcal{D})$). Note that the set of maximal points of $\bigcup_{P \in \mathrm{Sing}(\mathcal{D})} |\tilde{\mathcal{T}}_P(\mathcal{D})|$ is in bijection with $\overline{\mathcal{D}}^{[2]}$.

   From (2.11) and (2.12) one can rewrite the Noether formula for the genus –c.f. [**10**] p. 614– as follows
$$g(\mathcal{C}) = \frac{(d-1)(d-2)}{2} - \sum_{P \in \mathrm{Sing}\mathcal{C}} \deg(\mathcal{T}_P^{nul}(\mathcal{C})), \tag{2.13}$$
where $\mathcal{C}$ is an irreducible plane projective curve of degree d.

EXAMPLE 2.21. Let $f = (x^3 - y^5)(x - y^2)(y^2 - x^3)y$. Figure 1 is a possible graph for $\mathcal{T}_0(f)$. The dotted edges represent what has to be added for the extended tree $\tilde{\mathcal{T}}_0(f)$.

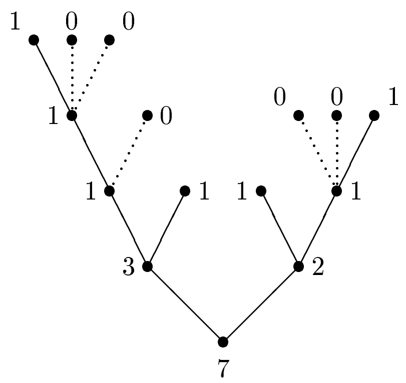


FIGURE 1. Multiplicity trees.

If $g = (y^2 + x^5)(x + y^3)$, then the restriction $\mathcal{T}_0(f)|_g$ is shown in Figure 2.

The following lemma will be strongly used throughout this section. It gives local conditions under which meromorphic forms are logarithmic. Such conditions can be expressed in terms of multiplicity trees.

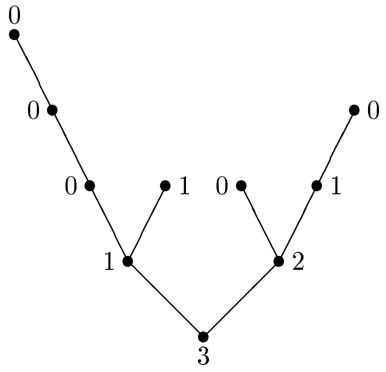


FIGURE 2. Restriction to multiplicity trees.

LEMMA 2.22. *Let f and h be germs in $\mathbb{C}\{x,y\}$ where x and y are local equations of P on a smooth germ of surface X_0. Let*

$$\psi := h\,\frac{dx\wedge dy}{f}$$

be a meromorphic 2-form on X_0. Let π be a resolution of $V_f \subset X_0$ such that

$$\mathcal{T}_P(f)|_h \;\geq\; \mathcal{T}_P^{nul}(f) \tag{2.14}$$

–c.f. Definition 2.20. Then the following results hold:

i) The 2-form ψ is logarithmic (with respect to V_f and π).

ii) Any pull-back $\pi_k^*\psi$ has poles along a subset of $\tilde{V}_{f,k}$, and if the equality in (2.14) holds, then the form $\pi_k^*\psi$ has poles exactly along $\tilde{V}_{f,k}$.

iii) The pull-back $\pi^*\psi$ of ψ has weight 1 –to recall the weight of a log form see Definition 1.18,

iv) The residues of $\pi^*\psi$ at any point of $\overline{V}_f^{[2]}$ vanish.

Proof. Let us calculate local equations for the pull-back of ψ after the first blow-up ε_1.

$$h\,\frac{dx\wedge dy}{f} \quad \overset{\substack{x=uv\\ y=v}}{\longleftarrow} \quad \tilde{h}_1\; v^{\nu_P(h)-1}\,\frac{v du\wedge dv}{v^{\nu_P}\tilde{f}_1} = \tilde{h}_1\; v^{\lambda}\,\frac{du\wedge dv}{\tilde{f}_1} \tag{2.15}$$

where $\lambda \geq 0$. Therefore $i)$ and $ii)$ follow immediately by induction.

Part $iii)$ is a consequence of the fact that $\tilde{V}_f$ is smooth.

Part $iv)$ is a consequence of $iii)$ and (1.13) for $k = 2$ and $\ell = 1$, since the residue of 2-forms of weight 1 is zero. □

The local conditions on h given in (2.14) define an ideal sheaf as follows.

DEFINITION 2.23. Let $\mathcal{D}$ be an arrangement and π a resolution of singularities for $\mathcal{D}$. The **basic logarithmic ideal sheaf of $\mathcal{D}$ with respect to π** shall be defined as follows

$$(\mathcal{I}_{\mathcal{D},\pi}^{nul})_P := \{h \in \mathcal{O}_P \mid \mathcal{T}_P(\mathcal{D},\pi)|_h \;\geq\; \mathcal{T}_P^{nul}(\mathcal{D},\pi)\}.$$

If no possible confusion results from the underlying resolution, the sheaf $\mathcal{I}^{nul}_{\mathcal{D},\pi}$ will be denoted by $\mathcal{I}^{nul}_{\mathcal{D}}$.

REMARK 2.24. By Lemma 2.22, it is obvious that the basic logarithmic ideal sheaf of $\mathcal{D}$ is in fact a logarithmic ideal sheaf.

Basic logarithmic ideal sheaves provide a way to construct global holomorphic 2-forms on the complement of the arrangement $\mathcal{D}$ as follows.

PROPOSITION 2.25. *Let $\mathcal{D}$ be an arrangement as in 2.6. Let us consider $\mathcal{C}_{ij} = \mathcal{C}_i \cup \mathcal{C}_j$ (the union of two components of $\mathcal{D}$), and denote by d_{ij} its degree –c.f. Definition 2.16. Let d be any natural number satisfying $d \leq d_{ij} - 2$. Then for any global section*

$$\varphi \in H^0(\mathbb{P}^2, \mathcal{I}^{nul}_{\mathcal{C}_{ij}}(d)),$$

the projective 2-form

$$\varphi s_{i,j} = \varphi \ \frac{Z^{d_{ij}-3}}{C_{ij}} \ \omega$$

is holomorphic, logarithmic with respect to $\mathcal{D}$, and its poles lie along $\mathcal{C}_0 \cup \mathcal{C}_{ij} \subset \mathcal{D}$.

Proof. The result follows from these two remarks:

i) In $U_Z = \mathbb{C}^2$, by Remark 2.17.2, the form $\varphi s_{i,j}$ can be written as

$$\varphi(x,y,1) \ \frac{dx \wedge dy}{C_{ij}(x,y,1)},$$

and hence one can apply the previous lemma.

ii) In the chart $U_Y = \{Y \neq 0\}$, the form $\varphi s_{i,j}$ becomes

$$z^{d_{ij}-d-2} \ \varphi(x,1,z) \ \frac{dx \wedge dz}{zC_{ij}(x,1,z)}.$$

Note that $d \leq d_{ij} - 2$ and that the line at infinity $\mathcal{C}_0 = \{Z = 0\}$ is transversal to $\mathcal{C}_i \cup \mathcal{C}_j$. Hence, $\varphi s_{i,j}$ has only logarithmic poles on $\mathcal{C}_0 \cap U_Y$. One can proceed analogously with the remaining chart $U_X = \{X \neq 0\}$. □

REMARK 2.26. In Proposition 2.25, the resolution of singularities of $\mathcal{D}$ is considered to be the underlying resolution of $\mathcal{C}_{ij}$.

The question of whether there are global sections for the twisted ideal sheaf of Proposition 2.25 has an affirmative answer in the following

PROPOSITION 2.27. *Let $\mathcal{D}$ be an arrangement as in 2.6, then*

$$\dim H^0(\mathbb{P}^2, \mathcal{I}^{nul}_{\mathcal{C}_{ij}}(d_{ij} - 2)) \geq d_{ij} - \#\{i,j\},$$

where $\#\{i,j\}$ denotes the cardinality of the set $\{i,j\}$.

Proof. Consider the exact sequence

$$0 \to \mathcal{I}^{nul}_{\mathcal{C}_{ij}}(d_{ij} - 2) \to \mathcal{O}_{\mathbb{P}^2}(d_{ij} - 2) \to \left(\mathcal{O}_{\mathbb{P}^2}/\mathcal{I}^{nul}_{\mathcal{C}_{ij}}\right)(d_{ij} - 2) \to 0.$$

Since $H^1(\mathcal{O}_{\mathbb{P}^2}(k)) = 0$ for any k, one has

$$\dim H^0(\mathbb{P}^2, \mathcal{I}^{nul}_{\mathcal{C}_{ij}}(d_{ij} - 2)) \geq \binom{d_{ij}}{2} - \dim H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}/\mathcal{I}^{nul}_{\mathcal{C}_{ij}}).$$

Note that the sheaf $\mathcal{O}_{\mathbb{P}^2}/\mathcal{I}^{nul}_{\mathcal{C}_{ij}}$ is supported on a finite number of points. Hence

$$\dim H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}/\mathcal{I}^{nul}_{\mathcal{C}_{ij}}(d_{ij} - 2)) =$$

$$= \dim H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}/\mathcal{I}^{nul}_{\mathcal{C}_{ij}}) = \sum_{P\in \mathrm{Sing}(\mathcal{C}_{ij})} \deg(\mathcal{T}^{nul}_P(\mathcal{C}_{ij}))$$

and thus

$$\dim H^0(\mathbb{P}^2, \mathcal{I}^{nul}_{\mathcal{C}_{ij}}(d_{ij}-2)) \geq \binom{d_{ij}}{2} - \sum_{P\in \mathrm{Sing}(\mathcal{C}_{ij})} \deg(\mathcal{T}^{nul}_P(\mathcal{C}_{ij})) =$$

$$= \binom{d_{ij}}{2} - \sum_{P\in \mathrm{Sing}(\mathcal{C}_{ij})} \sum_{P^\ell_k} \binom{\nu_{P^\ell_k}(\mathcal{C}_{ij})}{2}. \tag{2.16}$$

In the sequel, the following version of Vandermonde's convolution – c.f. [**46**] p. 29– will be repeatedly used:

$$\binom{a+b}{2} = \binom{a}{2} + \binom{b}{2} + ab \quad \text{for } a,b\in\mathbb{N}. \tag{2.17}$$

Assume now that $i \neq j$. Using

$$\nu_{P^\ell_k}(\mathcal{C}_{ij}) = \nu_{P^\ell_k}(\mathcal{C}_i) + \nu_{P^\ell_k}(\mathcal{C}_j)$$

and (2.17), the last term of (2.16) becomes

$$\binom{d_i}{2} + \binom{d_j}{2} + d_i d_j - \tag{2.18}$$

$$- \sum_{P\in \mathrm{Sing}(\mathcal{C}_{ij})} \sum_{P^\ell_k} \left(\binom{\nu_{P^\ell_k}(\mathcal{C}_i)}{2} + \binom{\nu_{P^\ell_k}(\mathcal{C}_j)}{2} + \nu_{P^\ell_k}(\mathcal{C}_i)\nu_{P^\ell_k}(\mathcal{C}_j) \right).$$

Combining the genus formula and (2.13), one has

$$\sum_{P\in \mathrm{Sing}(\mathcal{C}_{ij})} \sum_{P^\ell_k} \binom{\nu_{P^\ell_k}(\mathcal{C}_i)}{2} \leq \binom{d_i - 1}{2}.$$

Finally, note that

$$d_i d_j = \sum_{P\in \mathrm{Sing}(\mathcal{C}_{ij})} \sum_{P^\ell_k} \nu_{P^\ell_k}(C_i)\nu_{P^\ell_k}(\mathcal{C}_j)$$

because $(\mathcal{C}_i, \mathcal{C}_j)_P = \sum_{P^\ell_k} \nu_{P^\ell_k}(\mathcal{C}_i)\nu_{P^\ell_k}(\mathcal{C}_j)$.

Therefore, by (2.16) and (2.18), one has

$$\dim H^0(\mathbb{P}^2, \mathcal{I}^{nul}_{\mathcal{C}_{ij}}(d_{ij}-2)) \geq \binom{d_i}{2} + \binom{d_j}{2} - \binom{d_i - 1}{2} - \binom{d_j - 1}{2} =$$

$$= (d_i - 1) + (d_j - 1). \tag{2.19}$$

For the case $i = j$, the proof can be reconstructed from the preceding one using $\mathcal{C}_j = \emptyset$ and $d_j = 0$. Thus the last term in (2.19) becomes $d_i - 1$. □

COROLLARY 2.28. *If either $\mathcal{C}_i$ or $\mathcal{C}_j$ is not a line, then the logarithmic ideal sheaf $\mathcal{I}^{nul}_{\mathcal{C}_{ij}}$ twisted by $\mathcal{O}(d_{ij}-2)$ has non-trivial global sections.* □

REMARK 2.29. According to 2.17 and 2.22, if $f \in H^0(\mathbb{P}^2, \mathcal{I}^{nul}_{\mathcal{C}_{ij}}(d_{ij}-2))$ then $f^* s^*_{i,j}$ is a logarithmic affine 2-form and

$$\mathrm{Res}^{[2]}(f^*\ s^*_{i,j}) = 0.$$

2.6. Construction of logarithmic trees

In 2.23, a logarithmic ideal sheaf was defined based on the existence of a family of trees $\{\mathcal{T}_P\}_{P \in \mathbb{P}^2}$ satisfying $\mathcal{T}_P \cong \mathcal{T}_P(\mathcal{D})$. Such a logarithmic ideal sheaf led to the construction of new logarithmic 2-forms with respect to $\mathcal{D}$ –c.f. Proposition 2.25. This is the basic property of logarithmic trees. Our purpose in this section is to define new families of logarithmic trees.

Throughout this section, notations and definitions from 2.20, will be frequently used. To simplify notation, the use of the dual graph of a resolution is very convenient. The dual graph $\Gamma_P(f)$ of the resolution π at the point P satisfies the following three conditions:

i) There exists a bijection β_v between the set of vertices of $\Gamma_P(f)$ and the set of irreducible components of $\overline{V}_f$.
ii) There exists a bijection β_e between the set of edges of $\Gamma_P(f)$ and the set of singular points of $\overline{V}_f$.
iii) Two vertices δ and δ' in $\Gamma_P(f)$ are connected if $\beta_v(\delta)$ and $\beta_v(\delta')$ intersect.

Note that the set of vertices of $\Gamma_P(f)$ is hence in bijection with the set of maximal vertices of the extended multiplicity tree $\tilde{\mathcal{T}}_P(f)$.

NOTATION 2.30. Let ψ be a logarithmic 2-form (with respect to V_f and the resolution π). By Definition 2.20.7, and *ii*) above, it makes sense to consider the residue of the pull-back $\pi^*(\psi)$ of ψ at an edge γ of $\Gamma_P(f)$. We shall denote such a residue by

$$\left(\operatorname{Res}^{[2]}(\psi)\right)_\gamma .$$

Also, for the sake of simplification, we shall use the same notation δ for a local branch of V_f at P, for the strict transform of δ in $\tilde{V}_f$, for the vertex $\beta_v^{-1}(\delta)$ of the dual graph $\Gamma_P(f)$ and for the corresponding maximal vertices of the trees $\mathcal{T}_P(f)$ and $\tilde{\mathcal{T}}_P(f)$. Note also that any two local branches δ_1 and δ_2 of f are joined by a unique path in $\Gamma_P(f)$. Such a path, whether considered in $\tilde{\mathcal{T}}_P(f)$ or in $\Gamma_P(f)$, shall be denoted by $\gamma(\delta_1, \delta_2)$.

The purpose in the sequel is to describe conditions for a germ $h \in \mathbb{C}\{x, y\}$ under which the 2-form $h\,\frac{dx \wedge dy}{f}$ is logarithmic (with respect to V_f and π) and has non-zero residues only at the edges of the path $\gamma(\delta_1, \delta_2) \subset \Gamma_P(f)$. The following definition gives an idea of how trees come into play and what type of trees are useful for solving the problem.

DEFINITION 2.31. Let δ_1 and δ_2 be local branches of f at P. A weighted tree $\mathcal{T}$ will be said to be a **logarithmic tree for** δ_1 **and** δ_2 if it satisfies the following properties:

1. The trees $\mathcal{T}$ and $\mathcal{T}_P(f)$ are isomorphic as trees, i.e. $\mathcal{T} \cong \mathcal{T}_P(f)$,
2. the ideal $I := \{h \in \mathcal{O}_P \mid \mathcal{T}_P(f)|_h \geq \mathcal{T}\}$ is logarithmic, and
3. if φ belongs to $M_I := \{h \in \mathcal{O}_P \mid \mathcal{T}_P(f)|_h = \mathcal{T}\} \subset I$ then

$$\left(\operatorname{Res}^{[2]}(\varphi\,\frac{dx \wedge dy}{f})\right)_\gamma \neq 0$$

if and only if γ is an edge of $\gamma(\delta_1, \delta_2)$.

Next, a logarithmic tree for δ_1 and δ_2 is constructed. In order to do that, a recursive method is described, starting first with a non-weighted tree $\tilde{\mathcal{T}}$ ($\tilde{\mathcal{T}} \cong \tilde{\mathcal{T}}(f)_P$) and then assigning weights to its vertices. Some of the weights assigned during the procedure are definitive, others are subject to recursion. If definitive, they are denoted by $w(\bullet)$. If temporary, they are denoted by $w(\tilde{\mathcal{T}}^i_j, \bullet)$, where $\tilde{\mathcal{T}}^i_j$ is a subtree of $\tilde{\mathcal{T}}$. The finiteness of the recursion is based on the finiteness of the extended multiplicity trees.

Step 0. Let us start by considering a non-weighted tree $\tilde{\mathcal{T}} \cong \tilde{\mathcal{T}}_P(f)$ and distinguishing in it two maximal vertices δ_1 and δ_2. They shall be referred to as **distinguished points**.

Step 1. The greatest common descendant of the two distinguished vertices δ_1 and δ_2 is denoted by $P_2 = g.c.d.(\delta_1, \delta_2)$. Note that blowing up P_2 separates the branches δ_1 and δ_2.

Removing P_2 and its adjacent edges from $\tilde{\mathcal{T}}$ decomposes the tree into a finite number of subtrees $\tilde{\mathcal{T}}^2_0, \tilde{\mathcal{T}}^2_1, ..., \tilde{\mathcal{T}}^2_{n_2}$ as shown in Figure 3. From these subtrees clearly only one, say $\tilde{\mathcal{T}}^2_0$, contains descendants of P_2. From the remaining subtrees, only one, say $\tilde{\mathcal{T}}^2_1$, contains δ_1. Another one, say $\tilde{\mathcal{T}}^2_2$, contains δ_2. Any remaining subtrees are denoted by $\tilde{\mathcal{T}}^2_3, ..., \tilde{\mathcal{T}}^2_{n_2}$.

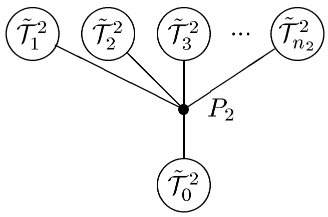


FIGURE 3. First decomposition.

From now on, this decomposition will be called **the first decomposition**.

Step 2. For technical reasons it is convenient to denote $\tilde{\mathcal{T}}_P(f)$ by $\tilde{\mathcal{T}}^1$. Definitive weights at $|\tilde{\mathcal{T}}^2_0| \cup \{P_2\} \subset |\tilde{\mathcal{T}}|$ are defined as follows:

$$w(\bullet) := \begin{cases} w(\tilde{\mathcal{T}}^1, \bullet) - 1 & \text{if } \bullet \in |\tilde{\mathcal{T}}^2_0| \\ w(\tilde{\mathcal{T}}^1, P_2) - 2 & \text{if } \bullet = P_2. \end{cases} \tag{2.20}$$

REMARK 2.32. Let us denote by X_{k_2} the smooth algebraic variety obtained after blowing up P_2. Remember that the exceptional divisor of such a blow-up is called E_{k_2} –c.f. Definition 2.20.1. Let $\psi = \varphi \, \frac{dx \wedge dy}{f}$ be a form satisfying,

$$\forall \ \bullet \in |\tilde{\mathcal{T}}^2_0|, \quad w(\mathcal{T}_P(f)|_{\varphi}, \bullet) \ \geq \ w(\tilde{\mathcal{T}}, \bullet) \quad \Big(= w(\tilde{\mathcal{T}}^1, \bullet) - 1 \text{ by } (2.20)\Big). \tag{2.21}$$

Note that, by 2.22, the pull-back $\pi^*_{k_2-1}\psi$ will have poles along the strict transform $\tilde{V}_{f,k_2-1}$. Assume also

$$w(\mathcal{T}_P(f)|_\varphi, P_2) \;\geq\; w(\tilde{\mathcal{T}}, P_2) \quad \Big(= w(\tilde{\mathcal{T}}^1, P_2) - 2 \text{ by } (2.20)\Big). \tag{2.22}$$

Then, by (2.15), the pull-back $\pi^*_{k_2}\psi = \varepsilon^*_{k_2}(\pi^*_{k_2-1}\psi)$ has a local equation

$$\pi^*_{k_2}(\varphi)\; u^\lambda \;\frac{du_{k_2} \wedge dv_{k_2}}{\tilde{f}_{k_2}} \tag{2.23}$$

where $\lambda \geq -1$ and E_{k_2} is defined by $u_{k_2} = 0$. Therefore, $\pi^*_{k_2}\psi$ has poles along $E_{k_2} \cup \tilde{V}_{f,k_2}$. Moreover, such a form will be logarithmic of weight 1 along the smooth part of $E_{k_2} \cup \tilde{V}_{f,k_2}$.

Step 3. Next some weights will be assigned to the vertices of $\tilde{\mathcal{T}}^2_1, \tilde{\mathcal{T}}^2_2, ..., \tilde{\mathcal{T}}^2_{n_2}$. Remember that only two of these subtrees, $\tilde{\mathcal{T}}^2_1$ and $\tilde{\mathcal{T}}^2_2$, have a distinguished point. The remaining subtrees have no distinguished points. Recall from Definition 2.20.2 that e_{ij} denoted the unique double point on $\pi^*(f)$, if any, lying in the intersection of E_i and E_j. Let us collect all the points of the form $e_{k_2 *}$ – that is, all singular non-infinitely near points in $\overline{V}_f$ lying on E_{k_2}. These points shall be called **secondary distinguished points**. Note that there cannot be more than one secondary distinguished point on each subtree $\tilde{\mathcal{T}}^2_i$. The two possible cases are considered separately:

ia) Suppose the subtree $\tilde{\mathcal{T}}^2_j$ does not contain any secondary distinguished point. This means that E_{k_2} is already transversal to $\tilde{V}_{f,k_2}$ at $P^2_j := g.c.d.(\tilde{\mathcal{T}}^2_j)$. Moreover, this implies that P^2_j is never the center of a blow-up. Hence, $\tilde{\mathcal{T}}^2_j$ is just a point. In this case, definitive weights are defined as:

$$w(\bullet) = \begin{cases} 0 & \text{if } \bullet \in |\tilde{\mathcal{T}}^2_j| \text{ is a distinguished point} \\ 1 & \text{otherwise.} \end{cases} \tag{2.24}$$

At this stage, the algorithm stops for $\tilde{\mathcal{T}}^2_j$.

Note that (2.24) is equivalent to

$$w(\bullet) = \begin{cases} \max\{w(\tilde{\mathcal{T}}^1, \bullet) - 1, 0\} & \text{if } \bullet \in |\tilde{\mathcal{T}}^2_j| \text{ is a distinguished point} \\ w(\tilde{\mathcal{T}}^1, \bullet) & \text{otherwise.} \end{cases} \tag{2.25}$$

iia) Suppose the subtree $\tilde{\mathcal{T}}^2_j$ contains one – and hence only one – secondary distinguished point $e_{k_2 m_j}$. As mentioned above, there is also a unique path γ_j joining $e_{k_2 m_j}$ and $P^2_j = g.c.d.(\tilde{\mathcal{T}}^2_j)$. Such a path shall be called **a distinguished path for** $\tilde{\mathcal{T}}^2_j$. In this case, temporary and definitive weights are defined as:

$$w(\tilde{\mathcal{T}}^2_j, \bullet) = \begin{cases} w(\tilde{\mathcal{T}}^1, \bullet) + 1 & \text{if } \bullet \in |\gamma_j|, & j = 1,2 \\ w(\tilde{\mathcal{T}}^1, \bullet) & \text{if } \bullet \in |\tilde{\mathcal{T}}^2_j| \setminus |\gamma_j|, & j = 1,2 \end{cases}$$

$$w(\bullet) = \begin{cases} w(\tilde{\mathcal{T}}^1, \bullet) & \text{if } \bullet \in |\gamma_j|, & j = 3, ..., n_2 \\ w(\tilde{\mathcal{T}}^1, \bullet) - 1 & \text{if } \bullet \in |\tilde{\mathcal{T}}^2_j| \setminus |\gamma_j|, & j = 3, ..., n_2. \end{cases} \tag{2.26}$$

At this stage, the algorithm stops for the subtrees $\tilde{\mathcal{T}}^2_3, ..., \tilde{\mathcal{T}}^2_{n_2}$.

REMARK 2.33. Let us consider the form $\psi = \varphi\ \frac{dx \wedge dy}{f}$ from Step 2, satisfying (2.21) and (2.22).

ib) As in case *ia*) above, let us assume E_{k_2} intersects transversally a branch of $\tilde{V}_{f,k_2} \subset X_{k_2}$, at P_j^2. Hence, according to (2.23), the form $\pi_{k_2}^* \psi$ is logarithmic at P_j^2. Moreover, if ψ satisfies (2.25), then $\pi_{k_2}^* \psi$ has a non-zero residue at P_j^2 if and only if P_j^2 is a distinguished point and ψ satisfies the equalities in (2.21) and (2.22).

iib) As in *iia*) above, let us assume E_{k_2} does not intersect $\tilde{V}_{f,k_2} \subset X_{k_2}$ transversally at P_j^2. Note that $u_{k_2} \tilde{f}_{k_2}$, where $u_{k_2} = 0$ defines E_{k_2}, is a reduced equation for the pull-back of $\tilde{f}_{k_2-1}$ after blowing up the point P_2. Let us consider the multiplicity tree $\tilde{\mathcal{T}}_{P_j^2}(u_{k_2} \tilde{f}_{k_2})$. Note that $\tilde{\mathcal{T}}_{P_j^2}(u_{k_2} \tilde{f}_{k_2}) \cong \tilde{\mathcal{T}}_j^2$. Moreover, from (2.26), it is readily seen that

$$\tilde{\mathcal{T}}_j^2 = \tilde{\mathcal{T}}_{P_j^2}^{nul}(u_{k_2} \tilde{f}_{k_2}) \quad \Big(= \tilde{\mathcal{T}}_{P_j^2}(u_{k_2} \tilde{f}_{k_2}) - 1\Big) \qquad \text{if } j = 3, \ldots, n_2$$

and

$$\tilde{\mathcal{T}}_j^2 = \tilde{\mathcal{T}}_{P_j^2}(u_{k_2} \tilde{f}_{k_2}) \quad \text{if } j = 1, 2.$$

As a simple consequence of *ib*) and *iib*) above, note that the subtrees $\tilde{\mathcal{T}}_j^2$ are isomorphic to extended multiplicity trees of $\tilde{V}_{f,k_2} \cup E_{k_2} \subset X_{k_2}$ at the points P_j^2.

Step 4. In order to use a recursive method, both secondary and non-secondary distinguished points will all be renamed as distinguished points. Consider now the subtrees $\tilde{\mathcal{T}}_1^2$ and $\tilde{\mathcal{T}}_2^2$ separately. Each of these subtrees has two distinguished points: one of type $e_{k_2 m_i}$ and the other one of type δ_i, for $i = 1, 2$. From Remark 2.33, $\tilde{\mathcal{T}}_1^2$ and $\tilde{\mathcal{T}}_2^2$ become multiplicity trees with two distinguished vertices. Therefore one can decompose $\tilde{\mathcal{T}}_1^2$ and $\tilde{\mathcal{T}}_2^2$ into subtrees as in Step 1. Let us denote their components as in Figure 4 (for $i = 3$) and start the process from Step 1. Formulæ (2.20), (2.25) and (2.26) can be updated using trees $\tilde{\mathcal{T}}_j^3$ and $\tilde{\mathcal{T}}_j^2$ instead of $\tilde{\mathcal{T}}_j^2$ and $\tilde{\mathcal{T}}_j^1$ as shown in (2.27), (2.28) and (2.29).

The decomposition of Figure 3 is called **the first decomposition**. In general, a decomposition such as the one shown in Figure 4 is called the $(i-1)$**-th decomposition**.

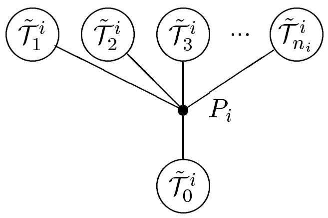


FIGURE 4. $(i-1)$-th decomposition.

A subtree of $\tilde{\mathcal{T}}$ is said **of type 0** if it is $\tilde{\mathcal{T}}_0^i$ at a certain step. Analogously, is said to be **of type 1** (or **2**) if it is $\tilde{\mathcal{T}}_j^i$, for some $j \in \{1, 2\}$ (or $\tilde{\mathcal{T}}_j^i$, for some $j \geq 3$ respectively). The point P_i on an $(i-1)$-th decomposition is called a **central**

point of $\tilde{\mathcal{T}}_P(f)$. Also, a subtree is called **simple** if it is of type 1 or 2 and is not decomposed further.

In the process, formulæ (2.20), (2.25) and (2.26) can be rewritten as

$$w(\bullet) := \begin{cases} w(\tilde{\mathcal{T}}_*^i, \bullet) - 1 & \text{if } \bullet \in |\tilde{\mathcal{T}}_0^{i+1}| \\ w(\tilde{\mathcal{T}}^i, P_{i+1}) - 2 & \text{if } \bullet = P_{i+1} \end{cases} \tag{2.27}$$

$$w(\bullet) = \begin{cases} \max\{w(\tilde{\mathcal{T}}_*^i, \bullet) - 1, 0\} & \text{if } \bullet \in |\tilde{\mathcal{T}}_*^{i+1}| \text{ is a distinguished point} \\ w(\tilde{\mathcal{T}}_*^i, \bullet) & \text{otherwise,} \end{cases} \tag{2.28}$$

$$w(\tilde{\mathcal{T}}_j^{i+1}, \bullet) = \begin{cases} w(\tilde{\mathcal{T}}_*^i, \bullet) + 1 & \text{if } \bullet \in |\gamma_j|, & j = 1, 2 \\ w(\tilde{\mathcal{T}}_*^i, \bullet) & \text{if } \bullet \in |\tilde{\mathcal{T}}_j^{i+1}| \setminus |\gamma_j|, & j = 1, 2 \end{cases}$$

$$w(\bullet) = \begin{cases} w(\tilde{\mathcal{T}}_*^i, \bullet) & \text{if } \bullet \in |\gamma_j|, & j = 3, ..., n_2 \\ w(\tilde{\mathcal{T}}_*^i, \bullet) - 1 & \text{if } \bullet \in |\tilde{\mathcal{T}}_j^{i+1}| \setminus |\gamma_j|, & j = 3, ..., n_2. \end{cases} \tag{2.29}$$

Clearly, this process will terminate, since going through Step 1 strictly decreases the cardinality of $|\tilde{\mathcal{T}}_j^i|$.

Step 5. Once the weighted tree $\tilde{\mathcal{T}}$ is constructed, one obtains the tree $\mathcal{T}_P^{\delta_1,\delta_2}(f)$ from $\tilde{\mathcal{T}}$ by deleting the vertices, edges and weights corresponding to points non-infinitely near to P.

By construction, one has the following.

LEMMA 2.34. *The tree $\mathcal{T}_P^{\delta_1,\delta_2}(f)$ is logarithmic for δ_1 and δ_2 at P.*

Proof. It follows directly from the construction of $\mathcal{T}_P^{\delta_1,\delta_2}(f)$ and remarks 2.32 and 2.33. □

The following lemma, with a very technical proof, shows an unexpected and somewhat surprising relation between the degree of $\mathcal{T}_P^{nul}(f)$ and the degree of $\mathcal{T}_P^{\delta_1,\delta_2}(f)$ constructed above. According to 2.20, the degree of a tree $\mathcal{T}$ is related to the number of conditions imposed to a germ g so that

$$\mathcal{T}|_g \ \geq \mathcal{T}.$$

According to this lemma it is "one condition easier" to construct germs g satisfying

$$\mathcal{T}_P^{\delta_1,\delta_2}(f)|_g \ \geq \mathcal{T}_P^{\delta_1,\delta_2}(f)$$

than satisfying

$$\mathcal{T}_P^{nul}(f)|_g \ \geq \mathcal{T}_P^{nul}(f).$$

LEMMA 2.35. $\deg(\mathcal{T}_P^{\delta_1,\delta_2}(f)) = \deg(\mathcal{T}_P^{nul}(f)) - 1.$

Proof. Let us write $\tilde{\mathcal{T}}^{\delta_1,\delta_2}$ for $\tilde{\mathcal{T}}_P^{\delta_1,\delta_2}(f)$ and $\tilde{\mathcal{T}}^{nul}$ for $\tilde{\mathcal{T}}_P^{nul}(f)$. The lemma is equivalent to

$$\deg(\tilde{\mathcal{T}}^{\delta_1,\delta_2}) = \deg(\tilde{\mathcal{T}}^{nul}) - 1,$$

since the addition of non-infinitely near points does not affect the degree of either tree. The purpose of the forthcoming lines is to obtain a formula for the weights of $\tilde{\mathcal{T}}^{\delta_1,\delta_2}$.

A vertex Q of a tree $\tilde{\mathcal{T}}$ will be said to have a label ℓ, denoted by $\ell(\tilde{\mathcal{T}},Q)$ if

$$w(\tilde{\mathcal{T}},Q)=w(\tilde{\mathcal{T}}_P(f),Q)+\ell.$$

When $\tilde{\mathcal{T}}=\tilde{\mathcal{T}}^{\delta_1,\delta_2}$, then it will be written $\ell(Q)$ for $\ell(\tilde{\mathcal{T}}^{\delta_1,\delta_2},Q)$.

A few remarks about simple subtrees of type $i\geq 1$ will be given in order to derive a formula for the labels of the vertices of the tree $\tilde{\mathcal{T}}^{\delta_1,\delta_2}$.

P1. Three obvious consequences of the construction of $\tilde{\mathcal{T}}^{\delta_1,\delta_2}$ are:
- (a) simple subtrees have either one or no distinguished vertices;
- (b) before a vertex belongs to a simple tree, it can never belong to any distinguished path;
- (c) before a vertex belongs to a simple tree, it always has to belong to trees of type 1.

P2. If a simple subtree $\tilde{\mathcal{T}}^i$ has no distinguished points, then it is of type 2, is a simple vertex ($\tilde{\mathcal{T}}^i=Q$) and has a label of 0. Moreover,

$$\deg(\tilde{\mathcal{T}})-\deg(\tilde{\mathcal{T}}^{nul}|_{\tilde{\mathcal{T}}})=\mathrm{mult}_e(\tilde{V}_{f,k},E_k). \tag{2.30}$$

Proof. Type 1 subtrees have, by definition, at least one distinguished point. Hence, the first part is trivial. The second part is proved in step 3 ia). Note also that, according to P1(c), one has the following chain

$$Q=\tilde{\mathcal{T}}^m_{j_2}\subset\tilde{\mathcal{T}}^{m-1}_{j_3}\subset\ldots\subset\tilde{\mathcal{T}}^2_{j_m}\quad ,$$

where $j_2\geq 3$, and $j_3,\ldots,j_m\in\{1,2\}$. By P1(b), the point Q never belongs to the distinguished path of any $\tilde{\mathcal{T}}^i_*$. Therefore, according to (2.29),

$$w(\tilde{\mathcal{T}}^{m-1}_{j_3},Q)=w(\tilde{\mathcal{T}}^{m-2}_{j_4},Q)=\ldots=w(\tilde{\mathcal{T}}^2_{j_m},Q)=$$
$$=w(\tilde{\mathcal{T}}^1,Q)=w(\tilde{\mathcal{T}}_P(f),Q).$$

Finally, applying (2.28), one has

$$w(\tilde{\mathcal{T}}^m_{j_2},Q)=w(\tilde{\mathcal{T}}_P(f),Q),$$

and hence its label is 0. Therefore,

$$\deg(\tilde{\mathcal{T}})-\deg(\tilde{\mathcal{T}}^{nul}|_{\tilde{\mathcal{T}}})=\binom{\omega(\tilde{\mathcal{T}}^{\delta_1,\delta_2},Q)+1}{2}-\binom{\omega(\tilde{\mathcal{T}}^{nul},Q)+1}{2}=$$

$$=\binom{\omega(\tilde{\mathcal{T}}_P(f),Q)+1}{2}-\binom{\omega(\tilde{\mathcal{T}}_P(f),Q)-1+1}{2}=\omega(\tilde{\mathcal{T}}_P(f),Q)=\mathrm{mult}_e(\tilde{V}_{f,k},E_k).$$

□

P3. Any point of the tree $\tilde{\mathcal{T}}^{\delta_1,\delta_2}$ can only belong to at most one distinguished path and be at most once a central point.
Proof. Both are clear consequences of the construction of $\tilde{\mathcal{T}}^{\delta_1,\delta_2}$. □

P4. Let $\tilde{\mathcal{T}}$ be a subtree of type 2 –and hence simple– with exactly one distinguished vertex, say Q. In this case,

$$\ell(\tilde{\mathcal{T}},\bullet)=\begin{cases}0 & \text{if } \bullet\in|\gamma|\\ -1 & \text{if } \bullet\in|\tilde{\mathcal{T}}|\setminus|\gamma|,\end{cases} \tag{2.31}$$

where γ is the path joining Q and $e = g.c.d.(\tilde{\mathcal{T}})$. Moreover,

$$\deg(\tilde{\mathcal{T}}) - \deg(\tilde{\mathcal{T}}_P^{nul}(f)|_{\tilde{\mathcal{T}}}) = \mathrm{mult}_e(\tilde{V}_{f,k}, E_k), \tag{2.32}$$

where $\tilde{\mathcal{T}}^{nul}|_{\tilde{\mathcal{T}}}$ denotes the corresponding subtree of $\tilde{\mathcal{T}}^{nul}$ isomorphic to $\tilde{\mathcal{T}}$ and where $e \in E_k$.

Proof. Combining P1(b) with (2.29), one has (2.31). Moreover,

$$\deg(\tilde{\mathcal{T}}) - \deg(\tilde{\mathcal{T}}^{nul}|_{\tilde{\mathcal{T}}}) = \sum_{\bullet\in|\gamma|} \binom{\nu_\bullet(f)+1}{2} - \binom{\nu_\bullet(f)}{2} =$$

$$= \sum_{\bullet\in|\gamma|} \nu_\bullet(f) = \mathrm{mult}_e(\tilde{V}_{f,k}, E_k). \quad \square$$

P5. A subtree of type 1 is simple if and only if it is a single vertex, say Q. In that case its label is

$$\ell(Q) = \begin{cases} -1 & \text{if } Q = \delta_i,\ i = 1,2 \\ 0 & \text{otherwise.} \end{cases} \tag{2.33}$$

Proof. For the first part, the "if" direction is obvious. There is only one direction left to prove, i.e. that any simple subtree of type 1 cannot have more than one vertex. Let $\tilde{\mathcal{T}}$ be a subtree of type 1. Let us denote by $e \in E_k \subset X_k$ the root of $\tilde{\mathcal{T}}$. If $\tilde{\mathcal{T}}$ has more than one vertex, then E_k is not transversal to $\tilde{V}_{f,k}$ at e and hence, according to step 3, the vertex $e_{k,*}$ in $\tilde{\mathcal{T}}$ has to be distinguished. Therefore, $\tilde{\mathcal{T}}$ has two distinguished points and hence it is not simple. The second part is a direct consequence of P1(a),(b) and (2.28). $\square$

In order to simplify the proof some more notation needs to be developed.

Let $A \subset \overline{V}_f^{[2]}$ be a set of maximal points of $\tilde{\mathcal{T}}^{\delta_1,\delta_2}$. A function χ_A shall be defined as follows:

$$\chi_A = \#A \cap \{\delta_1, \delta_2\}.$$

Note that, if A and B are disjoint, then $\chi_A + \chi_B = \chi_{A\cup B}$. Also note that

$$\sum_{d \text{ distinguished point}} \chi_{\{d\}} = 2.$$

Using the function χ, formula (2.33) can be rewritten as

$$\deg(\tilde{\mathcal{T}}^{\delta_1,\delta_2}|_Q) - \deg(\tilde{\mathcal{T}}^{nul}|_Q) = -\chi_Q.$$

Note that the difference of degrees between two isomorphic weighted trees can be calculated by decomposing each one into similar subtrees and then adding the differences of degrees for each pair of similar subtrees. Let $\tilde{\mathcal{T}}^i$ be a subtree of $\tilde{\mathcal{T}}^{\delta_1,\delta_2}$ whose decomposition produces only simple subtrees $\tilde{\mathcal{T}}_1^{i+1}$, $\tilde{\mathcal{T}}_2^{i+1}$, $\tilde{\mathcal{T}}_3^{i+1}$, ... ,$\tilde{\mathcal{T}}_{n_i}^{i+1}$. By $P5$, the subtrees $\tilde{\mathcal{T}}_1^{i+1}$ and $\tilde{\mathcal{T}}_2^{i+1}$ have only one vertex. Their labels are also described in $P5$. This, together with $P2$ and $P4$, allows for the calculation of the difference of degrees of $\tilde{\mathcal{T}}^i$ and $\tilde{\mathcal{T}}^{nul}|_{\tilde{\mathcal{T}}^i}$ as follows:

$$\deg(\tilde{\mathcal{T}}^i) - \deg(\tilde{\mathcal{T}}^{nul}|_{\tilde{\mathcal{T}}^i}) =$$

$$= \deg(\tilde{\mathcal{T}}^i|_{P_{i+1}}) - \deg(\tilde{\mathcal{T}}^{nul}|_{P_{i+1}}) + \sum_{\bullet\in E_k} \mathrm{mult}_\bullet(\tilde{V}_{f,k}, E_k) - \chi_{A^i} =$$

$$= \nu_{P_{i+1}}(f) - \chi_{A^i} + \deg(\tilde{\mathcal{T}}^i|_{P_{i+1}}) - \deg(\tilde{\mathcal{T}}^{nul}|_{P_{i+1}}) \quad ,$$

where P_{i+1} is the central point of $\tilde{\mathcal{T}}^i$, E_k is the exceptional divisor resulting after blowing up P_{i+1}, and

$$A^i = \bigcup_{j=0}^{n_{i+1}} |\tilde{\mathcal{T}}_j^{i+1}|.$$

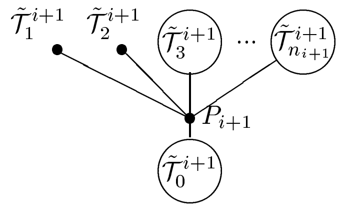
An obvious consequence of P3 is that all central points P_{i+1} belong to exactly one distinguished path, except for P_2, which belongs to none. Hence

$$\ell(\tilde{\mathcal{T}}^{\delta_1,\delta_2}, P_{i+1}) = \begin{cases} -1 & \text{if } i \geq 2 \\ -2 & \text{if } i = 1. \end{cases}$$

Using (2.28), the labels of $\tilde{\mathcal{T}}_0^{i+1}$ follow this pattern

$$\ell(\bullet) = \begin{cases} 0 & \text{if } \bullet \in |\gamma(e_i, P_{i+1})| \setminus \{P_{i+1}\} \quad i \geq 1 \\ -2 & \text{if } \bullet = P_2 \\ -1 & \text{otherwise,} \end{cases} \tag{2.34}$$

where $e_i :=$ g.c.d.$(\tilde{\mathcal{T}}_0^{i+1}) = g.c.d.(\tilde{\mathcal{T}}^i)$ and

$\tilde{\mathcal{T}}^i :=$ 

Therefore

$$\begin{gathered} \deg(\tilde{\mathcal{T}}_P^{\delta_1,\delta_2}|_{\tilde{\mathcal{T}}^i}) - \deg(\tilde{\mathcal{T}}_P^{nul}|_{\tilde{\mathcal{T}}^i}) = \\ = \deg(\tilde{\mathcal{T}}_P^{\delta_1,\delta_2}|_{\tilde{\mathcal{T}}^{i\,\prime}}) - \deg(\tilde{\mathcal{T}}_P^{nul}|_{\tilde{\mathcal{T}}^{i\,\prime}}) + \deg(\tilde{\mathcal{T}}_P^{\delta_1,\delta_2}|_{\tilde{\mathcal{T}}_0^{i+1}}) - \deg(\tilde{\mathcal{T}}_P^{nul}|_{\tilde{\mathcal{T}}_0^{i+1}}) = \\ = \nu_{P_{i+1}}(f) - \chi_{A^i} + \deg(\tilde{\mathcal{T}}_P^{\delta_1,\delta_2}(f)|_{P_{i+1}}) - \deg(\tilde{\mathcal{T}}_P^{nul}(f)|_{P_{i+1}}) + \\ + \sum_{\bullet \in |\gamma(e_i, P_{i+1})| \setminus P_{i+1}} \left(\binom{\nu_\bullet(f) + 1}{2} - \binom{\nu_\bullet(f)}{2} \right) = \\ = -\chi_{A^i} + \nu_{P_{i+1}}(f) + \sum_{\bullet \in |\gamma(e_i, P_{i+1})| \setminus P_{i+1}} \nu_\bullet(f) = \\ = \sum_{\bullet \in \gamma(e_i, P_{i+1})} \nu_\bullet(f) - \chi_{A^i} = \mathrm{mult}_{e_i}(\tilde{V}_{f,k_{i+1}}, E_{k_i}) - \chi_{A^i} \quad ; \end{gathered}$$

where e_i is the root of $\tilde{\mathcal{T}}^i$, E_{k_i} is the exceptional divisor containing e_i, and $\gamma(e_i, P_{i+1})$ is the path joining e_i and P_{i+1}.

Descending in the chain of subtrees $\tilde{\mathcal{T}}^i$, one will eventually reach $\tilde{\mathcal{T}}^1 = \tilde{\mathcal{T}}_P(f)$. From the above calculations, (2.34), and the fact that

$$\chi_{A^1} = \sum_{d \text{ distinguished point}} \chi_{\{d\}} = 2,$$

one can check that

$$\begin{gathered} \deg(\tilde{\mathcal{T}}^{\delta_1,\delta_2}) - \deg(\tilde{\mathcal{T}}^{nul}) = \\ = \deg(\tilde{\mathcal{T}}^{\delta_1,\delta_2}|_{\tilde{\mathcal{T}}^1}) - \deg(\tilde{\mathcal{T}}^{nul}|_{\tilde{\mathcal{T}}^1}) = \\ = \deg(\tilde{\mathcal{T}}^{\delta_1,\delta_2}|_{\mathrm{Asc}(P_2)}) - \deg(\tilde{\mathcal{T}}^{nul}|_{\mathrm{Asc}(P_2)}) + \deg(\tilde{\mathcal{T}}^{\delta_1,\delta_2}|_{\tilde{\mathcal{T}}_0^2}) - \deg(\tilde{\mathcal{T}}^{nul}|_{\tilde{\mathcal{T}}_0^2}) = \\ = \deg(\tilde{\mathcal{T}}^{\delta_1,\delta_2}|_{\mathrm{Asc}(P_2)}) - \deg(\tilde{\mathcal{T}}^{nul}|_{\mathrm{Asc}(P_2)}) = \end{gathered}$$

$$= \nu_{P_2}(f) - \chi_{A^1} + \deg(\tilde{\mathcal{T}}^{\delta_1,\delta_2}|_{P_2}) - \deg(\tilde{\mathcal{T}}^{nul}|_{P_2}) =$$
$$= \nu_{P_2}(f) + \binom{\nu_{P_2}(f)-1}{2} - \binom{\nu_{P_2}(f)}{2} - \chi_{A^1} =$$
$$= \nu_{P_2}(f) - (\nu_{P_2}(f)-1) - 2 = -1. \quad \square$$

According to lemmas 2.34 and 2.22 *ii*), logarithmic trees provide a simple way to construct logarithmic sheaves.

Definition 2.36. Let $\mathcal{D}$ be an arrangement and π a resolution of singularities for $\mathcal{D}$. Let $P \in \mathcal{D}$ be a point in $\mathcal{D}$, and let δ_1 and δ_2 be two local branches of $\mathcal{D}$ at P. The **ideal sheaf** $\mathcal{I}^{\delta_1,\delta_2}_{\mathcal{D},\pi}$ **associated with** δ_1 **and** δ_2 shall be defined as

$$(\mathcal{I}^{\delta_1,\delta_2}_{\mathcal{D},\pi})_Q = \left\{ h \in \mathcal{O}_Q \mid \begin{array}{lcll} \mathcal{T}_Q(\mathcal{D},\pi)|_h & \geq & \mathcal{T}^{\delta_1,\delta_2}_P(\mathcal{D},\pi) & \text{if } Q = P \\ \mathcal{T}_Q(\mathcal{D},\pi)|_h & \geq & \mathcal{T}^{nul}_Q(\mathcal{D},\pi) & \text{otherwise} \end{array} \right.$$

If no confusion about the underlying resolution seems likely to arise, the ideal sheaf $\mathcal{I}^{\delta_1,\delta_2}_{\mathcal{D},\pi}$ will simply be denoted by $\mathcal{I}^{\delta_1,\delta_2}_{\mathcal{D}}$. A global section s of $\mathcal{I}^{\delta_1,\delta_2}_{\mathcal{D}}(d)$ shall be called **essential** if $s_Q \in M_{I_Q}$ for every $Q \in \mathbb{P}^2$, where s_Q is the section s localized at Q, $I_Q = (\mathcal{I}^{\delta_1,\delta_2}_{\mathcal{D}})_Q$ and M_{I_Q} is as in Definition 2.31.3.

By Proposition 2.27 and Lemma 2.35, one has the following.

Proposition 2.37. *Let δ_1 and δ_2 be local branches of the arrangement $\mathcal{D}$ at a singular point P. Suppose δ_1 belongs to the irreducible component $\mathcal{C}_i$ and δ_2 to $\mathcal{C}_j$, then*

$$H^0(\mathbb{P}^2, \mathcal{I}^{\delta_1,\delta_2}_{C_{ij}}(d_{ij}-2)) > d_{ij} - \#\{i,j\}. \quad \square$$

2.7. A holomorphic presentation for $H^2(\mathbb{P}^2 \setminus \mathcal{R};\mathbb{C})$

Let $\mathcal{R}$ be a rational arrangement. Let us fix a resolution $\pi : \overline{X}_{\mathcal{R}} \to \mathbb{P}^2$ as in 2.1. The purpose of this section is to give a presentation of the space

$$H^2(W_2, d) \cong H^2(X_{\mathcal{R}};\mathbb{C})$$

of global C^∞ 2-forms of weight 2, module closed forms, on $\overline{X}_{\mathcal{R}}$, using only holomorphic forms as representatives.

Definitions 2.38.

1. For any affine component $\mathcal{C}_i$ of $\mathcal{R}$, a point at infinity $b(\mathcal{C}_i) \in \mathcal{C}_i \cap \mathcal{C}_0$ will be fixed. Such a point shall be called the **base point of** $\mathcal{C}_i$.
2. An **affine geometric set of 2-forms** $v_0(\mathcal{R})$ for $\mathcal{R}$ is constructed as follows. Take a point $P \in \mathrm{Sing}(\mathcal{R}^*) = \mathrm{Sing}(\mathcal{R}) \cap \mathbb{C}^2$. Consider two local branches δ_1 and δ_2 in the components $\mathcal{C}_i$ and $\mathcal{C}_j$ respectively. Take a 2-form $\psi^{\delta_1,\delta_2}_P := \varphi^{\delta_1,\delta_2}_P \, s_{i,j}$ satisfying
   i) $\varphi^{\delta_1,\delta_2}_P \in H^0(\mathbb{P}^2, \mathcal{I}^{\delta_1,\delta_2}_{C_{ij}}(d_{ij}-2))$
   ii) $\left(\varphi^{\delta_1,\delta_2}_P\right)_P \in M_I$, where $I = \left(\mathcal{I}^{\delta_1,\delta_2}_{C_{ij}}(d_{ij}-2)\right)_P$
   iii) $\varphi^{\delta_1,\delta_2}_P$ vanishes at the points $\mathcal{C}_{ij} \setminus \{b(\mathcal{C}_i), b(\mathcal{C}_j)\}$.

Note that Proposition 2.37 assures the existence of such a $\varphi_P^{\delta_1,\delta_2}$. In the case $i = j$, $\deg(\varphi_P^{\delta_1,\delta_2}) = d_i - 2$ and $\{\varphi_P^{\delta_1,\delta_2} = 0\}$ intersects $\mathcal{C}_0$ at the $d_i - 1$ points of $\mathcal{C}_i \cap \mathcal{C}_0$. Therefore, the linear system $H^0(\mathbb{P}^2, \mathcal{I}_{\mathcal{C}_i}^{\delta_1,\delta_2}(d_i - 2))$ contains global sections of type Zf, where f is a homogeneous polynomial of degree $d_i - 3$. Let us denote such a set by $v_0(\mathcal{R}, P)$. Note that $v_0(\mathcal{R}, P)$ is empty if $\mathcal{R}$ has only one branch at P. This way, the following set can be defined as

$$v_0(\mathcal{R}) := \bigcup_{P \in \mathrm{Sing}(\mathcal{R}^*)} v_0(\mathcal{R}, P).$$

3. A **geometric set of 2-forms for $\mathcal{R}$ at infinity** is constructed as follows. Take a component $\mathcal{C}_i \neq \mathcal{C}_0$ in $\mathcal{R}$. The d_i points $P_1^i, ..., P_{d_i}^i$ of $\mathcal{C}_i \cap \mathcal{C}_0$, will be denoted in such a way that $P_1^i = b(\mathcal{C}_i)$. For each point P_j^i, $j = 1, ..., d_i$ let $\psi_\infty^{P_j^i} := \varphi_\infty^{P_j^i} \; s_{i,i}$ be a 2-form such that
   i) $\varphi_\infty^{P_j^i} \in H^0(\mathbb{P}^2, \mathcal{I}_{\mathcal{C}_{ii}}^{nul}(d_{ii} - 2))$
   ii) $\varphi_\infty^{P_j^i}$ vanishes at the points P_k^i for $k \neq 1, j$.
   Proposition 2.37 also assures their existence. The set $v_\infty(\mathcal{R}, \mathcal{C}_i)$ is the collection of such forms when $j = 2, ..., d_i$. Note that $v_\infty(\mathcal{R}, \mathcal{C}_i)$ is empty if $\mathcal{C}_i$ is a line. Analogously to the affine geometric set, one can define the set

$$v_\infty(\mathcal{R}) := \bigcup_{i=1}^{n} v_\infty(\mathcal{R}, \mathcal{C}_i).$$

3. Finally, a **geometric set $v(\mathcal{R})$ of 2-forms for $\mathcal{R}$** is defined as the union of two geometric sets of 2-forms: one affine and one at infinity

$$v(\mathcal{R}) := v_0(\mathcal{R}) \cup v_\infty(\mathcal{R}).$$

REMARKS 2.39.

1. Let us study the residue map at infinity of forms in $v_0(\mathcal{R})$. By Remark 2.34, any form $\psi_P^{\delta_1,\delta_2} = \varphi_P^{\delta_1,\delta_2} \; s_{i,j}$ has non-zero residues only at the edges of the path $\gamma(\delta_1, \delta_2) \subset \Gamma_P(f)$ and possibly at infinity –for definitions see 2.30. Moreover, since $\varphi_P^{\delta_1,\delta_2}$ vanishes on every non-base point of $\mathcal{C}_{ij}$, the only non-zero residues at infinity are on the base points $b(\mathcal{C}_i)$ and $b(\mathcal{C}_j)$. Let us consider the coordinate of $\mathrm{Res}^{[1]}(\psi_P^{\delta_1,\delta_2})$ on $\mathcal{C}_0$. Such a coordinate is a 1-form on the projective line $\mathcal{C}_0$. Also note that each coordinate of the restriction map $R_1^{[1]}|_{\mathcal{C}_0}$ coincides –up to sign– with the usual residue map on $\mathbb{P}^1$ and hence the sum of such residues has to be zero. Therefore, by the commutativity of the residue maps given in 1.25, one has

$$\left(\mathrm{Res}^{[2]}(\psi_P^{\delta_1,\delta_2})\right)_Q = \begin{cases} \lambda & \text{if } Q = \pi^{-1}(b(\mathcal{C}_i)) \\ \pm\lambda & \textit{if } Q = \pi^{-1}(b(\mathcal{C}_j)) \\ 0 & \text{elsewhere} \end{cases}$$

   for any $Q \in \mathcal{C}_0$, where λ depends on the form $\psi_P^{\delta_1,\delta_2}$. It is not difficult to check that λ is a non-zero complex number if and only if $i \neq j$.
2. The residues at infinity of forms in a geometric set $v_\infty(\mathcal{R})$ are simpler to calculate. By Definition 2.38.2, one has

$$\left(\mathrm{Res}^{[2]}(\psi_\infty^{P_j^i})\right)_Q = \begin{cases} \lambda & \text{if } Q = b(\mathcal{C}_i) \\ \pm\lambda & \text{if } Q = P_j^i \\ 0 & \text{elsewhere} \end{cases}$$

for any $Q \in \mathcal{C}_0$.

A more detailed description of the forms of a geometric set will be given in what follows. The key to achieve this is the injectivity of the residue map for rational arrangements.

Let $\psi_P^{\delta_1,\delta_2} \in v_0(\mathcal{R})$ and let $\gamma(\delta_1, \delta_2)$ be the path joining δ_1 and δ_2 in Γ_P. Let us order the vertices

$$\alpha_1 = \delta_1, \alpha_2, ..., \alpha_k = \delta_2$$

and edges

$$\gamma_1, \gamma_2, ..., \gamma_{k-1}$$

of $\gamma(\delta_1, \delta_2)$ in the direction from δ_1 to δ_2. The edge γ_i joins α_i and α_{i+1}, $i = 1, ..., k-1$. Let us denote by v_i (resp. w_i) local equations of α_i (resp. α_{i+1}) in a neighborhood of γ_i. In particular, the divisor $\overline{\mathcal{R}}$ has local equations

$$\{v_i w_i = 0\}$$

in a neighborhood of $\gamma_i \in \overline{\mathcal{R}}^{[2]}$. The branch δ_1 (resp. δ_2) belongs to a global component, say $\mathcal{C}_i$ (resp. $\mathcal{C}_j$). Local equations of $\mathcal{C}_i$ (resp. of $\mathcal{C}_j$) at $b(\mathcal{C}_i)$ (resp. $b(\mathcal{C}_j)$) are denoted by z_i (resp. z_j). Note that a local equation of $\overline{\mathcal{R}}$ at $\overline{b}(\mathcal{C}_i) = \pi^{-1} b(\mathcal{C}_i)$ can be given by

$$\{z z_i = 0\},$$

since the line $\mathcal{C}_0$ is transversal. Under such conditions, one has the following.

LEMMA 2.40. *The 2-form $\pi^* \psi_P^{\delta_1,\delta_2}$ – c.f. 2.38.2 – has the following local expression in a neighborhood of γ_i*

$$\varphi_i \frac{dv_i \wedge dw_i}{v_i w_i}, \tag{2.35}$$

where $\varphi_i(\gamma_i) = \lambda \in \mathbb{C}^*$ for any $i = 1, ..., k-1$.

Moreover, its expressions in a neighborhood of $\overline{b}(\mathcal{C}_i)$ and $\overline{b}(\mathcal{C}_j)$ are

$$-\varphi_{1\infty} \tfrac{dz_i \wedge dz}{z_i z}, \quad and \quad -\varphi_{2\infty} \tfrac{dz \wedge dz_j}{z z_j} \tag{2.36}$$

respectively, where

$$\varphi_{1\infty}(\overline{b}(\mathcal{C}_i)) = \varphi_{2\infty}(\overline{b}(\mathcal{C}_j)) = \begin{cases} \lambda & if\ i \neq j \\ 0 & if\ i = j \end{cases} \tag{2.37}$$

and λ is the constant introduced above.

Proof. Let us prove (2.35) first. By Definition 2.38 and Lemma 2.34, any form $\pi^* \psi_P^{\delta_1,\delta_2}$ is logarithmic, has weight 2 and its affine poles lay at the edges of $\gamma(\delta_1, \delta_2)$. Therefore, Lemma 1.17 gives (2.35) Note that

$$\left(\mathrm{Res}_\pi^{[2]} \psi_P^{\delta_1,\delta_2} \right)_{\gamma_i} = (-1)^\sigma \varphi_i |_{\gamma_i} \quad ,$$

where σ is the orientation factor in the definition of $R_0^{[2]}$.

The poles of $\pi^* \psi_P^{\delta_1,\delta_2}$ on the divisor α_{i+1} are only at γ_i and γ_{i+1}. Since α_{i+1} is isomorphic to $\mathbb{P}^1$, the sum of residues on α_{i+1} has to be zero. Therefore

$$\varphi_i(\gamma_i) - \varphi_{i+1}(\gamma_{i+1}) = 0.$$

Hence, by induction

$$\varphi_1(\gamma_1) = ... = \varphi_{k-1}(\gamma_{k-1}) = \lambda.$$

The fact that λ is non-zero is given by 2.38.2.

Part (2.36) is an obvious consequence of Remark 2.39.1.

Finally, the poles on $\overline{\mathcal{C}}_0$ of $\pi^*\psi_P^{\delta_1,\delta_2}$ are $\overline{b}(\mathcal{C}_i)$ and $\overline{b}(\mathcal{C}_j)$ and their sum is zero. This yields (2.37). □

REMARKS 2.41.

1. Analogously, one can find a local description for vectors in $v_\infty(\mathcal{R})$. Any form $\pi^*\psi_\infty^{P_j^i}$ can be written locally at $P_k^i \in \mathcal{C}_i \cap \mathcal{C}_0$ as
$$\varphi \frac{dz \wedge dz_{P_k^i}}{zz_{P_k^i}},$$
where $z_{P_k^i}$ is a local equation for $\mathcal{C}_i$ at P_k^i. Let us assume $b(C_i) = P_1^i$ is the base point of $\mathcal{C}_i$, then
$$\varphi(P_j^i) = \begin{cases} -\varphi(P_1^i) = \lambda \neq 0 & if\ k = j \\ 0 & otherwise, \end{cases}$$
for any $j = 2, ..., d_i$.
2. In the case of line arrangements, all 2-forms $\frac{d\ell_1 \wedge d\ell_2}{\ell_1 \ell_2}$ generating $H^2(X_\mathcal{R}; \mathbb{C})$ automatically have constant coefficients.

DEFINITION 2.42. A 2-form $\tilde{\psi} \in v_0(\mathcal{R})$ is called **a normal 2-form** if the corresponding λ in Lemma 2.40 or 2.41.1 is equal to 1.

A geometric set of 2-forms $\tilde{v}(\mathcal{R})$ is said to be **normal** if every 2-form in $\tilde{v}(\mathcal{R})$ is normal.

Any geometric 2-form ψ can be normalized by multiplying it by $\frac{1}{\lambda}$. In that case, the resulting normalized 2-form will be denoted by $\tilde{\psi}$.

Even though the election of the forms of a normal geometric set $v(\mathcal{R})$ seems not to be unique, their classes are, as proven in the following

PROPOSITION 2.43. *In the above conditions,*

i) if $\tilde{\psi}_P^{\delta_1,\delta_2}$ and $\tilde{\psi}'^{\delta_1,\delta_2}_P$ are both normal, then they are cohomologically equivalent.

ii) if $\tilde{\psi}_\infty^{P_j^i}$ and $\tilde{\psi}'^{P_j^i}_\infty$ are both normal, then they are cohomologically equivalent.

Proof. Using the injectivity of $\mathrm{Res}^{[2]}$, it is enough to check that

$$\mathrm{Res}^{[2]}(\tilde{\psi}_P^{\delta_1,\delta_2} - \tilde{\psi}'^{\delta_1,\delta_2}_P) = 0$$

which is an immediate consequence of Lemma 2.40.

Analogously, using Remark 2.41.1, one can prove

$$\mathrm{Res}^{[2]}(\tilde{\psi}_\infty^{P_j^i} - \tilde{\psi}'^{P_j^i}_\infty) = 0. \quad \square$$

A set of relations in $H^2(X_\mathcal{R}; \mathbb{C})$ is given in the following.

Proposition 2.44. *i) For any two local branches δ_1 and δ_2 of $\mathcal{R}$ at a singular point P, one has*

$$\pi^*\tilde{\psi}_P^{\delta_1,\delta_2}+\pi^*\tilde{\psi}_P^{\delta_2,\delta_1}\approx 0. \tag{2.38}$$

ii) For any three branches δ_1, δ_2 and δ_3 of $\mathcal{R}$ at a singular point P, one has

$$\pi^*\tilde{\psi}_P^{\delta_1,\delta_2}+\pi^*\tilde{\psi}_P^{\delta_2,\delta_3}+\pi^*\tilde{\psi}_P^{\delta_3,\delta_1}\approx 0 \tag{2.39}$$

where the symbol $\approx$ means "cohomologically equivalent".

Proof. Part $i)$ follows immediately from Lemma 2.40.

For part $ii)$, let us denote by $\gamma_{i,j}$ the path in the dual graph Γ_P joining δ_i and δ_j. Denote also by $P'=g.c.d.(\delta_1,\delta_2,\delta_3)\in\mathcal{T}_P(\mathcal{R})$ the greatest common descendant of δ_1, δ_2 and δ_3, considered as maximal points in the multiplicity tree. After blowing up P', the exceptional divisor E separates the three branches δ_1, δ_2 and δ_3. It will first be proved that

$$\mathrm{Res}_\pi^{[2]}\left(\tilde{\psi}_P^{\delta_1,\delta_2}+\tilde{\psi}_P^{\delta_2,\delta_3}+\tilde{\psi}_P^{\delta_3,\delta_1}\right)_Q=0$$

for any $Q\in E$.

The exceptional divisor E, of equation $\{v=0\}$, is a vertex in $\gamma_{1,2}$, $\gamma_{2,3}$ and $\gamma_{3,1}$. This vertex has two neighboring vertices α_i and α_j of $\gamma_{i,j}$ where α_i is closer to δ_i and α_j is closer to δ_j as in Figure 5.

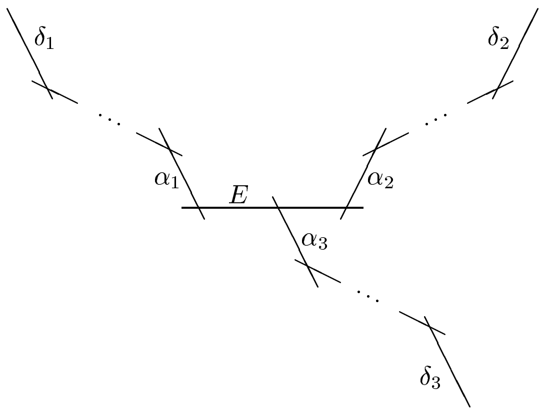


Figure 5. Paths $\gamma_{1,2},\gamma_{2,3},\gamma_{3,1}$.

By Lemma 2.40, the left-hand side of (2.39) can be written as

$$\left(\varphi_{1,1}\frac{d\alpha_1\wedge dv}{\alpha_1 v}+\varphi_{1,2}\frac{dv\wedge d\alpha_2}{v\alpha_2}\right)+\left(\varphi_{2,1}\frac{d\alpha_2\wedge dv}{\alpha_2 v}+\varphi_{2,2}\frac{dv\wedge d\alpha_3}{v\alpha_3}\right)+$$
$$+\left(\varphi_{3,1}\frac{d\alpha_3\wedge dv}{\alpha_3 v}+\varphi_{3,2}\frac{dv\wedge d\alpha_1}{v\alpha_1}\right)$$

in a neighborhood of E. Its residue, for example at the edge $\gamma_1\in\{\alpha_1=0\}\cap E$, is

$$\varphi_{1,1}(\gamma_1)-\varphi_{3,2}(\gamma_1)=1-1=0.$$

The same applies to $\{\alpha_2 = 0\} \cap E$ and $\{\alpha_3 = 0\} \cap E$.

Since all components in Figure 5 are rational and each α_i has only two singular points, it follows that our form has no residues on $\overline{\mathcal{R}}^{*[2]} = \overline{\mathcal{R}}^{[2]} \cap \mathbb{C}^2$.

Let us calculate its residues at infinity. Let Q be a point in $\mathcal{C}_0 \cap \mathcal{C}_i$, where $\mathcal{C}_i$ is the component containing δ_1. If $Q \neq b(\mathcal{C}_i)$, all residues at Q are trivially zero by Remark 2.39.1. If $Q = b(\mathcal{C}_i)$, then

$$\mathrm{Res}^{[2]}\left(\pi^* \tilde{\psi}_P^{\delta_1,\delta_2} + \pi^* \tilde{\psi}_P^{\delta_2,\delta_3} + \pi^* \tilde{\psi}_P^{\delta_3,\delta_1}\right)_Q = 1 + 0 - 1 = 0.$$

One proceeds analogously for the other points at infinity in the components of δ_2 and δ_3. □

Our next step is to give a presentation for

$$H^2(W_2, d) \cong H^2(X_{\mathcal{R}}; \mathbb{C}).$$

NOTATION AND REMARK 2.45. Let P be a point of a curve $\mathcal{C}$. The set of local branches of $\mathcal{C}$ at P, will be denoted by $\Delta_P(\mathcal{C})$. Note that a normal geometric set of 2-forms – as defined in Definition 2.42 – can be written as

$$\tilde{v}(\mathcal{R}) = \{\tilde{\psi}_P^{\delta,\delta'}\}_{\substack{P \in \mathrm{Sing}(\mathcal{C}_i \cup \mathcal{C}_j), \\ \delta \in \Delta_P(\mathcal{C}_i), \delta' \in \Delta_P(\mathcal{C}_j)}} \cup \{\tilde{\psi}_\infty^{i,k_i}\}_{k_i \in \{1,\dots,d_i-1\}}, \quad i,j \in \{1,...,n\},$$

The elements of $\tilde{v}(\mathcal{R})$ satisfy certain relations in $H^2(W_2, d)$ shown in (2.38) and (2.39).

Then one has the following.

THEOREM 2.46. *The vector space*

$$H^2(W_2, d) \cong H^2(X_{\mathcal{R}}; \mathbb{C})$$

is generated by a geometric set of 2-forms $\tilde{v}(\mathcal{R})$, *where*

$$\pi^* \tilde{\psi}_P^{\delta_1,\delta_2} + \pi^* \tilde{\psi}_P^{\delta_2,\delta_1} \approx 0. \tag{2.40}$$

$$\forall\ P \in \mathrm{Sing}(\mathcal{C}_{ij}), \delta_1 \in \Delta_P(\mathcal{C}_i), \delta_2 \in \Delta_P(\mathcal{C}_j)$$

and

$$\pi^* \tilde{\psi}_P^{\delta_1,\delta_2} + \pi^* \tilde{\psi}_P^{\delta_2,\delta_3} + \pi^* \tilde{\psi}_P^{\delta_3,\delta_1} \approx 0 \tag{2.41}$$

$$\forall\ P \in \mathrm{Sing}(\mathcal{C}_i \cup \mathcal{C}_j \cup \mathcal{C}_k), \delta_1 \in \Delta_P(\mathcal{C}_i), \delta_2 \in \Delta_P(\mathcal{C}_j), \delta_3 \in \Delta_P(\mathcal{C}_k)$$

constitute a complete set of relations. In other words, $H^2(X_{\mathcal{R}}; \mathbb{C})$ *is the quotient of the space generated by the 2-forms in* $\tilde{v}(\mathcal{R})$ *by the subspace generated by the vectors involved in* (2.40) *and* (2.41).

Proof. Let us choose a branch δ_P for each singular point P in $\mathcal{R}^*$. Let us define a subset $\tilde{v}_0'(\mathcal{R})$ of $\tilde{v}(\mathcal{R})$ collecting all normal forms of type $\tilde{\psi}_P^{\delta_P,\delta}$. Define now

$$\tilde{v}'(\mathcal{R}) := \tilde{v}_0'(\mathcal{R}) \cup \tilde{v}_\infty(\mathcal{R}).$$

The result is a straightforward consequence of the following two remarks:

i) Using (2.38) and (2.39) any form $\pi^* \tilde{\psi}_P^{\delta_1,\delta_2}$ can be written in terms of $\pi^* \tilde{\psi}_P^{\delta_P,\delta_1}$ and $\pi^* \tilde{\psi}_P^{\delta_P,\delta_2}$. Therefore, any vector of $\tilde{v}(\mathcal{R})$ can be written in terms of the vectors of $\tilde{v}'(\mathcal{R})$ using only (2.38) and (2.39).

ii) The elements of $\tilde{v}'(\mathcal{R})$ are linearly independent. This follows immediately from Remark 2.39.1 and 2, since all images by the residue map are linearly independent.

iii) The cardinality of $\tilde{v}'(\mathcal{R})$ coincides with the dimension of $H^2(X_{\mathcal{R}};\mathbb{C})$.

Proof. First, note that

$$\tilde{v}'(\mathcal{R}) = \tilde{v}_0'(\mathcal{R}) \cup v_\infty(\mathcal{R})$$

is a disjoint union. Hence, computing

$$\begin{aligned}\#\tilde{v}_0'(\mathcal{R}) &= \textstyle\sum_{P\in \mathrm{Sing}\mathcal{R}^*}(r_P(\mathcal{R})-1) = \\ &= \textstyle\sum_{P\in \mathrm{Sing}\mathcal{R}}(r_P(\mathcal{R})-1) - \sum_i d_i\end{aligned}$$

and

$$\#v_\infty(\mathcal{R}) = \sum_{i\neq 0}(d_i-1),$$

one obtains

$$\begin{aligned}\#\tilde{v}'(\mathcal{R}) = \left(\sum_{P\in\mathrm{Sing}\mathcal{R}}(r_P(\mathcal{R})-1) - \sum_{i=1}^n d_i\right) + \sum_{i=1}^n (d_i-1) = \\ = \sum_{P\in\mathrm{Sing}\mathcal{R}}(r_P(\mathcal{R})-1) - n.\end{aligned}$$

Finally, by Proposition 2.4, this number coincide with the dimension of $H^2(X_{\mathcal{R}};\mathbb{C})$. □

2.8. The product in $H^\bullet(\mathbb{P}^2 \setminus \mathcal{R};\mathbb{C})$

In order to describe the structure of the cohomology ring, all it is left to do is calculate the product

$$\frac{d\mathcal{C}_i}{\mathcal{C}_i} \wedge \frac{d\mathcal{C}_j}{\mathcal{C}_j} \tag{2.42}$$

in terms of the elements of $\tilde{v}(\mathcal{R})$.

THEOREM 2.47. *Let $\mathcal{C}_i$ and $\mathcal{C}_j$ be two irreducible components of $\mathcal{R}$ where $i,j\neq 0$, then*

$$\begin{aligned}\pi^*\left(\frac{d\mathcal{C}_i}{\mathcal{C}_i} \wedge \frac{d\mathcal{C}_j}{\mathcal{C}_j}\right) \approx \\ \approx \sum_{P\in\mathrm{Sing}(\mathcal{C}_{ij})} (\delta\cdot\delta')_P\ \pi^*\tilde{\psi}_P^{\delta,\delta'} + d_j\sum_{k=2}^{d_i}\pi^*\tilde{\psi}_\infty^{P_k^i} - d_i\sum_{k=2}^{d_j}\pi^*\tilde{\psi}_\infty^{P_k^j}\end{aligned}$$

where δ (resp. δ') spans over all local branches of $\mathcal{C}_i$ (resp. $\mathcal{C}_j$) at P.

Proof. Recall that the pull-back of $\tilde{\psi}_P^{\delta,\delta'}$ has a constant residue 1 along the edges of the path $\gamma(\delta,\delta')$, and possibly at the base points at infinity. Using the set of generators $\tilde{v}(\mathcal{R})$, one can write

$$\pi^*\left(\frac{d\mathcal{C}_i}{\mathcal{C}_i} \wedge \frac{d\mathcal{C}_j}{\mathcal{C}_j}\right) \approx$$

$$\approx \sum_{P\in \mathrm{Sing}(\mathcal{C}_i\cup\mathcal{C}_j)} \lambda_P(\delta,\delta')\ \pi^*\tilde{\psi}_P^{\delta,\delta'} + \sum_{k=2}^{d_i} \lambda_{k,i}\pi^*\tilde{\psi}_\infty^{P_k^i} + \sum_{k=2}^{d_j} \lambda_{k,j}\pi^*\tilde{\psi}_\infty^{P_k^j}.$$

By the injectivity of $\mathrm{Res}^{[2]}$, the coefficients in the previous expression can be found by calculating the residues along $\gamma(\delta,\delta')$ and at infinity.

Let us take a singular point P in $\mathcal{C}_{ij}^*$. At P, the equation C_{ij}^* can be decomposed as a product as follows

$$(C_i)_P = \delta_1 \ldots \delta_{k_i}$$
$$(C_j)_P = \delta'_1 \ldots \delta'_{k_j}$$

where, for instance $\delta_1 = \delta$ and $\delta'_1 = \delta'$. Note that

$$\left(\frac{d\mathcal{C}_i}{\mathcal{C}_i} \wedge \frac{d\mathcal{C}_j}{\mathcal{C}_j}\right)_P = \sum_{r,s} \frac{d\delta_r \wedge d\delta'_s}{\delta_r \delta'_s}.$$

Basically following the proof of Lemma 2.40, one can check that the residues of each form

$$\frac{d\delta_r \wedge d\delta'_s}{\delta_r \delta'_s}$$

are constant along the edges of $\gamma(\delta_r,\delta'_s)$. Hence, in order to calculate this residue, it is enough to do it at one edge. Let us consider the edge corresponding to the infinitely near point Q of δ. For simplicity, let us write $\delta_r = \delta$ and $\delta'_s = \delta'$. Then

$$\pi^*(\delta)_Q = uv^\alpha \varphi$$
$$\pi^*(\delta')_Q = v^{\alpha'}\varphi',$$

where $\{u=0\}$ is a local equation of $\tilde{\delta}$, $\{v=0\}$ is a local equation of the component of $\overline{\mathcal{R}}$ intersecting $\tilde{\delta}$, and φ and φ' are units in the local ring. Hence,

$$\frac{d\pi^*(\delta) \wedge d\pi^*(\delta')}{\pi^*(\delta)\pi^*(\delta')} =$$
$$= \frac{(\varphi + u\varphi_u)(\alpha'\varphi' + v\varphi'_v) - (\alpha u\varphi + uv\varphi_v)\varphi'_u}{\varphi\varphi'} \frac{du \wedge dv}{uv}$$

whose residue at Q is α'.

Finally we want to prove that α' is the multiplicity of intersection $(\delta \cdot \delta')_P$.

1. If δ and δ' are already transversal, then clearly $\alpha' = 1$.
2. Otherwise, let us denote by E_1 the exceptional divisor intersecting $\tilde{\delta}$. Then,

   $$\delta \cdot \delta' = \left(\tilde{\delta} + \sum \nu_k E_k\right) \cdot \left(\tilde{\delta}' + \sum \nu'_k E_k\right)$$

   where $\nu'_1 = \alpha'$. Therefore,

   $$\delta \cdot \delta' = \alpha'\tilde{\delta} \cdot E_1 + \left(\sum \nu_k E_k\right) \cdot \left(\tilde{\delta}' + \sum \nu'_k E_k\right).$$

   By the projection formula,

   $$\left(\sum \nu_k E_k\right) \cdot \left(\tilde{\delta}' + \sum \nu'_k E_k\right) = \left(\sum \nu_k E_k\right) \cdot \pi^*(\delta') = 0 \cdot \delta' = 0.$$

   Hence,

   $$(\delta \cdot \delta')_P = \alpha'.$$

To finish the proof one has to calculate the residues of (2.42) at the points at infinity. Note that

$$\left(\frac{d\mathcal{C}_i}{\mathcal{C}_i} \wedge \frac{d\mathcal{C}_j}{\mathcal{C}_j}\right) = \left(\frac{dC_i}{C_i} - d_i\frac{dz}{z}\right) \wedge \left(\frac{dC_j}{C_j} - d_j\frac{dz}{z}\right) =$$
$$= \frac{dC_i \wedge dC_j}{C_iC_j} + d_i\frac{dC_j \wedge dz}{C_j z} - d_j\frac{dC_i \wedge dz}{C_i z}.$$

Therefore, the only contributions at infinity come from

$$d_i\frac{dC_j \wedge dz}{C_j z} - d_j\frac{dC_i \wedge dz}{C_i z}.$$

In particular, if $P_k^i \in \mathcal{C}_i \cap \mathcal{C}_0$, then one has local equations

$$\left(\pi^*\frac{dC_i \wedge dC_j}{C_iC_j}\right)_{P_k^i} = -d_j\frac{dz_{P_k^i} \wedge dz}{z_{P_k^i} z}. \tag{2.43}$$

By Remark 2.41.1, this implies that

$$\lambda_{k,i} = -d_j \tag{2.44}$$

and

$$\lambda_{k,j} = d_i \ . \quad \square$$

REMARK 2.48. As a double-check, note what happens at the base points P_1^i and P_1^j. Let us consider $\{z_{P_1^i} = 0\}$, a local equation of $\mathcal{C}_i$ near P_1^i. The local contribution from $\pi^*\psi_\infty^{P_k^i}$ to

$$\frac{dz_{P_1^i} \wedge dz}{z_{P_1^i} z}$$

is d_j for each non-base point of $\mathcal{C}_i$ –c.f. Remark 2.41.1. Since there are $d_i - 1$ of them, the coefficient becomes

$$d_j(d_i - 1). \tag{2.45}$$

Meanwhile, there is another contribution to

$$\frac{dz_{P_1^i} \wedge dz}{z_{P_1^i} z}$$

coming from the forms $\pi^*\psi_P^{\delta_r,\delta'_s}$. Using Remark 2.39.1, it is easy to check that the sum of residues at the base point $b(\mathcal{C}_i)$ coming from the affine forms is

$$-\sum_{P \in \mathcal{C}_{ij}} (\delta_r, \delta'_s)_P = -d_id_j. \tag{2.46}$$

Hence, adding (2.45) and (2.46), one has

$$d_j(d_i - 1) - d_id_j = -d_j,$$

which is the coefficient calculated in (2.43) for the local equation

$$\frac{dz_{P_k^i} \wedge dz}{z_{P_k^i} z}$$

at P_k^i for any k.

CHAPTER 3

Generalized Aomoto Complexes

Associated with any arrangement $\mathcal{D} \subset \mathbb{P}^2$ and any resolution

$$\pi : \overline{X}_{\mathcal{D}} \longrightarrow \mathbb{P}^2$$

of $\mathcal{D}$, a family of complexes can be defined as follows. Consider W_i the vector space of global differential log forms of weight i on $\overline{\mathcal{D}} = \pi^*(\mathcal{D})_{red} \subset \overline{X}_{\mathcal{D}}$ and the exterior derivative d. Note that $(W_\bullet, d)$ has a differential graded algebra structure given by the exterior product of forms. According to the logarithmic de Rham theorem, the cohomology of the complex (W_i, d) is isomorphic, as an algebra, to the cohomology of the complement of $\overline{\mathcal{D}}$ in $\overline{X}_{\mathcal{D}}$. Any closed 1-form ω in W_1 defines a complex

$$0 \longrightarrow W_0 \cong \mathbb{C} \xrightarrow{\wedge\omega} W_1 \xrightarrow{\wedge\omega} W_2 \longrightarrow 0.$$

One can easily check that:

1. $dW_i \wedge \omega \subset dW_{i+1}$
2. if $\varphi \in \ker d_i$, then $\varphi \wedge \omega \in \ker d_{i+1}$
3. if $[\omega] = [\omega']$ and $\varphi \in \ker d_i$, then $(\wedge\omega - \wedge\omega')(\varphi) \in dW_{i+1}$.

Therefore, one has a family of complexes

$$0 \longrightarrow H^0(W_0, d) \cong \mathbb{C} \xrightarrow{\wedge\omega} H^1(W_1, d) \xrightarrow{\wedge\omega} H^2(W_2, d) \longrightarrow 0 \qquad (3.1)$$

parametrized by $H^1(W_1, d) \cong H^1(X_{\mathcal{D}}; \mathbb{C})$.

Definition 3.1. Let $\mathcal{D}$ be an arrangement, π a resolution of $\mathcal{D}$ and $\omega \in H^1(X_{\mathcal{D}}; \mathbb{C})$. Consider the complex $(A^\bullet, \omega)$ where $A^i := H^i(W_i, d)$ and the morphism $\wedge\omega$ is given in (3.1). Such a complex will be called a **Generalized Aomoto Complex (GAC)** for $\mathcal{D}$.

In the case of line arrangements, GAC's coincide with the Aomoto-Orlik-Solomon complexes. The latter depend only on the combinatorial type of the arrangement. They admit the following presentation –c.f. [**44**]–

$$\begin{aligned} A^0 &:= \mathbb{C} \\ A^1 &:= \sigma_1\mathbb{C} \oplus \ldots \oplus \sigma_n\mathbb{C} \\ \tilde{A}^2 &:= \wedge^2 A^1 \\ A^2 &:= \tilde{A}^2/R \end{aligned}$$

where R is the subspace of $\tilde{A}^2$ generated by

$$\sigma_i \wedge \sigma_j + \sigma_j \wedge \sigma_k + \sigma_k \wedge \sigma_i,$$

whenever $\mathcal{C}_i \cap \mathcal{C}_j \cap \mathcal{C}_k \neq \emptyset$.

For any

$$\omega = \sum_{i=1}^{n} \lambda_i \sigma_i,$$

the differential $\wedge\omega$ is defined by linearity from

$$\sigma_j \wedge \omega := \sum_{i=1}^{n} \lambda_i \sigma_j \wedge \sigma_i.$$

In this chapter, GAC's of rational arrangements will be described. As for line arrangements, such complexes will only depend on the combinatorial type of the arrangement – c.f. Definition 3.2.

3.1. Combinatorial type of an arrangement

The concept of combinatorial type was first defined by E.Artal in [**3**]. It can be defined as follows

DEFINITION 3.2. Two reduced curves $\mathcal{D}$ and $\mathcal{D}'$ in $\mathbb{P}^2$ have the **same combinatorial type** if they satisfy the following three conditions:

1. Both have the same number of components, and up to reordering
$$\mathcal{C} = \mathcal{C}_1 \cup ... \cup \mathcal{C}_n$$
$$\mathcal{C}' = \mathcal{C}'_1 \cup ... \cup \mathcal{C}'_n$$
with $\deg(\mathcal{C}_i) = \deg(\mathcal{C}'_i)$.
2. There is a bijection
$$\phi : \operatorname{Sing}(\mathcal{D}) \longrightarrow \operatorname{Sing}(\mathcal{D}')$$
between the sets of singular points, and also
$$\phi^i_P : \Delta_P(\mathcal{C}_i) \longrightarrow \Delta_Q(\mathcal{C}'_i)$$
between the sets of local branches at any $P \in \operatorname{Sing}(\mathcal{D})$, $Q = \phi(P)$, such that the local type of the singularity P at $\mathcal{C}_i$ coincides with the local type of Q at $\mathcal{C}'_i$ for any $i = 1, .., n$.
3. For any two different local branches at P, δ_1 and δ_2, in $\mathcal{C}_i$ and $\mathcal{C}_j$ respectively,
$$(\delta_1 \cdot \delta_2)_P = (\phi^i_P(\delta_1) \cdot \phi^j_P(\delta_2))_{\phi(P)}.$$

Obviously, for any arrangement $\mathcal{D}$, the vector spaces A^i of a GAC are determined by the combinatorial type, as shown in Corollary 2.4. The point, then, is to prove that the coboundary map ω depends only on the combinatorics in the case of rational arrangements. The key to this is the technical work developed in Chapter 2 that computes the structure of the cohomology ring.

To help in the understanding of this chapter, an interpretation of $\tilde{A}^2 = \wedge^2 A^1$ will be given.

DEFINITION 3.3. Let us consider $\mathcal{D}_t$ a deformation of $\mathcal{D}$ where $\mathcal{D}_0 = \mathcal{D}$ and, for any $t > 0$ sufficiently small, the curve $\mathcal{D}_t$ is nodal with irreducible components $\mathcal{C}_0 \cup \mathcal{C}_{t,1} \cup ... \cup \mathcal{C}_{t,n}$ and $\deg(\mathcal{C}_{t,i}) = \deg(\mathcal{C}_i) = d_i$. Such a deformation is called a **nodal deformation of the arrangement** $\mathcal{D}$. Restricting a nodal deformation of $\mathcal{D}$ to $\mathbb{C}^2$, one obtains a family of affine arrangements $\mathcal{D}_t^*$ called an **affine nodal deformation of** $\mathcal{D}^*$.

If $\mathcal{L}$ is a line arrangement, then the set of generators of $\tilde{A}^2 = \wedge^2 A_i^1$ is in bijection with the set of double points in an affine nodal deformation $\mathcal{L}_t^*$ of $\mathcal{L}^*$. Therefore, there are maps

$$\begin{array}{rccc} \psi_t : & \mathrm{Sing}(\mathcal{L}_t^*) & \longrightarrow & \mathrm{Sing}(\mathcal{L}^*) \\ & Q_t & \mapsto & \lim_{t \to 0} Q_t \end{array} .$$

For any ψ_t, there is yet another map

$$\psi_{t,P} : \psi_t^{-1}(P) \longrightarrow \Delta_P \times \Delta_P$$

from the set of points degenerating into P to the pairs of lines passing through P. Since $\mathcal{L}_t^*$ has only double points, any point degenerating into P will be the intersection of exactly two lines. These degenerate into lines of $\mathcal{L}^*$ passing through P. Hence, they can be considered as an element of $\Delta_P \times \Delta_P$. This pair will be, by definition, $\psi_{t,P}(Q)$. Note that all these maps are determined by the combinatorial type of $\mathcal{L}$. The main idea in the present chapter is an attempt to generalize this intuition about GAC's for lines to the rational case. The main problem is the following. In the case of lines, every two components will intersect only at one point. This is not the case for general rational arrangements. Our next aim is to obtain a vector space from A^1 which allows for an encoding of the higher number of intersections.

3.2. The complex $(A_c^\bullet, \omega_c)$

Our aim in this section is to define a family of complexes $(A_c^\bullet, \omega_c)$ based on the combinatorial data of the arrangement $\mathcal{D}$. Such a family will be related to the GAC's for $\mathcal{D}$.

Consider the following vector spaces over $\mathbb{C}$

$$\begin{aligned} &\tilde{A}^0 := \mathbb{C} \ , \\ &\tilde{A}_i^1 := \sigma_i \mathbb{C} \qquad\qquad i = 1, ..., n \ , \\ &\tilde{A}^1 := \tilde{A}_1^1 \oplus ... \oplus \tilde{A}_n^1 \\ &\tilde{A}_i^{1,r} := \oplus_{k=1}^r \tilde{A}_i^1(k) \\ &\tilde{A}^2 := \bigwedge^2 \left(\oplus_{i=1}^n \tilde{A}_i^{1,d_i} \right) \end{aligned} \tag{3.2}$$

where $\tilde{A}_i^1(k)$ denotes the k-th component of the direct sum.

The vector spaces of chains A_c^* will be quotients of $\tilde{A}^*$. In order to define the subspaces of relations we will start decomposing $\tilde{A}^2$ in a convenient way. Note the following fact from commutative algebra. Let V and W be complex vector spaces. Then,

$$\bigwedge^2(V \oplus W) \cong \bigwedge^2 V \oplus \bigwedge^2 W \oplus \frac{V \otimes W \oplus W \otimes V}{R_{VW}}$$

where R_{VW} is generated by $(v \otimes w + w \otimes v)\ \ \forall v \in V, \forall w \in W$.

It is not difficult to see that, in general,

$$\bigwedge^2 \left(\bigoplus V_i\right) \cong \bigoplus_i \left(\bigwedge^2 V_i\right) \oplus \bigoplus_{i<j} \frac{V_i \otimes V_j \oplus V_j \otimes V_i}{R_{ij}} \tag{3.3}$$

where R_{ij} is generated by

$$v_i \otimes v_j + v_j \otimes v_i\ ,\ \forall v_i \in V_i,\ \forall v_j \in V_j. \tag{3.4}$$

One can define

$$\tilde{A}^1_{ij} := \bigoplus_{k=1}^{d_i} \bigoplus_{k'=1}^{d_j} \frac{\tilde{A}^1_i(k) \otimes \tilde{A}^1_j(k') \oplus \tilde{A}^1_j(k') \otimes \tilde{A}^1_i(k)}{R_{ij}}$$

with the convention $i < j$, and

$$\tilde{A}^1_i(k,k') := \frac{\tilde{A}^1_i(k) \otimes \tilde{A}^1_i(k') \oplus \tilde{A}^1_i(k') \otimes \tilde{A}^1_i(k)}{R_{kk'}}$$

where the spaces R_{ij} and $R_{k,k'}$ are defined using (3.4). Then, applying (3.2), (3.3) and the fact that $\wedge^2 \tilde{A}^1_i = \{0\}$, one has

$$\tilde{A}^2 = \bigoplus_{i<j} \tilde{A}^1_{ij} \oplus \bigoplus_{i=1}^{n} \bigoplus_{1<k<k' \leq d_i} \tilde{A}^1_i(k,k') \oplus \bigoplus_{i=1}^{n} \bigoplus_{k=2}^{d_i} \tilde{A}^1_i(1,k).$$

Next let us define

$$\tilde{A}^2_0 := \bigoplus_{i<j} \tilde{A}^1_{ij} \oplus \bigoplus_{i=1}^{n} \bigoplus_{1<k<k' \leq d_i} \tilde{A}^1_i(k,k') \tag{3.5}$$

and

$$\tilde{A}^2_\infty := \bigoplus_{i=1}^{n} \bigoplus_{k=2}^{d_i} \tilde{A}^1_i(1,k). \tag{3.6}$$

Clearly,

$$\tilde{A}^2 = \tilde{A}^2_0 \oplus \tilde{A}^2_\infty.$$

Comparing the dimensions of each summand, note that

$$\dim \tilde{A}^2 = \binom{\sum d_i}{2} = \sum d_i d_j + \sum \binom{d_i}{2} = \tag{3.7}$$

$$= \left(\sum d_i d_j + \sum \binom{d_i - 1}{2}\right) + \sum (d_i - 1).$$

Consider a nodal deformation $\mathcal{D}_t$ of $\mathcal{D}$. Note that $\dim \tilde{A}^2$ coincides with the number of double points of $\mathcal{D}_t$ minus n. The first summand of the last term in (3.7) is $\dim \tilde{A}^2_0$ and corresponds to the number of affine double points. The remaining summand,

$\dim \tilde{A}^2_\infty$, is the number of points at infinity minus one, $d_i - 1$, per component. One can construct a map

$$\begin{array}{cccc} \overline{\psi}_t : & \mathrm{Sing}(\mathcal{D}_t) & \longrightarrow & \mathrm{Sing}(\mathcal{D}) \\ & Q_t & \mapsto & \lim_{t\to 0} Q_t \end{array}.$$

Note that $\psi_t^{-1}(Q)$ is unique if and only if Q is a double point of $\mathcal{D}$. A base point at infinity for $\mathcal{C}_i$ shall be defined as follows

$$b(\mathcal{C}_{i,t}) := \overline{\psi}_t^{-1}(b(\mathcal{C}_i)).$$

The set of base points of $\mathcal{D}_t$ will be denoted by $B(\mathcal{D}_t)$. Consider also

$$\mathcal{G}(\mathcal{D}_t) := \mathrm{Sing}(\mathcal{D}_t) \setminus B(\mathcal{D}_t). \tag{3.8}$$

There is a natural restriction map ψ_t

$$\begin{array}{ccc} \mathrm{Sing}(\mathcal{D}_t) & \xrightarrow{\overline{\psi}_t} & \mathrm{Sing}(\mathcal{D}) \\ \bigcup & & \bigcup \\ \mathcal{G}(\mathcal{D}_t) & \xrightarrow{\psi_t} & \mathcal{G}(\mathcal{D}) \end{array}.$$

Note that the cardinality of $\psi_t^{-1}(P)$ is the degree of the tree $\mathcal{T}_P^0(\mathcal{D})$ described in Definition 2.20.6, that is, the δ-invariant of $\mathcal{D}$ at P. As mentioned above, we want to be able to identify which branches intersect at each double point of $\mathcal{G}(\mathcal{D}_t)$. For that purpose, one constructs maps

$$\psi_{t,P} : \quad \psi_t^{-1}(P) \subset \mathcal{G}(\mathcal{D}_t) \quad \longrightarrow \quad \Delta_P(\mathcal{D}) \times \Delta_P(\mathcal{D})$$

where $\Delta_P(\mathcal{D})$ is the set of local branches of $\mathcal{D}$ at P. Any double point Q in $\psi_t^{-1}(P)$ is determined by a pair of local branches $(\delta_{1,t}, \delta_{2,t})$ of $\mathcal{D}_t$. Each local branch $\delta_{k,t}$ will degenerate into a local branch $\delta_{k,0}$ of $\mathcal{D}$ at P. Using the notation above, one can define

$$\psi_{t,P}(Q) := (\delta_{1,0}, \delta_{2,0}). \tag{3.9}$$

The cardinality of

$$\psi_{t,P}^{-1}(\delta_i, \delta_j) \cup \psi_{t,P}^{-1}(\delta_j, \delta_i)$$

equals the number of double points near P in the intersection of δ_i and δ_j after a small deformation. Note also that there is a choice involving ordering in the image of a point –one can choose (δ_i, δ_j) or (δ_j, δ_i). For this reason we will extend both definitions to maps

$$\tilde{\psi}_t : \quad \mathcal{G}(\mathcal{D}_t) \sqcup -\mathcal{G}(\mathcal{D}_t) \quad \longrightarrow \quad \mathcal{G}(\mathcal{D})$$

$$\tilde{\psi}_{t,P} : \quad \tilde{\psi}^{-1}(P) \quad \longrightarrow \quad \Delta_P(\mathcal{D}) \times \Delta_P(\mathcal{D})$$

where $\mathcal{G}(\mathcal{D}_t) \sqcup -\mathcal{G}(\mathcal{D}_t)$ denotes the disjoint union of $\mathcal{G}(\mathcal{D}_t)$ and $-\mathcal{G}(\mathcal{D}_t)$. The latter being the set whose elements are the points of $\mathcal{G}(\mathcal{D}_t)$ with a negative sign attached. These maps are defined by

$$\tilde{\psi}_t|_{\mathcal{G}(\mathcal{D})} = \psi_t,$$

$$\tilde{\psi}_t(-Q) := \psi_t(Q) \quad \forall Q \in \mathcal{G}(\mathcal{D}_t)$$

and

$$\tilde{\psi}_{t,P}(-Q) = i(\psi_{t,P}(Q)) \text{ where } i(x,y) = (y,x).$$

EXAMPLE 3.4. It might happen that $\delta_{1,0} = \delta_{2,0}$ in (3.9). This is, for example, the case in a singularity of local equation $x(y^2 - x^3)$ at the origin. In this case $\mathcal{D}_t$ can have, for example, a local equation $x(y^2 - x(x-t)^2)$. Then

$$\{P_t = (t,0), Q_{1,t} = (0,t), Q_{2,t} = (0,-t)\} \subset \mathcal{G}(\mathcal{D}_t).$$

Let δ_1 (resp. δ_2) be the branch of $\mathcal{D}$ at P whose tangent is $\{y=0\}$ (resp. $\{x=0\}$), then

$$\begin{array}{ccc} \{\pm P_t, \pm Q_{1,t}, \pm Q_{2,t}\} & \xrightarrow{\tilde{\psi}_{t,P}} & \Delta_P(\mathcal{D}) \times \Delta_P(\mathcal{D}) \\ \pm P_t & \mapsto & (\delta_1, \delta_1) \\ Q_{k,t} & \mapsto & (\delta_1, \delta_2) \\ -Q_{k,t} & \mapsto & (\delta_2, \delta_1) \end{array}$$

for $k = 1, 2$.

One can summarize the previous construction in the following

DEFINITIONS 3.5.
Let $\mathcal{D} = \mathcal{C}_1 \cup \ldots \cup \mathcal{C}_n$ be an arrangement. Let us consider $\mathcal{D}_t$ a nodal deformation of $\mathcal{D}$. Such a deformation defines maps

$$\begin{array}{cccccc} \overline{\psi}_{t,\mathcal{D}} : & \mathrm{Sing}(\mathcal{D}_t) & \longrightarrow & \mathrm{Sing}(\mathcal{D}) & & \\ & P_t & \mapsto & P = & \lim_{t\to 0} P_t \end{array}$$

and

$$\psi_{t,P} : \quad \overline{\psi}^{-1}_{t,\mathcal{D}}(P) \quad \longrightarrow \quad \Delta_P(\mathcal{D}) \times \Delta_P(\mathcal{D})$$

from the set of points degenerating into P to the pairs of local branches of $\mathcal{D}$ at P. The image of a double point P_t in $\mathcal{D}_t$ is the limit of the pair of branches that intersect in P_t.
Two curves $\mathcal{D}$ and $\mathcal{D}'$ in $\mathbb{P}^2$ have the **same weak combinatorial type** if there are nodal deformations, $\mathcal{D}_t$ and $\mathcal{D}'_t$, of $\mathcal{D}$ and $\mathcal{D}'$ respectively, such that:

1. both curves have the same number of components and, up to reordering,
$$\mathcal{D} = \mathcal{C}_{t,1} \cup \ldots \cup \mathcal{C}_{t,n}$$
$$\mathcal{D}' = \mathcal{C}'_{t,1} \cup \ldots \cup \mathcal{C}'_{t,n}$$
with $\deg(\mathcal{C}_{t,i}) = \deg(\mathcal{C}'_{t,i}) = d_i$,
2. there are bijections
$$\phi_t : \mathrm{Sing}(\mathcal{D}_t) \longrightarrow \mathrm{Sing}(\mathcal{D}'_t)$$
$$\phi_{t,P} : \Delta_P(\mathcal{D}) \longrightarrow \Delta_Q(\mathcal{D}')$$
where $Q = \psi_0(P)$ such that the following diagrams
$$\begin{array}{ccc} \mathrm{Sing}(\mathcal{D}_t) & \xrightarrow{\phi_t} & \mathrm{Sing}(\mathcal{D}'_t) \\ \downarrow \overline{\psi}_{t,\mathcal{D}} & & \downarrow \overline{\psi}_{t,\mathcal{D}'} \\ \mathrm{Sing}(\mathcal{D}) & \xrightarrow{\phi_0} & \mathrm{Sing}(\mathcal{D}') \end{array}$$
and
$$\begin{array}{ccc} \overline{\psi}^{-1}_{t,\mathcal{D}}(P) & \xrightarrow{\phi_t|} & \overline{\psi}^{-1}_{t,\mathcal{D}'}(Q) \\ \downarrow \psi_{t,P} & & \downarrow \psi_{t,Q} \\ \Delta_P(\mathcal{D}) \times \Delta_P(\mathcal{D}) & \xrightarrow{\phi_{t,P}\times\phi_{t,P}} & \Delta_Q(\mathcal{D}') \times \Delta_Q(\mathcal{D}') \end{array}$$
commute.

REMARK 3.6. Obviously, same combinatorial type implies same weak combinatorial type, but the converse is in general not true. For instance, consider the following irreducible curves $\mathcal{D}$ and $\mathcal{D}'$

$$\mathcal{D} = \{[X:Y:Z] \in \mathbb{P}^2 \mid (XZ - Y^2)^2 - X^3Y = 0\}$$
$$\mathcal{D}' = \{[X:Y:Z] \in \mathbb{P}^2 \mid ZX^3 = Y^4\}.$$

Both have only one singularity at $[0:0:1]$, of types A_6 and E_6 respectively, and hence different combinatorial types. Since they have only one branch at the singular point, the maps $\psi_{t,P}(\mathcal{D})$ and $\psi_{t,P}(\mathcal{D}')$ are constant. Moreover, since

$$\#\overline{\psi}_{t,\mathcal{D}}^{-1}(P) = \#\overline{\psi}_{t,\mathcal{D}'}^{-1}(P) = 3$$

they have in fact the same weak combinatorial type.

Note that, according to (3.7), the dimension of $\tilde{A}^2$ coincides with the cardinality of $\mathcal{G}(\mathcal{D}_t)$ for any $t > 0$ sufficiently small. Hence, one can fix such a $t > 0$ and denote the generators of $\tilde{A}^2$ by σ^Q, where $\pm Q \in \mathcal{G}(\mathcal{D}_t)$ and $\sigma^{-Q} = -\sigma^Q$ –these relations appear naturally from (3.4). Form now on, the maps $\tilde{\psi}_t$ and $\tilde{\psi}_{t,P}$ will simply be denoted by $\tilde{\psi}$ and $\tilde{\psi}_P$ respectively.

Going back to the definition of the complex, let us define $R^0 := R^1 := \{0\}$ and R^2 the subspace of $\tilde{A}^2$ generated by

$$\begin{cases} \sigma^{\ell} - \sigma^{\ell'} & \text{if } \tilde{\psi}(\ell) = \tilde{\psi}(\ell') = P \quad \text{and } \tilde{\psi}_P(\ell) = \tilde{\psi}_P(\ell') \\ & \\ \sigma^{\ell_1} + \sigma^{\ell_2} + \sigma^{\ell_3} & \text{if } \tilde{\psi}(\ell_r) = P, \qquad \text{and } \begin{cases} \tilde{\psi}_P(\ell_1) = (\delta_i, \delta_j), \\ \tilde{\psi}_P(\ell_2) = (\delta_j, \delta_k), \\ \tilde{\psi}_P(\ell_3) = (\delta_k, \delta_i). \end{cases} \end{cases}$$

Since $R^\bullet$ are subspaces of $\tilde{A}^\bullet$, one can define the following vector spaces

$$A_c^\bullet := \tilde{A}^\bullet / R^\bullet.$$

Let us consider a diagonal map

$$\begin{array}{rccll} \delta_i: & \tilde{A}_i^1 & \longrightarrow & \tilde{A}^{1,d_i} & \\ & \sigma_i & \mapsto & (\sigma_i, \sigma_i, ..., \sigma_i) & = \sigma_{i1} + \sigma_{i2} + ... + \sigma_{id_i} \end{array}$$

and denote by δ the sum morphism

$$\begin{array}{rccc} \delta: & \tilde{A}^1 \cong A^1 & \longrightarrow & \oplus_{i=1}^n \tilde{A}^{1,d_i} \\ & \sum_i \sigma_i & \mapsto & \sum_i \delta_i \sigma_i \end{array}$$

Hence, any generator of A_c^1 defines a map

$$\begin{array}{rccc} \wedge\sigma_i: & A_c^1 & \longrightarrow & A_c^2 \\ & \sigma_j & \mapsto & \delta\sigma_i \wedge \delta\sigma_j \end{array} . \tag{3.10}$$

Hence, any element

$$\omega_c = \sum_{i=1}^n \lambda_i \sigma_i \in A_c^1 \tag{3.11}$$

defines a homomorphism $A_c^1 \xrightarrow{\wedge\omega_c} A_c^2$.

REMARKS 3.7.

1. Let $(A^\bullet, \omega)$ be a GAC for $\mathcal{D}$ as in Definition 3.1. Note that $A^0 \cong A^0_c \cong \mathbb{C}$ and $A^1 \cong A^1_c$ via the following map
$$\begin{array}{ccc} A^1 & \longrightarrow & A^1_c \\ \frac{d\mathcal{C}_i}{\mathcal{C}_i} & \mapsto & \sigma_i \end{array} . \tag{3.12}$$
2. The complex $(A^\bullet_c, \omega_c)$ depends only on the weak combinatorial type of $\mathcal{D}$, whereas $(A^\bullet, \omega)$ depends not only on the choice of certain polynomials $\varphi_P^{\delta_i,\delta_j}$ –c.f. Definition 2.38– but also on the jacobians of the components of the arrangement –c.f. (2.7)– as well as on the choice of certain base points.
3. For line arrangements, the complex $(A^\bullet_c, \omega_c)$ describes the classical Aomoto complex, first introduced in [**44**] –see also [**45**]. Note that
$$A^2_c = \langle s_{ij} : \begin{array}{l} s_{ij} = -s_{ji}, \\ s_{ij} + s_{jk} + s_{ki} = 0 \quad \text{if } \mathcal{C}_i \cap \mathcal{C}_j \cap \mathcal{C}_k \neq \emptyset \end{array} \rangle.$$
Moreover, the map described in (3.10) clearly coincides with the one in the classical Aomoto complex. Hence, we have just shown that $(A^\bullet, \omega)$ and $(A^\bullet_c, \omega_c)$ are isomorphic as complexes, where ω_c is the image of ω by the map (3.12). Therefore, by Remark 3.7.2 above, the cohomology of $(A^\bullet, \omega)$ depends only on the weak combinatorial type of $\mathcal{D}$.

3.3. An equivalence of complexes

In general, GAC's do not coincide with the complex A_c defined in Section 3.2. Still, they carry the same information. We will in fact prove that they are equivalent.

LEMMA 3.8. *If $\mathcal{R}$ is a rational complex, then*
$$\dim A^2_c = \dim A^2.$$

Proof. From the subspace of relations R^2 one can note the following. Given any singular point $P \in \mathcal{G}(\mathcal{R}) = \operatorname{Sing}(\mathcal{R}) \setminus B(\mathcal{R})$ let us fix a local branch δ_P. All generators of the form σ^Q with $\psi_P(Q) = (\delta_i, \delta_j)$ can be written in terms of $\sigma^{Q'}$ and $\sigma^{Q''}$ where $\psi_P(Q') = (\delta_P, \delta_i)$ and $\psi_P(Q'') = (\delta_P, \delta_j)$ using
$$\sigma^Q + \sigma^{Q'} - \sigma^{Q''} = 0.$$
Hence, the set
$$\overline{v}_c(\mathcal{R}) := \bigcup_{P \in \mathcal{G}(\mathcal{R})} \{\sigma^Q \mid \psi_P(Q) = (\delta_P, \delta)\}$$
generates A^2_c.

Using the relations
$$\sigma^Q = \sigma^{Q'} \quad \text{if } \psi_P(Q) = \psi_P(Q')$$
let us select a representative, denoted by $\sigma^{(\delta_P,\delta)}$ in the class
$$[\sigma^{(\delta_P,\delta)}] := \{\sigma^Q \mid \psi_P(Q) = \psi_P(Q')\}.$$

Therefore, the set

$$v_c(\mathcal{R}) := \bigcup_{P\in\mathcal{G}(\mathcal{R})} \{\sigma^{(\delta_P,\delta)} \mid \delta_P \neq \delta \in \Delta_P\}$$

still generates A^2_c.

It is also easy to check that the elements of $v_c(\mathcal{R})$ are linearly independent. Therefore, the dimension of A^2_c coincides with the cardinality of $v_c(\mathcal{R})$, which is

$$\sum_{P\in\mathcal{G}(\mathcal{R})} (r_P - 1) = \sum_{P\in\mathrm{Sing}(\mathcal{R})} (r_P - 1) - n = \dim H^2(X_{\mathcal{R}};\mathbb{C}\,) \ = \dim A^2$$

–c.f. Proposition 2.4 for the second equality. $\square$

THEOREM 3.9. *The complexes* $(A^\bullet, \omega)$ *and* $(A^\bullet_c, \omega_c)$ *are equivalent, where* ω_c *is the image of* ω *via the morphism* (3.12).

Proof. First of all, let us define a morphism of complexes. We can define the following map

$$\begin{array}{ccc} A^2 & \longrightarrow & A^2_c \\ \psi_P^{\delta,\delta'} & \mapsto & \sigma^Q \end{array} \tag{3.13}$$

where Q is the only nodal point in the deformation $\mathcal{D}_1$ such that $\phi_1(Q) = P$ and $\phi_P(Q) = (\delta, \delta')$. One can check that this map is an isomorphism.

We will denote by π^0 the identity map between A^0 and A^0_c; by π^1 the map defined in (3.12) and by π^2 the composition of (3.13) with the natural projection of A^2_c onto its subspace $\tilde{A}^2_0/R^2$ –c.f. (3.5).

Using Theorems 2.46 and 2.47 it is readily seen that

$$(A^\bullet, \omega) \xrightarrow{\pi^\bullet} (A^\bullet_c, \omega_c)$$

is a morphism of complexes.

By Remarks 3.7.1 and 3.8, it only remains to check that

$$\ker\left(A^1 \overset{\wedge\omega}{\to} A^2\right) \cong \ker\left(A^1_c \overset{\wedge\omega_c}{\to} A^2_c\right). \tag{3.14}$$

Since

$$\chi(A^\bullet, \omega) = \chi(A^\bullet_c, \omega_c) = \chi(X_{\mathcal{R}}),$$

(3.14) is equivalent to

$$\mathrm{im}\left(A^1 \overset{\wedge\omega}{\to} A^2\right) \cong \mathrm{im}\left(A^1_c \overset{\wedge\omega_c}{\to} A^2_c\right). \tag{3.15}$$

One has the following commutative diagram

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & & & A^2_\infty \arrow[d] & \\
A^1_c \arrow[r, phantom, "\overset{\pi^1}{\cong}"] \arrow[d, "id"] & A^1 \arrow[r, "\wedge\omega"] & A^2 \arrow[r, phantom, "\overset{\pi^2}{\cong}"] & A^2_c \arrow[d] & \\
A^1_c \arrow[rrr, "\wedge\omega_c"] & & & A^2_0 \arrow[d] & \subset A^2_c \\
 & & & 0 &
\end{tikzcd}$$

where $A^2_0 := \tilde{A}^2_0/R^2$ and $A^2_\infty := \tilde{A}^2_\infty/R^2$. Hence,

$$\begin{aligned}\operatorname{im}\left(A^1 \xrightarrow{\wedge\omega} A^2\right) &\cong \operatorname{im}\left(A^1_c \xrightarrow{(\pi^1)^{-1}} A^1 \xrightarrow{\wedge\omega} A^2 \xrightarrow{\pi^2} A^2_c\right) \cong \\ &\cong \operatorname{im}\left(A^1_c \xrightarrow{\wedge\omega_c} A^2_c\right)\end{aligned}$$

which proves (3.15). □

COROLLARY 3.10. *In the case of rational arrangements, the cohomology of the complex* $(A^\bullet, \omega)$ *described in* Definition (3.1) *is determined only by the weak combinatorial type of the arrangement.*

Proof. It is immediate from Theorem 3.9 and Remark 3.7.2. □

CHAPTER 4

Characteristic varieties, local systems and rational arrangements

Our purpose in this chapter is to relate GAC's to cohomology with coefficients in a local system and ultimately to characteristic varieties. The main two results are stated in Theorems 4.2 and 4.7. The first one is based on a similar result for hyperplane arrangements –c.f. [**20**]. It claims that, under certain conditions on ω, the GAC $(A^\bullet, \omega)$ calculates the cohomology with coefficients in a certain local system ρ_ω given by ω. The second theorem describes the relationship between a stratification of the family of GAC's and the components of Char_i passing through the origin.

4.1. GAC's and cohomology of local systems

Let $\mathcal{R} = \mathcal{C}_0 \cup \mathcal{C}_1 \cup \ldots \cup \mathcal{C}_n$ be a rational arrangement, where $\mathcal{C}_0$ is a transversal line. As usual, its complement in $\mathbb{P}^2$ will be denoted by $X_\mathcal{R}$. Let

$$\pi : \overline{X}_\mathcal{R} \to \mathbb{P}^2$$

be a resolution of $\mathcal{R}$. As usual, $\overline{\mathcal{R}} := (\pi^*(\mathcal{R}))_{red}$ denotes the reduced divisor with normal crossings in $\overline{X}_\mathcal{R}$ after resolution. Also let us denote by $\Omega^p_{\overline{X}_\mathcal{R}}(\log\langle\overline{\mathcal{R}}\rangle)$ (resp. $\mathcal{A}^p_{\overline{X}_\mathcal{R}}(\log\langle\overline{\mathcal{R}}\rangle)$) the sheaf of holomorphic (resp. C^∞) log p-forms on $\overline{X}_\mathcal{R}$ along the divisor with normal crossings $\overline{\mathcal{R}}$.

LEMMA 4.1.

$$H^q(\overline{X}_\mathcal{R}, \Omega^p_{\overline{X}_\mathcal{R}}(\log\langle\overline{\mathcal{R}}\rangle)) \cong H^q(\overline{X}_\mathcal{R}, \mathcal{A}^p_{\overline{X}_\mathcal{R}}(\log\langle\overline{\mathcal{R}}\rangle)),$$

that is,

$$H^q(\overline{X}_\mathcal{R}, \Omega^p_{\overline{X}_\mathcal{R}}(\log\langle\overline{\mathcal{R}}\rangle)) = 0 \quad \forall q > 0.$$

Proof. Note that the generators of $H^0(\overline{X}_\mathcal{R}, \mathcal{A}^p_{\overline{X}_\mathcal{R}}(\log\langle\overline{\mathcal{R}}\rangle))$ are in fact holomorphic closed forms, as shown in Theorem 2.11 and Theorem 2.46. Hence,

$$\begin{array}{ccccc} & & \Gamma(\overline{X}_\mathcal{R}, \Omega^q_{\overline{X}_\mathcal{R}}(\log\langle\overline{\mathcal{R}}\rangle)) & \cong & H^q(\Gamma(\overline{X}_\mathcal{R}, \Omega^\bullet_{\overline{X}_\mathcal{R}}(\log\langle\overline{\mathcal{R}}\rangle)), d) \\ & & & & \downarrow \\ H^q(X_\mathcal{R}; \mathbb{C}) & \cong & \Gamma(\overline{X}_\mathcal{R}, \mathcal{A}^q_{\overline{X}_\mathcal{R}}(\log\langle\overline{\mathcal{R}}\rangle)) & \cong & H^q(\Gamma(\overline{X}_\mathcal{R}, \mathcal{A}^\bullet_{\overline{X}_\mathcal{R}}(\log\langle\overline{\mathcal{R}}\rangle)), d). \end{array} \tag{4.1}$$

On the other hand, the logarithmic version of the de Rham Theorem ([**18**] 3.2.13) assures that the Hodge-Deligne spectral sequence

$$E_1^{p,q} = H^q(\overline{X}_\mathcal{R}, \Omega^p_{\overline{X}_\mathcal{R}}(\log\langle\overline{\mathcal{R}}\rangle)) \Rightarrow H^{p+q}(X_\mathcal{R}; \mathbb{C}) \tag{4.2}$$

degenerates in E_1. Therefore, there is an injection

$$H^q(\overline{X}_{\mathcal{R}}, \Omega^q_{\overline{X}_{\mathcal{R}}}(\log\langle\overline{\mathcal{R}}\rangle)) \hookrightarrow H^{p+q}(X_{\mathcal{R}}; \mathbb{C})$$

for any $p, q \in \mathbb{N}$.

Combining this and (4.1), one has

$$H^q(\Gamma(\overline{X}_{\mathcal{R}}, \Omega^\bullet_{\overline{X}_{\mathcal{R}}}(\log\langle\overline{\mathcal{R}}\rangle)), d) = H^q(\Gamma(\overline{X}_{\mathcal{R}}, \mathcal{A}^\bullet_{\overline{X}_{\mathcal{R}}}(\log\langle\overline{\mathcal{R}}\rangle)), d) \cong H^q(X_{\mathcal{R}}; \mathbb{C}). \tag{4.3}$$

Hence,

$$H^q(\overline{X}_{\mathcal{R}}, \Omega^p_{\overline{X}_{\mathcal{R}}}(\log\langle\overline{\mathcal{R}}\rangle)) = \begin{cases} \Gamma(\overline{X}_{\mathcal{R}}, \mathcal{A}^p_{\overline{X}_{\mathcal{R}}}(\log\langle\overline{\mathcal{R}}\rangle)) & \text{if } q = 0 \\ 0 & \text{if } q > 0. \end{cases} \qquad \square$$

Note that any 1-form

$$\omega = \sum_{i=1}^{n} \alpha_i \otimes \frac{d\mathcal{C}_i}{\mathcal{C}_i} \in \operatorname{End}(\mathcal{O}^r_{X_{\mathcal{R}}}) \otimes H^0(\overline{X}_{\mathcal{R}}, \Omega^1_{X_{\mathcal{R}}}) \tag{4.4}$$

defines a connection ∇^0_ω on the trivial bundle as follows

$$\begin{array}{rccc} \nabla^0_\omega : & \mathcal{O}^r_{X_{\mathcal{R}}} & \longrightarrow & \Omega^1_{X_{\mathcal{R}}}(\mathcal{O}^r_{X_{\mathcal{R}}}) = \Omega^1_{X_{\mathcal{R}}} \otimes \mathcal{O}^r_{X_{\mathcal{R}}} \\ & f & \mapsto & df - \omega \otimes f. \end{array}$$

All one has to check is that ∇^0_ω is a $\mathbb{C}$-linear homomorphism verifying the Leibniz rule. That is, for any $\lambda \in \mathcal{O}_{X_{\mathcal{R}}}$ and $f \in \mathcal{O}^r_{X_{\mathcal{R}}}$, we have

$$\nabla^0_\omega(\lambda f) = d(\lambda f) - \lambda\omega \otimes f = \lambda df + d\lambda \otimes f - \lambda\omega \otimes f = d\lambda \otimes f + \lambda \nabla^0_\omega f.$$

This connection extends to the higher exterior powers as

$$\begin{array}{rccc} \nabla^p_\omega : & \Omega^p_{X_{\mathcal{R}}}(\mathcal{O}^r_{X_{\mathcal{R}}}) = \Omega^p_{X_{\mathcal{R}}} \otimes \mathcal{O}^r_{X_{\mathcal{R}}} & \longrightarrow & \Omega^{p+1}_{X_{\mathcal{R}}}(\mathcal{O}^r_{X_{\mathcal{R}}}) = \Omega^{p+1}_{X_{\mathcal{R}}} \otimes \mathcal{O}^r_{X_{\mathcal{R}}} \\ & \alpha \otimes f & \mapsto & d\alpha \otimes f + (-1)^p \alpha \wedge \nabla^0_\omega f. \end{array}$$

Again, one has to check that

$$\nabla^p_\omega(\lambda\alpha \otimes f) = d(\lambda\alpha) \otimes f + (-1)^p \lambda\alpha \wedge \nabla^0_\omega f = d\lambda \wedge \alpha \otimes f + \lambda d\alpha \otimes f +$$
$$+(-1)^p \lambda\alpha \wedge \nabla^0_\omega f = d\alpha \otimes \lambda f + (-1)^p \alpha \wedge \nabla^0_\omega(\lambda f) = \nabla^p_\omega(\alpha \otimes \lambda f).$$

Moreover, since

$$\nabla^{p+1}_\omega \circ \nabla^p_\omega f = \nabla^{p+1}_\omega \left(d\alpha \otimes f + (-1)^p \alpha \wedge \nabla^0_\omega f\right) =$$
$$= d^2\alpha \otimes f + (-1)^{p+1} d\alpha \wedge df - (-1)^{p+1}(d\alpha \wedge \omega) \otimes f +$$
$$+(-1)^p d\alpha \wedge df - (-1)^p(-1)^{p+1} \alpha \wedge df \wedge \omega - (-1)^p (d\alpha \wedge \omega) \otimes f -$$
$$-(-1)^p \alpha \wedge d\omega \otimes f - (-1)^p(-1)^{p+1} \alpha \wedge \omega \wedge df = 0,$$

for any $f \in \mathcal{O}^r_{X_{\mathcal{R}}}$, one can define a de Rham-type complex of twisted coefficients $(\Omega^\bullet_{X_{\mathcal{R}}}, \nabla_\omega)$ with differential ∇_ω, sometimes also denoted as $(\Omega^\bullet_{X_{\mathcal{R}}}, d + \omega)$.

Note that, in our case, the complex is concentrated in degrees 0, 1 and 2, but these complexes exist in a more general context –c.f. [**17**].

The hypercohomology of this complex degenerates to the cohomology of $X_{\mathcal{R}}$ with coefficients in the local system ρ_ω defined by the flat sections of ∇_ω –c.f. [**17**] Proposition I.2.19. Such a local system is given by

$$\begin{array}{rccc} \rho_\omega : & \pi_1(X_{\mathcal{R}}, x) & \longrightarrow & \operatorname{GL}(\mathbb{C}, r) \\ & \gamma_i & \mapsto & \exp(2\pi\sqrt{-1} P_i \alpha_i(x) P_i^{-1}) \end{array}$$

where γ_i is a meridian around $\mathcal{C}_i$ and $P_i \in \mathrm{GL}(\mathbb{C}, r)$ corresponds to the trivialization of the vector bundle of flat sections along the path γ_i. The matrix $P_i \alpha_i(x) P_i^{-1}$ is also referred to as the **matrix of the connection at** γ_i – c.f. [**17**] Theorem II.1.17.

Let ω be as in (4.4), where α_i are constant coefficients. In this case, ω can be seen as an element of $H^1(X_{\mathcal{R}}; \mathbb{C}) = \mathrm{Hom}(H_1(X_{\mathcal{R}}, \mathbb{Z}), \mathbb{C})$ and hence, the local system ρ_ω corresponds to the following representation

$$\begin{array}{rccc} \rho_\omega : & H_1(X_{\mathcal{R}}; \mathbb{C}) & \longrightarrow & \mathbb{C}^* \\ & \gamma_i & \mapsto & \exp(2\pi\sqrt{-1}\omega(\gamma_i)). \end{array} \tag{4.5}$$

Deligne studied the relationship between the complex $(\Omega^\bullet_{X_{\mathcal{R}}}, d+\omega)$ and the complex $(\Omega^\bullet_{\overline{X}_{\mathcal{R}}}(\log\langle\overline{\mathcal{R}}\rangle), \overline{\nabla}_\omega)$, where $\overline{\nabla}_\omega$ is a meromorphic extension of the connection ∇_ω to $\overline{X}_{\mathcal{R}}$. He proved that

$$\mathbb{H}^k(\overline{X}_{\mathcal{R}}, \Omega^\bullet_{\overline{X}_{\mathcal{R}}}(\log\langle\overline{\mathcal{R}}\rangle) \otimes \mathcal{O}^r_{\overline{X}_{\mathcal{R}}}, \overline{\nabla}_\omega) \cong \mathbb{H}^k(X_{\mathcal{R}}, \Omega^\bullet_{X_{\mathcal{R}}} \otimes \mathcal{O}^r_{X_{\mathcal{R}}}, \nabla_\omega) \tag{4.6}$$

provided the residues of the connection along $\overline{\mathcal{R}}$ have no positive integer eigenvalues – c.f. [**17**] Corollary II.3.15. In the sequel, these conditions shall be referred to as **non-resonance conditions** – c.f. [**37**].

The complex of twisted coefficients $(\Omega^\bullet_{\overline{X}_{\mathcal{R}}}(\log\langle\overline{\mathcal{R}}\rangle), \overline{\nabla}_\omega)$ has a spectral sequence associated with it. Deligne's Theorem implies that such a spectral sequence degenerates in a finite number of steps, provided the non-resonance conditions are satisfied, as follows

$$E_1^{p,q} = H^q(\overline{X}_{\mathcal{R}}, \Omega^p_{\overline{X}_{\mathcal{R}}}(\log\langle\overline{\mathcal{R}}\rangle) \otimes \mathcal{O}^r_{\overline{X}_{\mathcal{R}}}, \overline{\nabla}_\omega) \Rightarrow H^{p+q}(X_{\mathcal{R}}; \rho_\omega).$$

By Lemma 4.1, the double complex $E_1^{*,*}$ results in

$$\begin{aligned} 0 \to H^0(\overline{X}_{\mathcal{R}}, \Omega^0_{\overline{X}_{\mathcal{R}}}(\log\langle\overline{\mathcal{R}}\rangle)) &\xrightarrow{\wedge\omega} H^0(\overline{X}_{\mathcal{R}}, \Omega^1_{\overline{X}_{\mathcal{R}}}(\log\langle\overline{\mathcal{R}}\rangle)) \xrightarrow{\wedge\omega} \\ &\xrightarrow{\wedge\omega} H^0(\overline{X}_{\mathcal{R}}, \Omega^2_{\overline{X}_{\mathcal{R}}}(\log\langle\overline{\mathcal{R}}\rangle)) \to 0, \end{aligned} \tag{4.7}$$

that is, the GAC for $\mathcal{R}$.

Hence, we have proven the following

THEOREM 4.2. *If $\mathcal{R}$ is a rational arrangement, then*

$$H^k(A^\bullet, \omega) \cong H^k(X_{\mathcal{R}}; \rho_\omega)$$

provided ω satisfies the non-resonance conditions. □

The power of Theorem 4.2 obviously depends on how restrictive the non-resonance conditions are. The following result is analogous to the one given by H. Esnault, V. Schechtman, E.Viehweg in [**20**].

PROPOSITION 4.3. *If $\mathcal{R}$ is a rational arrangement, then the non-resonance conditions are given by*

$$\sum_{i \in I_P} a_{k,i}(P)\alpha_i \notin \mathbb{N}^+ \tag{4.8}$$

and

$$\alpha_i \notin \mathbb{N}^+, \tag{4.9}$$

where $P \in \operatorname{Sing}(\mathcal{R})$, $I_P = \{i \in \{1, ..., n\} \mid P \in \mathcal{C}_i\}$, $a_{k,i}(P) = mult_{E_k}\pi^*(C_i)$ *and* E_k *is an exceptional divisor in the resolution* π *of* $\mathcal{R}$.

Proof. The proof basically follows the same idea used in Proposition 2.10. Let C be the local equation of a component of $\mathcal{R}$ at P and let m be the multiplicity of C at that point, then the pull-back of the form $\frac{dC}{C}$, after blowing up P becomes

$$\frac{d\tilde{C}}{\tilde{C}} + m\frac{dv}{v}$$

where $\tilde{C}$ is the pull-back of C and v is the equation of the exceptional divisor. Adding up the multiplicities at each infinitely near point for every component $\mathcal{C}_i$ passing through P, we obtain the following equation for the residue of $\pi^*(\nabla_w)$ at each E_k

$$\operatorname{Res}(\pi^*(\nabla_\omega))_{E_k} = \sum_i a_{k,i}\alpha_i \tag{4.10}$$

and

$$\operatorname{Res}(\pi^*(\nabla_\omega))_{\tilde{\mathcal{C}}_i} = \alpha_i. \tag{4.11}$$

Finally, not allowing either (4.10) or (4.11) to be positive integers gives rise to the non-resonance conditions. □

REMARKS 4.4.

1. In [**20**], non-resonance conditions for hyperplane arrangements are obtained. For line arrangements, the coefficients $a_{k,i}$ described in the previous Proposition are all 1 and hence, our conditions coincide with those in [**20**].
2. Note that the set of resonance conditions – the complement of non-resonance conditions – provided in Proposition 4.3 contains the faces of quasiadjunction of the arrangement $\mathcal{R} \subset \mathbb{P}^2$ – c.f. [**37**].

4.2. GAC's and characteristic varieties

The results presented in this section, applied to the case of line arrangements, can be found in [**37**].

Let

$$\omega = \sum_{i=1}^{n} \alpha_i \otimes \frac{d\mathcal{C}_i}{\mathcal{C}_i}$$

be a logarithmic 1-form. Recall there is a local system

$$\begin{array}{rccc} \rho_\omega : & H_1(X_{\mathcal{R}};\mathbb{C}) & \to & \mathbb{C}^* \\ & (\gamma_1, ..., \gamma_n) & \mapsto & (\exp(2\pi\sqrt{-1}\alpha_1), ..., \exp(2\pi\sqrt{-1}\alpha_n)) \end{array}$$

associated to ω as in (4.5). So far, by section 4.1, we know that, except for a discrete set of 1-forms ω, the GAC $(A^\bullet, \omega)$ calculates the cohomology of $X_{\mathcal{R}}$ with coefficients in the local system ρ_ω. In fact, one can even prove a stronger result, basically following the one proved by Libgober in [**37**], Thm. 5.1.5.

THEOREM 4.5. *With the preceding notation, the following isomorphism*

$$H^*(A^\bullet, \omega) \cong H^*(X_{\mathcal{R}}, \rho_\omega)$$

takes place, provided that

$$(\exp(2\pi\sqrt{-1}\alpha_1), ..., \exp(2\pi\sqrt{-1}\alpha_n))$$

does not belong to the first characteristic variety of the complement, $\mathrm{Char}_1(X_{\mathcal{R}})$.

Proof. First of all, note that $H^*(A^\bullet, \lambda\omega) = H^*(A^\bullet, \omega)$ for any $\lambda \in \mathbb{C}^*$. Therefore it can be assumed that $\lambda\omega$ satisfies the non-resonance conditions for a certain λ and that, by hypothesis, it lies outside the first characteristic variety. Such a point satisfies $H^1(X_{\mathcal{R}}, \rho_{\lambda\omega}) = 0$. Hence, by Theorem 4.2, one has $H^1(A^\bullet, \omega) = 0$. Note that $H^0(A^\bullet, \omega) = 0$. Hence, all that is left to do is to prove the isomorphism in the case $* = 2$. This is a consequence of the following. The rank of each $C^*(X_{\mathcal{R}}, \rho_\omega)$ does not depend on ω –c.f. Definition 1.2. Then, one has $\chi(\rho_\omega) = \chi(X_{\mathcal{R}})$. Hence,

$$h^2(X_{\mathcal{R}}, \rho_\omega) = \chi(X_{\mathcal{R}}) = h^2(A^\bullet, \omega). \quad \square$$

DEFINITION 4.6. Let us define the following subset in $H^0(\overline{X}_{\mathcal{R}}, \Omega^1_{\overline{X}_{\mathcal{R}}}(\log\langle\overline{\mathcal{R}}\rangle))$,

$$\mathcal{V}_i(\mathcal{R}) := \{\omega \in H^0(\overline{X}_{\mathcal{R}}, \Omega^1_{\overline{X}_{\mathcal{R}}}(\log\langle\overline{\mathcal{R}}\rangle)) \mid h^1(A^\bullet, \omega) \geq i\}.$$

One has the following

THEOREM 4.7. *For $i \geq 1$ the sets $\mathcal{V}_i = \mathcal{V}_i(\mathcal{R})$ consist of a finite union of linear spaces of dimension $i+1$. In fact, there is a bijection between the irreducible components of $\mathcal{V}_i$ and of $\mathrm{Char}_i^{\overline{1}}$, the subspace of connected subgroups in Char_i. Moreover, for each irreducible component of $\mathcal{V}_i$, the exponential map*

$$(\exp(2\pi\sqrt{-1}\alpha_1), ..., \exp(2\pi\sqrt{-1}\alpha_n))$$

is the universal cover of the corresponding component $V_i \subset \mathrm{Char}_i^{\overline{1}}$.

Proof. Suppose that $\mathcal{V}_i \neq \emptyset$, then there exists a 1-form ω for which $h^1(A^\bullet, \omega) \geq i > 0$. Therefore, there is a form η linearly independent from ω and such that $\omega \wedge \eta = 0$. Take one irreducible component $\mathcal{V}$ in $\mathcal{V}_i$ containing the linear space generated by ω and η. Since the set of non-resonance conditions is discrete, we can take $\omega' \in \mathcal{V}$ such that none of its residues are positive integers. By Theorem 4.2, this implies that the GAC $(A^\bullet, \omega')$ can be used to calculate $\dim H^1(X, \rho_{\omega'})$. That is,

$$\dim H^1(X, \rho_{\omega'}) = \dim H^1(A^\bullet, \omega') \geq i.$$

This means that all such $\rho_{\omega'}$ belong to a characteristic variety V of depth i passing through the origin. Moreover, since the exponential map is a local homeomorphism, the dimension of V, say k, is at least the dimension of $\mathcal{V}$, say κ, i.e. $k \geq \kappa$. By [**1**], Theorem V.1.7, there is a map

$$f : X_{\mathcal{R}} \longrightarrow \mathbb{P}^1 \setminus \cup_{r=1}^{k+1}\{p_r\}$$

where $f^*H^1(\mathbb{P}^1 \setminus \cup_{r=1}^{k+1}\{p_r\}; \mathbb{C}^*)$ is V. Note that the map f^* is injective and hence, there is a subspace of $H^0(\overline{X}_{\mathcal{R}}, \Omega^1(\log\langle\overline{\mathcal{R}}\rangle))$ generated by the pull-back of the k linearly independent 1-forms of $H^0(\mathbb{P}^1, \Omega^1(\log\langle\cup_{r=1}^{k+1}\{p_r\}\rangle))$. Any of these forms η satisfy $\omega \wedge \eta = 0$. Hence, there is a subspace of $\mathcal{V}$ of dimension k. This implies $k \leq \kappa \leq k$, i.e. $k = \kappa$. Finally note that if $\omega \in \mathcal{V}_i \setminus \mathcal{V}_{i+1}$, then

$$i = \dim H^1(A^\bullet, \omega) = \dim H^1(X_{\mathcal{R}}, \rho_\omega) = k - 1.$$

Conversely, let ρ_ω belong to an irreducible component V of $\mathrm{Char}_i^{\overline{1}}$. By [**37**], ω belongs to the union of the affine subspaces $Q_k = \sum_{j \in I_P} \nu_{k,j}(P)x_j = n_k$, for certain $n_k \in \mathbb{Z}$, where $\nu_{k,j}(P) = \sum_{P_k^\ell} \nu_{P_k^\ell}(\mathcal{C}_j)$ for all infinitely near points to P on the exceptional divisor E_k and $j \in I_P$ if $P \in \mathcal{C}_j$. Since $h^1(A^\bullet, \omega) = h^1(A^\bullet, \lambda\omega)$ and $\rho_{\lambda\omega} \in V \subset \mathrm{Char}_i^{\overline{1}}$ for any $\lambda \in \mathbb{C}^*$, one has $n_k = 0$ and thus, ω belongs to a

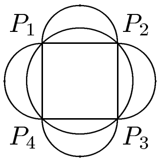


FIGURE 1. $\mathcal{Q}_1 \cup \mathcal{Q}_2 \cup \mathcal{Q}_3$

linear subspace of dimension i. Finally, we want to prove that $Q := \cap\{Q_k = 0\}$ coincides with the $\mathcal{V}_i$ we started with. Since Q has dimension i and ω belongs to it, one indeed has $Q = \mathcal{V}_i$. □

EXAMPLE 4.8. Let $\mathcal{R}^*$ be three conics in a general pencil of conics in $\mathbb{C}^2$.

Since the three conics lie in the pencil $\mathcal{P}$ of conics passing through $P_1, ..., P_4$, there is a surjective map

$$\mathbb{C}^2 \setminus \mathcal{R}^* \to \mathbb{P}^1 \setminus \{3 \text{ points}\}$$

whose fibers are conics in $\mathcal{P}$ with four points removed. By the already mentioned result in [**1**] Proposition 1.7, such a map gives rise to a component of dimension 2 in the first characteristic variety of $X_{\mathcal{R}} = \mathbb{C}^2 \setminus \mathcal{R}^*$. Following the notation of Chapter 3, note that one can write

$$A_c^1 = \sigma_1 \oplus \sigma_2 \oplus \sigma_3,$$

$$\tilde{A}^2 = \bigoplus_{i=1}^{4} \left(\sigma_{1,2}^i \oplus \sigma_{2,3}^i \oplus \sigma_{1,3}^i \oplus \sigma_{\infty}^i\right),$$

and

$$R_2 = \{\sigma_{2,3}^i = \sigma_{1,2}^i - \sigma_{1,3}^i\}_{i=1,\ldots,4}.$$

The matrix of the linear map $(\bullet \wedge \sum a_i \sigma_i)$ with respect to the basis

$$\{\sigma_{1,2}^1, \sigma_{1,3}^1, \sigma_{\infty}^1, \sigma_{1,2}^2, \sigma_{1,3}^2, \sigma_{\infty}^2, \sigma_{1,2}^3, \sigma_{1,3}^3, \sigma_{\infty}^3, \sigma_{1,2}^4, \sigma_{1,3}^4, \sigma_{\infty}^4\}$$

looks like

$$M_\omega := \left[\begin{array}{c|c|c|c} \quad M \quad & \quad M \quad & \quad M \quad & \quad M \quad \end{array}\right],$$

where

$$M := \begin{bmatrix} -a_2 - a_3 & -a_3 & 0 \\ a_1 & -a_3 & 0 \\ a_1 & a_1 + a_2 & 0 \end{bmatrix}.$$

Hence,

$$\operatorname{rank}(M_\omega) = \operatorname{rank}(M) = \begin{cases} 0 & \text{if } a_1 = a_2 = a_3 = 0 \\ 1 & \text{if } a_1 + a_2 + a_3 = 0 \text{ and } (a_1, a_2, a_3) \neq (0,0,0) \\ 2 & \text{otherwise} \end{cases}$$

and therefore $\mathcal{V}_1 = \{a_1 + a_2 + a_3 = 0\}$ and $\dim \mathcal{V}_1 = 2$. Moreover, there is only one component in $\operatorname{Char}_1(X_{\mathcal{R}})$ passing through the origin.

CHAPTER 5

Examples

The examples given in this chapter pretend to focus on different aspects of the relation between arrangements, their characteristic varieties and the subtori of components passing through the origin. A list of conjectures related to this invariant for line arrangements – or more generally for hyperplane arrangements – can be found in ([**38**], section 5). Some of them, in connection with rational arrangements, are exposed here. The behavior of characteristic varieties under degeneration and Cremona transformations is also described below. For some of the tedious calculations, especially Fox calculus, a computer package[1] that combines Macaulay and Maple has been used to calculate the radicals of the Fitting ideals.

REMARK 5.1. Our purpose is to show that asking our arrangement to have a transversal line at infinity is merely a technicality. More specifically, if $\mathcal{D}$ is an arrangement, a transversal line $\mathcal{L}$ can always be added to $\mathcal{D}$ and the characteristic varieties $\mathrm{Char}_i(\mathcal{D})$ can be reconstructed from $\mathrm{Char}_i(\mathcal{D} \cup \mathcal{L})$.

Note that there is an embedding of $\mathrm{Char}_i(\mathcal{D})$ into $\mathrm{Char}_i(\mathcal{D} \cup \mathcal{L})$ corresponding to the surjection of fundamental groups

$$\pi_1(\mathbb{P}^2 \setminus (\mathcal{D} \cup \mathcal{L})) \quad \twoheadrightarrow \quad \pi_1(\mathbb{P}^2 \setminus \mathcal{D})$$

whose kernel is exactly the subgroup $\mathbb{Z}$ generated by the meridian of the line. Hence $\mathrm{Char}_i(\mathcal{D})$ coincides with the subtorus $t_1^{d_1} \cdots t_n^{d_n} = 1$ inside $\mathrm{Char}_i(\mathcal{D} \cup \mathcal{L})$, where n is the number of components in $\mathcal{D}$ and d_j is the degree of the j-th component – c.f. [**37**] 1.2.3.

EXAMPLE 5.2. The following issue is mentioned in ([**38**], 5.1) for line arrangements. Is it true that every positive dimensional irreducible component of $\mathrm{Char}_1(X_{\mathcal{D}})$ contains the origin $\overline{1}$? Suciu [**50**] has recently proved this conjecture to be wrong. In the following we provide an analogous example for general rational arrangements.

Let us construct an arrangement of four conics and one line as follows. Let us consider the covering

$$\begin{array}{ccc} [X : Y : Z] & \in & \mathbf{P}^2 \\ \downarrow & & \downarrow \;\; p \\ [X^4 : Y^4 : Z^4] & \in & \mathbf{P}^2 \end{array}$$

[1] This program was written by Vladimir Egorin
University of Illinois at Chicago
851 S. Morgan Street
Chicago, IL, 60607
e-mail: vladimir@math.uic.edu

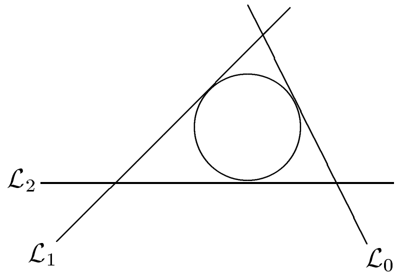


FIGURE 1. $\mathcal{Q} \cup \mathcal{L}_0 \cup \mathcal{L}_1 \cup \mathcal{L}_2$

branched along the lines $\mathcal{L}_0 := \{X = 0\}$, $\mathcal{L}_1 := \{Y = 0\}$ and $\mathcal{L}_2 := \{Z = 0\}$. Consider a conic $\mathcal{Q}$ tangent to the lines $\mathcal{L}_i$, $i = 0, 1, 2$ as in Figure 1.

The pull-back of $\mathcal{Q} \cup \mathcal{L}_0$ is the union of a line $\tilde{\mathcal{L}}_0$ and four conics $\tilde{\mathcal{Q}}_1$, $\tilde{\mathcal{Q}}_2$, $\tilde{\mathcal{Q}}_3$ and $\tilde{\mathcal{Q}}_4$, which intersect pairwise at two tacnodes with the following property. If one takes any two conics and considers the line joining the tacnodes, then such a line contains the tacnodes of the two remaining conics. The line $\tilde{\mathcal{L}}_0$ is one of these lines and passes through $\tilde{\mathcal{Q}}_1 \cap \tilde{\mathcal{Q}}_2$ and $\tilde{\mathcal{Q}}_3 \cap \tilde{\mathcal{Q}}_4$. A more detailed description of this arrangement, as well as the fundamental group, can be found in [**15**]. We will consider the rational arrangement

$$\mathcal{R} = \tilde{\mathcal{L}}_0 \cup \tilde{\mathcal{Q}}_1 \cup \tilde{\mathcal{Q}}_2 \cup \tilde{\mathcal{Q}}_3 \cup \tilde{\mathcal{Q}}_4.$$

Note that $H_1(X_\mathcal{R}) = \gamma_1\mathbb{Z} \oplus \cdots \oplus \gamma_4\mathbb{Z}$, where γ_i represents a meridian about $\tilde{\mathcal{Q}}_i$. Hence $\mathbb{T} = \operatorname{Spec}(\mathbb{C}[t_1^{\pm1}, ..., t_4^{\pm1}]) = (\mathbb{C}^*)^4$, where each t_i corresponds to the action of γ_i on the universal abelian cover $\tilde{X_\mathcal{R}}$. Using Fox Calculus the way it is described in 1.12.3, one can calculate the first characteristic variety

$$\operatorname{Char}_1(X_\mathcal{R}) \cup \overline{1} = \bigcup_{1 \le i < j \le 4} (V_{ij} \cup W_{ij}) \cup \{(t_1, t_2, 1, 1)\} \cup \{(1, 1, t_3, t_4)\},$$

where

$$V_{ij} := \{(t_1, t_2, t_3, t_4) \mid t_i = t_j = 1,\ t_k = -t_\ell^{-1},\ \text{ if } \mathcal{C}_i \cap \mathcal{C}_j \cap \mathcal{L} = \mathcal{C}_k \cap \mathcal{C}_\ell \cap \mathcal{L} = \emptyset\ \}$$

and

$$W_{ij} := \{(t_1, t_2, t_3, t_4) \mid t_i = t_j = t,\ t_k = t_\ell = -t^{-1},\ \ \{i, j, k, \ell\} = \{1, 2, 3, 4\}\ \}.$$

Note that none of the seven 1-dimensional tori contains the origin $\overline{1} = (1, 1, 1, 1)$.

EXAMPLE 5.3. The complexes $(A_c^\bullet, \omega)$ and Remark 5.1 allow us to detect the components passing through the origin from Example 5.2.

First of all, let us consider the arrangement

$$\mathcal{R}' = \mathcal{R} \cup \mathcal{L} = \mathcal{C}_0 \cup \mathcal{C}_1 \cup ... \cup \mathcal{C}_4 \cup \mathcal{C}_5$$

where $\mathcal{L} = \mathcal{C}_0$ is a transversal line, $\mathcal{C}_1, ..., \mathcal{C}_4$ are the conics $\tilde{\mathcal{Q}}_1, ..., \tilde{\mathcal{Q}}_4$ and $\mathcal{C}_5$ is the line $\tilde{\mathcal{L}}_0$ passing through the tacnodes of $\mathcal{C}_1 \cap \mathcal{C}_2$ and $\mathcal{C}_3 \cap \mathcal{C}_4$. Note that one can write

$$H_1(X_{\mathcal{R}'}) = (\gamma_0\mathbb{Z} \oplus \gamma_1\mathbb{Z} \oplus \cdots \oplus \gamma_5\mathbb{Z})/(2\gamma_1 + \cdots + 2\gamma_4 + \gamma_5 - \gamma_0)\mathbb{Z}$$

where each γ_i represents a meridian about $\mathcal{C}_i$. Hence

$$\mathbb{T}' = \operatorname{Spec}\left(\frac{\mathbb{C}\,[t_0, t_1^{\pm 1}, \ldots, t_5^{\pm 1}]}{(t_1 \ldots t_4)^2 t_5 = t_0}\right) = (\mathbb{C}^*)^5.$$

The subtorus associated with $X_{\mathcal{R}}$ corresponds to the following immersion

$$\mathbb{T} = (\mathbb{C}^*)^4 = \operatorname{Spec}\left(\frac{\mathbb{C}\,[t_0, t_1^{\pm 1}, \ldots, t_5^{\pm 1}]}{(t_1 \ldots t_4)^2 t_5 = t_0 = 1}\right) \subset \mathbb{T}'.$$

One can determine the components of $\operatorname{Char}_i(\mathcal{R})$ by restricting those of $\operatorname{Char}_i(\mathcal{R}')$ to the subtorus $\mathbb{T}$.

One can describe the complex (A_c, ω_c) as follows.

$$A_c^0 := \mathbb{C}$$

$$A_c^1 := \sigma_1\mathbb{C} \oplus ... \oplus \sigma_4\mathbb{C} \oplus \sigma_5\mathbb{C}$$

$$\tilde{A}^2 := \bigoplus_{1\leq i<j\leq 5} (\sigma_{ij}\mathbb{C} \oplus \sigma'_{ij}\mathbb{C}) \oplus \bigoplus_{i=1}^{4} \sigma_\infty^i.$$

The relations in A_c^2 are given by the tacnodes of $\mathcal{C}_1 \cap \mathcal{C}_2$ and $\mathcal{C}_3 \cap \mathcal{C}_4$ as follows

$$\sigma_{15} = \sigma_{12} + \sigma_{25}$$

$$\sigma'_{15} = \sigma'_{12} + \sigma'_{25}$$

$$\sigma_{35} = \sigma_{34} + \sigma_{45}$$

$$\sigma'_{35} = \sigma'_{34} + \sigma'_{45}.$$

Note also that

$$\sigma_i \wedge \sigma_j = \begin{cases} 2(\sigma_{ij} + \sigma'_{ij}) & \text{if } \mathcal{C}_i \neq \mathcal{C}_j \text{ are conics} \\ \sigma_{i5} + \sigma'_{i5} & \text{if } \mathcal{C}_i \text{ is a conic} \\ 0 & \text{if } i = j \quad . \end{cases}$$

Let us consider the following basis for A_c^2

$$\{\sigma_{12}, \sigma_{13}, \sigma_{14}, \sigma_{23}, \sigma_{24}, \sigma_{34}, \sigma_{25}, \sigma_{45}, \sigma'_{12}, ..., \sigma'_{45}, \sigma_\infty^1, ..., \sigma_\infty^4\}.$$

If

$$\omega = \sum_{i=1}^{5} a_i \sigma_i,$$

then the matrix M_ω for

$$A_c^1 \xrightarrow{\wedge\omega_c} A_c^2$$

can be written as

$$M_\omega = \left[\begin{array}{c|c|ccc} & & 0 & ... & 0 \\ M & M & & ... & \\ & & 0 & ... & 0 \end{array}\right],$$

where M is defined as

$$\begin{bmatrix} 2a_2 + a_5 & 2a_3 & 2a_4 & 0 & 0 & 0 & a_5 & 0 \\ -2a_1 & 0 & 0 & 2a_3 & 2a_4 & 0 & a_5 & 0 \\ 0 & -2a_1 & 0 & -2a_2 & 0 & 2a_4 + a_5 & 0 & a_5 \\ 0 & 0 & -2a_1 & 0 & -2a_2 & -2a_3 & 0 & a_5 \\ -a_1 & 0 & 0 & 0 & 0 & -a_3 & -a_1 - a_2 & -a_3 - a_4 \end{bmatrix}.$$

The rank of M_ω can be computed in terms of the coefficients a_i having

$$\operatorname{rank}(M_\omega) = \begin{cases} 0 & \text{if } a_1 = \dots = a_5 = 0 \\ 3 & \begin{cases} \text{if } a_1 = a_2 = a_5 + 2a_3 + 2a_4 = 0 \\ \text{if } a_3 = a_4 = a_5 + 2a_1 + 2a_2 = 0 \end{cases} \\ 4 & \text{otherwise} \quad . \end{cases}$$

Since $\dim\ker(\wedge\omega_c) = 4 - \operatorname{rank}(M_\omega)$ and

$$\dim\operatorname{im}(A_c^1 \xrightarrow{\wedge\omega_c} A_c^2) = \begin{cases} 0 & \text{if } \omega = 0 \\ 1 & \text{otherwise} \end{cases} \quad ,$$

one can see that

$$H^1(A_c^\bullet, \omega_c) = \begin{cases} 4 & \text{if } \omega = 0 \\ 1 & \begin{cases} \text{if } a_1 = a_2 = a_5 + 2a_3 + 2a_4 = 0 \\ \text{if } a_3 = a_4 = a_5 + 2a_1 + 2a_2 = 0 \end{cases} \\ 0 & \text{otherwise} \quad . \end{cases}$$

This means that

$$\mathcal{V}_1(\mathcal{R}') = \{(a_1, a_2, 0, 0, -2a_1 - 2a_2)\} \cup \{(0, 0, a_3, a_4, -2a_3 - 2a_4)\}$$

where $\mathcal{V}_1(\mathcal{R}')$ represents the union of linear spaces in Definition 4.6 for the arrangement $\mathcal{R}'$. As mentioned above, one can recover the components of $\operatorname{Char}_1^{\overline{1}}(\mathcal{R})$ from those of $\operatorname{Char}_1^{\overline{1}}(\mathcal{R}')$ as follows. Let us first exponentiate $\mathcal{V}_1(\mathcal{R}')$ as in Theorem 4.7. Then one keeps only those components that pass through the origin and intersects them with the subtorus $\mathbb{T}$ given by the equations $(t_1t_2t_3t_4)^2t_5 = t_0 = 1$. Hence, the components are

$$\{(t_1, t_2, 1, 1)\} \cup \{(1, 1, t_3, t_4)\}.$$

EXAMPLE 5.4. The following result shows that $\operatorname{Char}_1^{\overline{1}}$ is non-trivial only when the arrangement has singularities with more than two branches.

Let $\mathcal{D} = \mathcal{C}_0 \cup \mathcal{C}_1 \cup \dots \cup \mathcal{C}_n$ be a rational arrangement with a transversal line at infinity $\mathcal{C}_0$. Assume also that, at any singular point $P \in \operatorname{Sing}(\mathcal{D})$, the arrangement has at most two local branches. Then the origin $\overline{1}$ is an isolated point of $\operatorname{Char}_1(\mathcal{D})$.

Proof. Note that the underlying torus in this case is

$$\mathbb{T} = \operatorname{Spec}(\mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]) = (\mathbb{C}^*)^n$$

where each t_i represents the action of a meridian of the component $\mathcal{C}_i$ on $H_1(\tilde{X}_{\mathcal{D}})$.

By Section 3.2, the complex $(A_c^\bullet, \omega_c)$ has the following structure

$$A_c^0 := \mathbb{C}$$

$$A_c^1 := \langle \sigma_1, \dots, \sigma_n \rangle$$

$$A_c^2 := \bigoplus_{P \in \mathcal{G}(\mathcal{D}) \cap \operatorname{Doub}(\mathcal{D})} \sigma^P \mathbb{C} \oplus \bigoplus_{P \in \mathcal{G}(\mathcal{D}) \setminus \operatorname{Doub}(\mathcal{D})} \sigma^P \mathbb{C}$$

where $\operatorname{Doub}(\mathcal{D})$ is the set of singular points of $\mathcal{D}$ at which $\mathcal{D}$ has exactly two local branches and $\mathcal{G}(\mathcal{D})$ is defined as in (3.8). Let us denote by M_ω the matrix corresponding to the linear morphism $\bullet \wedge \omega_c$, where

$$\omega := \sum_{i=1}^{n} a_i \sigma_i.$$

The matrix M_ω has dimension $n \times m$, where $m = \dim A_c^2$. We claim that, for any $i = 1, ..., n$, there exists a minor of order $n-1$ in M_ω whose determinant is λa_i^{n-1}. If the claim is true, then $\dim \ker(\bullet \wedge \omega_c) > 1$ if and only if $\omega = 0$. Hence,

$$\mathcal{V}_1(\mathcal{D}) := \{\omega \in \mathbb{C}^n \mid H^1(A^\bullet, \omega) > 0\} = \{0\}$$

and the point $(e^0, ..., e^0) = (1, ..., 1)$ is isolated in Σ^1 by Theorem 4.7. Therefore, the claim would follow after applying Proposition 1.13.

All that is left to do is to prove the claim. Note that for each $j = 1, ..., n$ we can consider the $n-1$ points $P_{1,j}, ..., P_{j-1,j}, P_{j+1,j}, ..., P_{n,j}$ where $P_{i,j} \in \mathcal{C}_i \cap \mathcal{C}_j$. Since the arrangement $\mathcal{D}$ has exactly two local branches at $P_{i,j}$, then all of these points are different. To simplify, let us denote by $\sigma_{i,j} := \sigma^{P_{i,j}}$ the corresponding element in the basis of A_c^2 shown above. By the definition of $\bullet \wedge \omega_c$ – c.f. (3.10) – the projection of $\sigma_i \wedge \omega$ on $\sigma_{i,j}$ is $\lambda_{ij} a_j$, where

$$\lambda_{ij} = \pm(\mathcal{C}_i \cdot \mathcal{C}_j)_{P_{i,j}}.$$

Let us fix j and consider the vectors

$$v_j := \{\sigma_{1,j}, ..., \sigma_{j-1,j}, \sigma_{j+1,j}, ..., \sigma_{n,j}\}.$$

The minor of A_ω corresponding to the coefficients of $\sigma_i \wedge \omega_c$ $(i \neq j)$ on v_j is

$$A_j := \begin{bmatrix} \lambda_{1j} a_j & 0 & ... & 0 \\ 0 & \lambda_{2j} a_j & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & \lambda_{nj} a_j \end{bmatrix}$$

whose determinant is $\lambda_{1j} ... \lambda_{nj} a_j^{n-1}$. □

REMARK 5.5. Note that, if the result is true for arrangements with a transversal line, then it is also true for any kind of arrangement by Remark 5.1.

EXAMPLE 5.6. In this example the behavior of characteristic varieties under degeneration of curves will be studied. Let us start by stating the following classical result about degeneration and fundamental groups – c.f. [**54**] p. 616. Let $\mathcal{D}_t$, $t \in [1, 0)$ be a smooth equisingular family of arrangements degenerating into another arrangement $\mathcal{D} = \mathcal{D}_0$. Denote the complements $X_{\mathcal{D}_t}$ simply by X_t. Then, one has a surjection of fundamental groups

$$\pi_1(X_0) \twoheadrightarrow \pi_1(X_1)$$

– for a simple proof of this well-known result see for example [**19**] Corollary 4.3.2. Hence, there is a surjection of the Alexander invariants

$$H_1(\tilde{X}_0; \mathbb{C}) \twoheadrightarrow H_1(\tilde{X}_t; \mathbb{C}) \tag{5.1}$$

where $\tilde{X}_t$ is the universal abelian cover of X_t. This surjection is in fact a morphism of $\mathbb{Z}[H_1(X_0)]$-modules. Therefore, one has

$$\mathrm{Char}_i(X_t) \hookrightarrow \mathrm{Char}_i(X_0) \subset \mathrm{Spec}(X_0).$$

We will use this property to study possible degenerations of arrangements.

Let $\mathcal{C}_6$ denote the sextic of Zariski given by the equation

$$f_2^3 + f_3^2 = 0,$$

where f_i is a homogeneous polynomial in $\mathbb{C}[X, Y, Z]$ of degree i. This curve has six cusps as sole singularities and they lie on $\{f_2 = 0\} \cap \{f_3 = 0\}$. Zariski [**52**] proved the fundamental group of $X_{\mathcal{C}_6}$ to be isomorphic to $\mathbb{Z}_2 * \mathbb{Z}_3$ and Libgober [**35**] calculated its Alexander polynomial $(t^2 - t + 1)$. Note that the set of zeroes of the Alexander polynomial (in the irreducible case) coincides with the first characteristic variety. Hence

$$\operatorname{Char}_1(\mathcal{C}_6) = \{\omega, \overline{\omega}\} \subset \mathbb{C}^* = \operatorname{Spec}(\mathbb{C}[t^{\pm 1}]) = \mathbb{T}_{\mathcal{C}_6}$$

where ω is a primitive 6-th root of unity. On the other hand, there is another sextic, $\mathcal{C}'_6 := \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3$ obtained from the arrangement $\mathcal{R}$ in Example 5.2 by removing the line and one of the conics. This process of deletion corresponds to the following inclusion of tori

$$\mathbb{T}_{\mathcal{C}'_6} = \operatorname{Spec}\left(\frac{\mathbb{C}[t_1^{\pm 1}, \ldots, t_4^{\pm 1}]}{(t_4 = 1, t_1 t_2 t_3 = 1)}\right) \subset \operatorname{Spec}(\mathbb{C}[t_1^{\pm 1}, \ldots, t_4^{\pm 1}]) = \mathbb{T},$$

–using the notation from Example 5.2.

Hence

$$\operatorname{Char}_1(\mathcal{C}'_6) = \{(1, -1, -1, 1), (-1, 1, -1, 1), (-1, -1, 1, 1)\} \cup$$
$$\cup \{(1, t, -t^{-1}, 1)\} \cup \{(t, 1, -t^{-1}, 1)\} \cup \{(t, -t^{-1}, 1, 1)\}.$$

One can ask oneself whether $\mathcal{C}_6$ admits a degeneration to $\mathcal{C}'_6$. In that case, the embedding of $\operatorname{Char}_1(\mathcal{C}_6)$ into $\operatorname{Spec}(H_1(X_{\mathcal{C}'_6}))$ would send any point $\rho \in \mathbb{C}^*$ to $(\rho, \rho, \rho, 1)$. Hence, if $\mathbb{T}_{\mathcal{C}_6} \overset{i}{\hookrightarrow} \mathbb{T}_{\mathcal{C}'_6}$ denotes such an embedding, then

$$i(\operatorname{Char}_1(\mathcal{C}_6)) = \{(\omega, \omega, \omega, 1), (\overline{\omega}, \overline{\omega}, \overline{\omega}, 1)\}$$

which is clearly not contained in $\operatorname{Char}_1(\mathcal{C}'_6) \subset \mathbb{T}_{\mathcal{C}'_6}$. Therefore there is no such degeneration. Incidentally, note that the points of $i(\operatorname{Char}_1(\mathcal{C}_6))$ are contained in the translated torus

$$\{(t_1, t_2, t_3, t_4) \in \mathbb{T} \mid t_4 = 1, t_1 t_2 t_3 = -1\}.$$

Remarks 5.7.

1. The construction of Char_i in the previous example might seem artificial in the sense that $H_1(X_{\mathcal{C}_6})$ has torsion, but note that in Definition 1.11 of characteristic varieties we never asked the first homology of the complement to be free abelian.
2. The fact that the six ordinary cusps of $\mathcal{C}_6$ lie on a conic is crucial for this example. It was also proved by Zariski in [**55**] that if the six cusps do not lie on a conic, then the fundamental group of such a sextic is abelian of order 6. Hence, it has an empty characteristic variety. In fact, there exists a family of such sextics that degenerates into $\mathcal{C}'_6$.

Example 5.8. Another interesting example is the study of Cremona transformations and their influence on characteristic varieties.

Any homeomorphism of the complement to an arrangement will produce an isomorphism of its cohomology, resulting in a change of coordinates in $\operatorname{Spec}(H_1(X_{\mathcal{D}};\mathbb{Z}))$. In this example, the change of coordinates in $\operatorname{Spec}(H_1(X_{\mathcal{D}};\mathbb{Z}))$ will be calculated for Cremona transformations. Cremona transformations are finite sequences of simple Cremona transformations. A simple Cremona transformation is a sequence of three blow-ups and three blow-downs as follows. Consider three lines $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$ in $\mathbb{P}^2$. For simplicity, they will be assumed to be in general position. First, one blows up the three points of intersection. Since each line $\mathcal{L}_i$ has been blown up at two points, the strict transforms have a self intersection of -1, so they can be blown down. The new surface obtained is a $\mathbb{P}^2$, and the exceptional divisors of the blow-up E_1, E_2 and E_3 become lines. One has a homeomorphism

$$\mathbb{P}^2 \setminus \cup\mathcal{L}_i \longrightarrow \mathbb{P}^2 \setminus \cup E_i.$$

Analogously, let $\mathcal{D}$ be an arrangement containing $\mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3$. Let $\mathcal{D}'$ denote the image of $\mathcal{D}$ via the simple Cremona transformation based on these lines. Then one has a homeomorphism

$$\mathbb{P}^2 \setminus \mathcal{D} \longrightarrow \mathbb{P}^2 \setminus \mathcal{D}'.$$

The surface obtained after blowing-up the three points in $\mathbb{P}^2$ is denoted by S_1. The pre-image of the curve

$$\mathcal{D} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{C}_4 \cup \ldots \cup \mathcal{C}_n,$$

where $\mathcal{C}_i = \mathcal{L}_i$ $(i = 1, 2, 3)$ is denoted by $\overline{\mathcal{D}} = \tilde{\mathcal{D}} \cup E_1 \cup E_2 \cup E_3$. In order to describe the map

$$H_1(S_1 \setminus \overline{\mathcal{D}}) \longrightarrow H_1(\mathbb{P}^2 \setminus \mathcal{D})$$

a presentation for the homology $H_1(S_1 \setminus \overline{\mathcal{D}})$ is required.

Let $\mathcal{T}\overline{\mathcal{D}}$ be a regular neighborhood of $\overline{\mathcal{D}}$. Using Lefschetz duality, excision and retraction, one has

$$H_1(S_1 \setminus \overline{\mathcal{D}}) \overset{L.}{\cong} H^3(S_1 \setminus \overline{\mathcal{D}}, \partial\mathcal{T}\overline{\mathcal{D}}) \overset{exc.}{\cong} H^3(S_1, \mathcal{T}\overline{\mathcal{D}}) \overset{ret.}{\cong} H^3(S_1, \overline{\mathcal{D}}).$$

To calculate $H^3(X_1, \overline{\mathcal{D}})$ one can use the exact sequence of pairs

$$H^2(S_1, \overline{\mathcal{D}}) \longrightarrow H^2(S_1) \xrightarrow{j^*} H^2(\overline{\mathcal{D}}) \longrightarrow H^3(S_1, \overline{\mathcal{D}}) \longrightarrow H^3(S_1) \cong H_1(S_1) = 0.$$

Therefore,

$$H^3(S_1, \overline{\mathcal{D}}) \cong \frac{H^2(\overline{\mathcal{D}})}{j^*(H^2(S_1))}.$$

Note that $H^2(S_1)$ is generated by a general line $\mathcal{L}$ and the exceptional divisors E_i. The morphism j^* is given by restriction. The map

$$H^2(\overline{\mathcal{D}}) \xrightarrow{\varphi} H_1(S_1 \setminus \overline{\mathcal{D}})$$

is given by the multiplicities of intersection as follows

$$\begin{array}{ccc} H^2(\overline{\mathcal{D}}) & \xrightarrow{\varphi} & H_1(S_1 \setminus \overline{\mathcal{D}}) \\ \mathcal{C}_i & \mapsto & \sum_{j=1}^n (\mathcal{C}_i \cdot \mathcal{C}_j) c_j + \sum_{j=1}^3 (\mathcal{C}_i \cdot E_j) e_j, \end{array}$$

where c_i are meridians around $\mathcal{C}_i$, and e_j meridians around E_j. Hence,

$$H_1(S_1 \setminus \overline{\mathcal{D}}) = \{c_i, e_j; \varphi(j^*(\mathcal{L})) = \varphi(E_j) = 0\} =$$

$$= \{c_i, e_j; \sum d_i c_i = \sum (\mathcal{C}_i, E_j) c_i = 0\}.$$

Finally, in order to obtain a presentation for $H_1(\mathbb{P}^2 \setminus \mathcal{D}')$, one only needs to write $H_1(S_1 \setminus \overline{\mathcal{D}})$ in terms of $e_1, e_2, e_3, c_4, ..., c_n$.

EXAMPLE 5.9. The tori $\{t_i = 1\}$ in $\mathrm{Spec}(\mathbb{C}[t_1^{\pm 1}, ..., t_n^{\pm 1}])$ are usually called **coordinate tori**. A component of $\mathrm{Char}_i(\mathcal{D})$ is called **essential** if it is not a coordinate torus, that is, if it does not result from removing a component in $\mathcal{D}$ –c.f. [**37**] Lemma 1.4.3. Essential tori have been studied for affine arrangements –c.f. [**21**], [**50**] [**38**] and [**37**]. Next, it will proved that, for general arrangements, essential tori depend on the compactification of the complement.

Let us consider the Ceva arrangement in $\mathbb{C}^2$, that is, the arrangement consisting of three lines in general position and the medians of the inner triangle as shown in Figure 2.

Let us perform a simple Cremona transformation of $\mathcal{D} = \mathcal{L}_\infty \cup \mathcal{L}_1 \cup ... \cup \mathcal{L}_6$ based on the lines $\mathcal{L}_1, \mathcal{L}_2$ and $\mathcal{L}_3$. After blowing-up, one obtains a surface S and a divisor

$$\overline{\mathcal{D}} = \tilde{\mathcal{D}} \cup E_1 \cup E_2 \cup E_3,$$

where E_i is the exceptional divisor obtained after blowing-up the point P_i, $i = 1, 2, 3$.

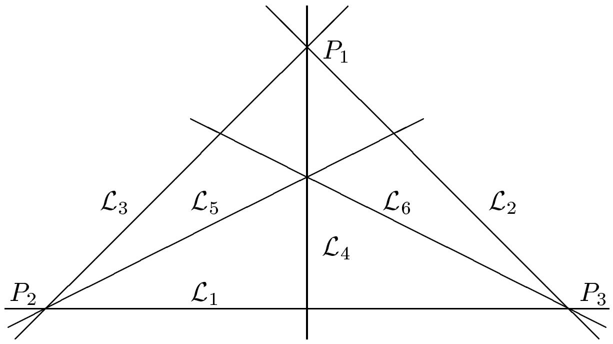


FIGURE 2. Ceva arrangement.

Denote by ℓ_i $i = 1, \ldots, 6, \infty$ the meridians of $\tilde{\mathcal{L}}_i$, and by e_j $j = 1, 2, 3$ those around E_j. Then, according to the previous example, one has the following presentation for $H_1(S \setminus \overline{\mathcal{D}})$

$$\left\langle\ \ell_1, \ldots, \ell_6, \ell_\infty, e_1, e_2, e_3 \ ; \begin{array}{l} \ell_2 + \ell_3 + \ell_4 = e_1 \\ \ell_1 + \ell_2 + \ell_6 = e_3 \\ \ell_1 + \ell_3 + \ell_5 = e_2 \\ \ell_1 + \ell_2 + ... + \ell_6 = \ell_\infty \end{array} \right\rangle. \tag{5.2}$$

Therefore, one can take a free system of generators

$$\{\ell_1, ..., \ell_6\}$$

for the free abelian group $H_1(X_{\mathcal{D}}) = \mathbb{Z}^6$ and denote by t_i the corresponding generators in

$$\mathbb{C}[H_1(X_{\mathcal{D}})] = \mathbb{C}[t_1^{\pm 1}, ..., t_6^{\pm 1}] = \mathbb{C}[\mathbb{Z}^6].$$

Analogously,

$$\{e_1, e_2, e_3, \ell_4, \ell_5, \ell_\infty\}$$

is a free system of generators of $H_1(X_{\mathcal{D}'}) = \mathbb{Z}^6$. Let us denote by

$$t_1', t_2', t_3', t_4, t_5, t_\infty$$

the corresponding generators in

$$\mathbb{C}[H_1(X_{\mathcal{D}})] = \mathbb{C}[t_1', t_2', t_3', t_4, t_5, t_\infty] = \mathbb{C}[\mathbb{Z}^6].$$

From (5.2), the isomorphism of rings is given by

$$\begin{array}{ccc}
\mathbb{C}[\mathbb{Z}^6] & \stackrel{\psi}{\longrightarrow} & \mathbb{C}[\mathbb{Z}^6] \\
t_1 & \mapsto & t_\infty t_5^{-1} t_6^{-1} t_1'^{-1} \\
t_2 & \mapsto & t_\infty t_4^{-1} t_6^{-1} t_2'^{-1} \\
t_3 & \mapsto & t_\infty t_4^{-1} t_5^{-1} t_3'^{-1} \\
t_4 & \mapsto & t_4 \\
t_5 & \mapsto & t_5 \\
t_6 & \mapsto & t_6 = t_\infty^2 (t_1' t_2' t_3' t_4 t_5)^{-1}.
\end{array}$$

The first characteristic varieties of $\mathcal{D}$ are calculated in [**21**]. There is only one essential torus $\mathbf{T}$. To describe it, one can consider the following equations

$$t_1 t_3 t_5 = 1 \ , \ t_1 t_2 t_6 = 1 \ , \ t_2 t_3 t_4 = 1 \ , \ t_4 t_5 t_6 = 1.$$

This variety has two connected components: one of them, denoted by $\mathbf{T}$, is a connected torus; the other one is a translation of $\mathbf{T}$ by $(-1, -1, -1, 1, 1, 1)$. Note that $\psi(\mathbf{T}) = \mathbf{T}'$ has equation

$$\{t_2' = 1 \ , \ t_3' = 1 \ , \ t_1' = 1 \ , \ t_\infty = 1\}.$$

Hence, it is non-essential for $\mathcal{D}'$.

EXAMPLE 5.10. Two plane curves $\mathcal{C}$ and $\mathcal{D}$ form a Zariski pair –c.f. [**3**]– if they satisfy the following conditions:

1. both have the same number of components, and up to reordering
$$\mathcal{C} = \mathcal{C}_1 \cup ... \cup \mathcal{C}_n$$
$$\mathcal{D} = \mathcal{D}_1 \cup ... \cup \mathcal{D}_n$$
with $\deg(\mathcal{C}_i) = \deg(\mathcal{D}_i)$,
2. there exist bijections
$$\phi : \mathrm{Sing}(\mathcal{C}) \longrightarrow \mathrm{Sing}(\mathcal{D})$$
and
$$\phi_P^i : \Delta_P(\mathcal{C}_i) \longrightarrow \Delta_Q(\mathcal{D}_i)$$
for any $P \in \mathrm{Sing}(\mathcal{C})$, $Q = \phi(P)$, such that the local type of the singularity P at $\mathcal{C}_i$ coincides with the local type of Q at $\mathcal{D}_i$ for any $i = 1, .., n$. Remember that $\Delta_P(\mathcal{C}_i)$ is the set of local branches of $\mathcal{C}_i$ at P,
3. any two different local branches at P, say δ_1 and δ_2, of $\mathcal{C}_i$ and $\mathcal{C}_j$ respectively, have the same intersection number as $\phi_P^i(\delta_1)$ and $\phi_P^j(\delta_2)$ at $Q = \phi(P)$, and
4. the pairs $(\mathbb{P}^2, \mathcal{C})$ and $(\mathbb{P}^2, \mathcal{D})$ are not homeomorphic.

Since Zariski [**52**],[**53**],[**55**], several examples of Zariski pairs have been found [**43**],[**3**],[**4**],[**6**], [**49**],[**51**]. Different strategies have been used to prove Condition 4. One can show directly that the fundamental groups are not isomorphic [**52**],[**43**],[**3**], [**4**],[**49**], that the Alexander polynomials are different [**3**],[**6**], or the existence of certain non-abelian coverings for one and not for the other [**51**],[**6**]. The following is an attempt to show the use of characteristic varieties to distinguish Zariski pairs.

This example is due to E.Artal, J.Carmona, H.-O Tokunaga and the author in [**5**]. Consider a rational quartic $\mathcal{C}_1$ with singularities $\mathbb{A}_1$, $\mathbb{A}_3$ and a smooth conic $\mathcal{C}_2$ intersecting the quartic in only one point P, which is smooth in $\mathcal{C}_1$. Then, $\mathcal{C}_1 \cup \mathcal{C}_2$ has a singularity at P of local type $\mathbb{A}_{15}$. Such a configuration can be realized. One obtains in principle two members $\mathcal{D}_1$ and $\mathcal{D}_2$ satisfying the following additional condition. The tangent line at P in $\mathcal{D}_2$ (resp. $\mathcal{D}_1$) does (resp. does not) contain the tacnode. One can calculate a presentation for the fundamental group of $\mathcal{D}_i \cup \mathcal{L}$, where $\mathcal{L}$ is a transversal line. One has

$$\pi_1(X_{\mathcal{D}_2}) = \langle a, b\, ; [a, b^{-1}a(ba)^3] = [b, b^{-1}a(ba)^3] = 1 \rangle$$

and

$$\pi_1(X_{\mathcal{D}_1}) = \langle a, b\, ;\ [a, b^2] = [a, a^2(ab)^2] = [b, a^2(ab)^2] = 1 \rangle.$$

Comparing these two groups does not seem to be an easy task. One-variable Alexander polynomials are in both cases trivial. The first characteristic varieties can be computed as

$$\mathrm{Char}_1(\mathcal{D}_1) = \{(1,1)\},$$

whereas

$$\mathrm{Char}_1(\mathcal{D}_2) = \{(1,1), (1,-1), (-1,i), (-1,-i)\}$$

both inside the subtorus $\mathbb{T} = \mathrm{Spec}(\mathbb{C}\,[t_1^{\pm 1}, t_2^{\pm 1}])$, where t_i corresponds to the action of a meridian about $\mathcal{C}_i$ on the first cohomology of the universal abelian covers of $X_{\mathcal{D}_1}$ and $X_{\mathcal{D}_2}$. Hence, $\mathcal{D}_1$ and $\mathcal{D}_2$ form a Zariski pair.

Example 5.11. According to Example 5.2, not all components of positive dimension of $\mathrm{Char}_1(\mathcal{R})$ for rational arrangements contain the origin. The question whether they still depend on the combinatorics of the arrangement still remains open. In this example, two non-rational arrangements will be shown to have the same combinatorics, non-homeomorphic complements and different characteristic varieties of positive dimension not passing through the origin.

Consider the following two families $\mathcal{M}_1$ (resp. $\mathcal{M}_2$) of curves

$$\mathcal{D} = \mathcal{C} \cup \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3 \cup \mathcal{L}_4$$

of degree 7, where $\mathcal{C}$ is a smooth cubic, and $\mathcal{L}_i$ are lines such that:

1. the arrangement $\cup_{i=1}^4 \mathcal{L}_i$ has only double points,
2. each $\mathcal{L}_i$ is tangent to $\mathcal{C}$ at an inflexion point P_i $(i = 1, ..., 4)$,
3. there are (resp. are not) three aligned points in $\{P_1, ..., P_4\}$.

This example is shown in [**4**], Example 4. Both families are non-vacuous and connected. The authors also calculate presentations for the fundamental group of curves $\mathcal{D}_1 \in \mathcal{M}_1$ and $\mathcal{D}_2 \in \mathcal{M}_2$ and prove they are not isomorphic. From each presentation one can compute the characteristic varieties. Note that

$$\mathrm{Spec}(\mathbb{C}\,[H_1(X_{\mathcal{D}_1})]) \cong \mathrm{Spec}(\mathbb{C}\,[H_1(X_{\mathcal{D}_2})]) \cong (\mathbb{C}^*)^4.$$

It turns out that

$$\begin{gathered}\mathrm{Char}_1(\mathcal{D}_1) \cup \overline{1} = \\ = \overline{1} \cup \{(1, t, -t^{-1}, 1)\} \cup \{(1, t, t, 1) \mid p(t) = 0\} \cup \\ \cup \{(t_1, t_2, -t_2^{-1}, 1) \mid p(t_1 t_2) = 0\} \cup \{(t, -1, -1, 1) \mid p(t) = 0\}\end{gathered}$$

where $p(t) = t^2 + t + 1$ and

$$\mathrm{Char}_i(\mathcal{D}_1) \cup \overline{1} = \overline{1} \quad \text{for } i > 1.$$

Meanwhile,

$$\mathrm{Char}_1(\mathcal{D}_2) \cup \overline{1} = \overline{1} \cup \{(t, 1, 1, 1) \mid p(t) = 0\}$$

and

$$\mathrm{Char}_i(\mathcal{D}_2) \cup \overline{1} = \overline{1} \quad \text{for } i > 1.$$

Bibliography

[1] D. Arapura, ‘Geometry of cohomology support loci for local systems I’, *J. Algebraic Geom.* **6** (1997), 563–597.

[2] V. I. Arnold ‘The cohomology ring of the colored braid group’, *Math. Notes Acad. Sci. USSR* **5** (1969), 138–140.

[3] E. Artal-Bartolo, ‘Les couples de Zariski’, *J. Algebraic Geom.* **3** (1994), 223–247.

[4] E. Artal-Bartolo, J. Carmona-Ruber, ‘Zariski pairs, fundamental groups and Alexander polynomials’, *J. Math. Soc. Japan* **50** (1998), No. 3, 521–543.

[5] E. Artal-Bartolo, J. Carmona-Ruber, J. Cogolludo-Agustín, H.-O Tokunaga, ‘Sextics with singular points in special position’, *J. Knot Theory Ramifications* `to appear`, available at `http://xxx.lanl.gov/abs/math.AG/0007152`.

[6] E. Artal-Bartolo, H.-O Tokunaga, ‘Zariski pairs of index 19 and Mordell-Weil groups of $K3$ surfaces’, *Proc. London Math. Soc. (3)* **80** (2000), No. 1, 127–144.

[7] C. Berge, ‘Graphes et hypergraphes’, *Monographies universitaires de Mathématiques*, DUNOD, Paris, 1970.

[8] J. S. Birman, ‘Braids, links and mapping class groups’, *Annals of Math. Studies* **82**, Princeton University Press, 1975.

[9] E. Brieskorn, ‘Sur les groups des tresses’, *Lecture Notes in Math.* vol. 317, Berlin, Heidelberg, New York, 1973.

[10] E. Brieskorn, H. Knörrer, ‘Plane algebraic curves’, *Birkhäuser Verlag*, Basel, Boston, Stuttgart, 1986.

[11] D. A. Buchsbaum, D. Eisenbud, ‘What annihilates a module?’, *J. Algebra* **47** (1977), 231–243.

[12] H. Cartan, S. Eilenberg, ‘Homological algebra’, Princeton University Press, 1956.

[13] D. Cohen, P. Orlik, ‘Arrangements and local systems’, *Math. Res. Lett.* **7** (2000), 299–316.

[14] J. I. Cogolludo-Agustín, ‘Invariants of the complement to rational arrangements’, *C. R. Acad Sci. Paris Sér. I Math.* **330** (2000), 697–701.

[15] J. I. Cogolludo-Agustín, ‘Fundamental groups for some cuspidal curves’, *Bull. London Math. Soc.* **31** (1999) No. 2, 136–142.

[16] D. Cohen, A. Suciu, ‘Characteristic varieties of arrangements’, *Math. Proc. Cambridge Philos. Soc.* **127** (1999), No. 1, 33–53.

[17] P. Deligne, ‘Equations différentielles à points singuliers réguliers’, *Lecture Notes in Math.* 163, Berlin, Heidelberg, New York, 1970.

[18] P. Deligne, ‘Théorie de Hodge II’, *Inst. Hautes Études Sci. Publ. Math.* **40** (1971), 5–58.

[19] A. Dimca, ‘Topology of hypersurface singularities’, Springer-Verlag, New York, 1992.

[20] H. Esnault, V. Schechtman, E.Viehweg, ‘Cohomology of local systems of the complement of hyperplanes’, *Invent. Math.* **109** (1992), No. 3, 557–561; Erratum, ibid. **112** (1993), No. 2, 447.

[21] M. Falk, ‘Arrangements and cohomology’, *Ann. Comb.* **1** (1997), 135–157.

[22] R. H. Fox, ‘Free differential calculus I’, *Ann. of Math.* **57** (1975), 547–560.

[23] M. Green, R. Lazarsfeld, ‘Higher obstructions to deforming cohomology groups of line bundles’, *J. Amer. Math. Soc.* **4** (1991), 87–103.

[24] P. Griffiths, J. Harris, ‘Principles of algebraic geometry’, John Wiley & sons, New York Chichester Brisbane Toronto, 1978.

[25] P. Griffiths, W. Schmid, ‘Recent developments in Hodge theory: a discussion of techniques and results’, *Discrete subgroups of Lie groups and applicatons to moduli* (International Colloquium, Bombay, 1973), pp. 31–127. Oxford University Press.

[26] A. Grothendieck, ‘On the de Rham cohomology of algebraic varieties’, *Inst. Hautes Études Sci. Publ. Math.* **29** (1966), 351–358.

[27] J. A. Hillman, 'Alexander ideals of links', *Lecture Notes in Math.* vol. 895, Springer-Verlag, Berlin Heidelberg New York, (1981).

[28] E. Hironaka, 'Abelian coverings of the complex projective plane branched along configurations of real lines', *Mem. Amer. Math. Soc.*, Providence, RI, 1993.

[29] E. Hironaka, 'Alexander stratifications of character varieties', *Ann. Inst. Fourier (Grenoble)* **47** (1997), No. 2, 555–583.

[30] S. Iitaka, 'Logarithmic forms of algebraic varieties', *J. Fac. Sci. University of Tokyo. Section IA. Mathematics* **23** (1976), No. 3, 525–544.

[31] E. R. van Kampen, 'On the fundamental group of an algebraic curve', *Amer. J. Math.* **55**, (1933), 255–260.

[32] S. Kobayashi, K. Nomizu, 'Found. Diff. Geom.', vol. I,II, Interscience Publ., New York, 1969.

[33] A. Libgober, 'On the homotopy type of the complement to plane algebraic curves', *J. Reine Angew. Math.***367** (1986), 103–114.

[34] A. Libgober, 'On the homology of finite abelian coverings', *Topology Appl.* **43** (1992), 157–166.

[35] A. Libgober, 'Homotopy groups of the complements to singular hypersurfaces II', *Ann. of Math. (2)* **139** (1994), No. 1, 117–144.

[36] A. Libgober, 'First order deformations for rank one local systems with non vanishing cohomology', *Topology Appl.* (to appear).

[37] A. Libgober, 'Characteristic varieties of algebraic curves', *preprint*, available at `http://xxx.lanl.gov/abs/math.AG/9801070`.

[38] A. Libgober, S. Yuzvinsky, 'Cohomology of local systems', *Arrangements-Tokyo 1998* (M.Falk, H.Terao, eds.) Adv. Stud. Pure Math., vol. 27, Kinokuniya, Tokyo, 2000, pp. 169–184.

[39] A. Libgober, S. Yuzvinsky, 'Cohomology of the Orlik-Solomon algebras and local systems', *Compositio Math.* **121** (2000), No. 3, 337–361.

[40] D. Matei, A. Suciu, 'Homotopy types of complements of 2-arrangements in $\mathbb{R}^4$', *Topology* **39** (2000), No. 1, 61–88.

[41] D. Matei, A. Suciu, 'Cohomology rings and nilpotent quotients of real and complex arrangements', *Arrangements-Tokyo 1998* (M.Falk, H.Terao, eds.), Adv. Stud. Pure Math., vol. 27, Kinokuniya, Tokyo, 2000, pp. 185–215.

[42] B. R. McDonald, 'Linear Algebra over commutative rings', *Pure Appl. Math.* Marcel Dekker, Inc., New York and Basel, 1984.

[43] M. Oka, 'Two transforms of plane curves and their fundamental groups', *J. Math. Sci. University Tokyo* **3** (1995), No. 2, 399–443.

[44] P. Orlik, L. Solomon, 'Combinatorics and topology of complements of hyperplanes', *Invent. Math.* **56** (1980), 167–189.

[45] P. Orlik, H. Terao, 'Arrangements of hyperplanes', *Grundlehren Math. Wiss.*, vol. 300, Springer-Verlag, New York Berlin Heidelberg, 1992.

[46] S. Roman, 'The Umbral Calculus', Academic Press, New York, 1984.

[47] K. Saito, 'Theory of logarithmic differential forms and logarithmic vector fields', *J. Math. Sci. University Tokyo* **27** (1981), 265–291.

[48] J. P. Serre, 'Algèbre locale. Multiplicités', *Lecture Notes in Math.* vol. 11, Springer Verlag, 1975.

[49] I. Shimada, 'A note on Zariski pairs', *Compositio Math.* **104** (1996), No. 2, 125–133.

[50] A. Suciu, 'Translated tori in the characteristic varieties of complex hyperplane arrangements', *Topology Appl.* (to appear), available at `http://xxx.lanl.gov/abs/math.AG/9912227`.

[51] H.-O Tokunaga, 'Some examples of Zariski pairs arising certain elliptic $K3$ surfaces', *Math. Z.* **227** (1998), No. 3, 465–477.

[52] O. Zariski, 'On the problem of existence of algebraic functions of two variables possesing a given branch curve', *Amer. J. Math.* **51** (1929), 305–328.

[53] O. Zariski, 'On the irregularity of cyclic multiple planes', *Ann. of Math.* **32** (1931), 445–489.

[54] O. Zariski, 'On the Poincaré group of rational plane curves', *Amer. J. Math.* **58** (1936), 607–619.

[55] O. Zariski, 'The topological discriminant group of a Riemann surface of genus p', *Amer. J. Math.* **59** (1937), 335–358.

[56] O. Zariski, 'Algebraic surfaces', *Ergeb. Math. Grenzgeb.*, vol. 61 Springer-Verlag, Berlin Heidelberg New York, 1971.