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On the topology of hypocycloids

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Abstract. Algebraic geometry has many connections with physics: string theory, enumerative geometry, and mirror symmetry, among others. In particular, within the topological study of algebraic varieties physicists focus on aspects involving symmetry and non-commutativity. In this paper, we study a family of classical algebraic curves, the hypocycloids, which have links to physics via the bifurcation theory. The topology of some of these curves plays an important role in string theory [3] and also appears in Zariski's foundational work [9]. We compute the fundamental groups of some of these curves and show that they are in fact Artin groups.

Keywords: hypocycloid curve, cuspidal points, fundamental group.

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1. Introduction

Hypocycloid curves have been studied since the Renaissance (apparently Dürer in 1525 described epitrochoids in general and then Roemer in 1674 and Bernoulli in 1691 focused on some particular hypocycloids, like the astroid, see [5]). Hypocycloids are described as the roulette traced by a point P attached to a circumference S of radius r rolling about the inside of a fixed circle C of radius R, such that $0 < \rho = \frac{r}{R} < \frac{1}{2}$ (see Figure 1). If the ratio ρ is rational, an algebraic curve is obtained. The simplest (non-trivial) hypocycloid is called the deltoid or the *Steiner curve* and has a history of its own both as a real and complex curve.

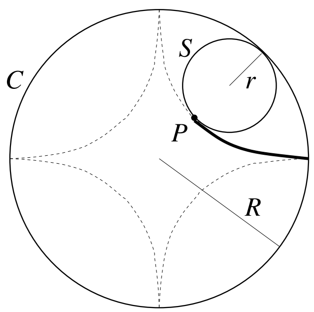

Figure 1: Hypocycloid

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Hypocycloids first appeared as trajectories of motions or integral solutions of vector fields, describing physical phenomena. Modern physics also finds these objects useful. For instance, in the context of superstring compactifications of Calabi-Yau threefolds, certain Picard-Fuchs equations arise naturally. The monodromy group of such equations is the *target duality group* acting on the moduli of string theory and can be computed as the fundamental group of the complement of the bifurcation locus in a deformation space.

Using the celebrated Lefschetz-Zariski theorems of hyperplane sections (in the homotopy setting), these monodromy groups can be recovered in the context of complements of complex algebraic projective curves.

For this reason, braid monodromies and fundamental groups of complements of plane curves have been intensively studied not only by by mathematicians, but also by physicists in the past decades.

Our purpose is to investigate the topology of the complement of some of those interesting hypocycloids using their symmetries and the structure of their affine and projective singularities in a very effective way. In order to do so, we need to introduce Zariski-van Kampen method [9, 7], braid monodromies and Chebyshev polynomials and exploit their properties.

2. First properties of complex hypocycloids

Let us construct a parametrization of a hypocycloid as a real curve. Since ρ is a positive rational number, it admits the irreducible form $\rho := \frac{\ell}{N}$, where ℓ and N are coprime positive integers. Also note that ρ and $1-\rho$ define the same curve, hence $\rho \in (0, \frac{1}{2})$, i.e., $k := N-\ell > \ell$, will be assumed. For simplicity, the external circle C can be assumed to have radius 1. If $C^{\mathbb{R}}_{k,\ell}$ denotes the real hypocycloid given by k and ℓ, it is not difficult to prove that

$$X_{k,\ell}(\theta) = \frac{\ell \cos k\theta + k \cos \ell\theta}{N}, \quad Y_{k,\ell}(\theta) = \frac{\ell \sin k\theta - k \sin \ell\theta}{N} \tag{1}$$

provides a parametrization of $C^{\mathbb{R}}_{k,\ell}$. This parametrization is useful for drawing the hypocycloid but a rational one is preferred. Given $n \in \mathbb{N}$ we denote by T_n, U_n, and W_n the Chebyshev polynomials defined by

$$\cos n\theta = T_n(\cos\theta), \quad \sin\,(n+1)\theta = \sin\theta\, U_n(\cos\theta), \quad \sin\left(n+\frac{1}{2}\right)\theta = \sin\frac{1}{2}\theta\, W_n(\cos\theta). \tag{2}$$

Let us recall that T_n, U_n, and W_n have degree n and they have zeroes at:

$$\begin{array}{llll} T_n(x) = 0 & \Leftrightarrow & x = \cos\left(\frac{(2r-1)\pi}{2n}\right) & r = 1, \ldots, n, \\ U_n(x) = 0 & \Leftrightarrow & x = \cos\left(\frac{r\pi}{n+1}\right) & r = 1, \ldots, n, \\ W_n(x) = 0 & \Leftrightarrow & x = \cos\left(\frac{r\pi}{n+\frac{1}{2}}\right) & r = 1, \ldots, n. \end{array} \tag{3}$$

The following rational parametrization is obtained:

$$x_{k,\ell}(t) := \frac{P_{k,\ell}\left(\frac{1-t^2}{1+t^2}\right)}{N}, \quad y_{k,\ell}(t) := \frac{2t\, Q_{k,\ell}\left(\frac{1-t^2}{1+t^2}\right)}{N(1+t^2)}, \tag{4}$$

where $P_{k,\ell}(x) := \ell T_k(x) + kT_\ell(x)$, and $Q_{k,\ell}(t) := \ell U_{k-1}(x) - kU_{\ell-1}(x)$.

If the parameter t is allowed to run along the complex numbers outside $\{\pm\sqrt{-1}\}$ one obtains a complex plane curve, which will be called the complex hypocycloid, or simply hypocycloid for short, and denoted by $C_{k,\ell} \subset \mathbb{C}^2$. Note that $C^{\mathbb{R}}_{k,\ell} \subset C_{k,\ell} \cap \mathbb{R}^2$.

Moreover, let us recall that any affine complex curve in $\mathbb{C}^2$ defined by rational parametric equations $t \mapsto \left(\frac{p_1(t)}{p_3(t)}, \frac{p_2(t)}{p_3(t)}\right)$ can be embedded in $\mathbb{P}^2 := \mathbb{CP}^2$, the complex projective plane, by homogenizing its parametric equations and removing denominators. This way, the parameter space becomes $\mathbb{P}^1$ and the new projective parametric equations become $[t : s] \mapsto [s^{d-d_1}p_1\left(\frac{t}{s}\right) : s^{d-d_2}p_2\left(\frac{t}{s}\right) : s^{d-d_3}p_3\left(\frac{t}{s}\right)]$, where $d_i := \deg p_i(t)$, and $d := \max\{d_1, d_2, d_3\}$. The complex projective hypocycloid will be denoted by $\bar{C}_{k,\ell} \subset \mathbb{P}^2$. Note that the former parameters $\{\pm\sqrt{-1}\}$ can be interpreted as the *points at infinity* of the complex hypocycloid.

Proposition 2.1. *The complex projective hypocycloid $\bar{C}_{k,\ell}$ is a rational curve of degree $2k$ with the following properties:*

(i) *The curve $C_{k,\ell}$ is invariant by an action of the dihedral group $\mathbb{D}_{2N}$.*

(ii) *The singular points of $C_{k,\ell}$ are only ordinary nodes and ordinary cusps arranged as follows: N cusps, $N(\ell-1)$ (real) nodes, and $N(k-\ell-1)$ (non-real) nodes.*

(iii) *The intersection with the line at infinity consists of two points with local equations $u^k - v^{k-\ell} = 0$ tangent to the line at infinity (these points are singular if $k-\ell > 1$).*

(iv) *The parametrization given in* (4) *is an immersion (outside the N cusps and eventually the points at infinity if $k-\ell > 1$).*

Proof. First, let us show that the rotation of angle $\frac{2\pi}{N}$ and the reflection with respect to the horizontal axis given by the equation $\{y = 0\}$ globally fix $C_{k,\ell}$. These two symmetries generate a dihedral group of order $2N$, denoted by $\mathbb{D}_{2N}$.

The reflection is an immediate consequence of the fact that the first coordinate of the affine parametrization is even $x_{k,\ell}(-t) = x_{k,\ell}(t)$, while the second coordinate is odd $y_{k,\ell}(-t) = -y_{k,\ell}(t)$, see (4).

In order to visualize the rotation, it is more convenient to use trigonometric notation. One can check that

$$\begin{pmatrix} X_{k,\ell}\left(\theta + \frac{2\pi}{N}\right) \\ \\ Y_{k,\ell}\left(\theta + \frac{2\pi}{N}\right) \end{pmatrix} = \begin{pmatrix} \cos\frac{2\pi k}{N} & -\sin\frac{2\pi k}{N} \\ \\ \sin\frac{2\pi k}{N} & \cos\frac{2\pi k}{N} \end{pmatrix} \begin{pmatrix} X_{k,\ell}(\theta) \\ \\ Y_{k,\ell}(\theta) \end{pmatrix},$$

Since this is a rotation of degree $\frac{2\pi k}{N}$ and $\gcd(k, N) = 1$ then part (i) follows.

Note that

$$\varphi([t : s] := \left[p_{k,\ell}(t,s) : 2tsq_{k,\ell}(t,s) : N(s^2+t^2)^k\right] \tag{5}$$

is a parametrization of the projective hypocycloid $\bar{C}_{k,\ell}$, where

$$p_{k,\ell}(t,s) := (s^2+t^2)^k P_{k,\ell}\left(\frac{s^2-t^2}{s^2+t^2}\right), \quad q_{k,\ell}(t,s) := (s^2+t^2)^{k-1} Q_{k,\ell}\left(\frac{s^2-t^2}{s^2+t^2}\right).$$

Note that $p_{k,\ell}$ is homogeneous of degree k and $q_{k,\ell}$ is homogeneous of degree $k-1$. Since the leading coefficient of $P_{k,\ell}$ is $2^{k-1}\ell$ and the one of $Q_{k,\ell}$ is $2^{k-2}\ell$, see [8], then $\varphi([1 : \pm\sqrt{-1}]) =$

$[1:\pm\sqrt{-1}:0]$ and hence this parametrization induces a well-defined surjective map from $\mathbb{P}^1$ to $\bar{C}_{k,\ell}$. Hence, outside a finite number of points (those where the parametrization is not injective) $\bar{C}_{k,\ell}$ is isomorphic to $\mathbb{P}^1$, which implies that $\bar{C}_{k,\ell}$ is a rational curve. Also, its degree corresponds with the degree of any of its parametric equations, namely $2k$.

It is a straightforward computation that

$$\frac{d\varphi_1}{dt}=-\frac{4k\ell ts^2}{N(s^2+t^2)^2}\left(U_{k-1}+U_{\ell-1}\right),\quad\text{and}\quad \frac{d\varphi_2}{dt}=\frac{2k\ell s}{N(s^2+t^2)}\left(T_k-T_\ell\right).$$

One should consider two different cases:

- If N is even, then one can use the following two formulas (see [8]): $U_{k-1}+U_{\ell-1}=2T_{\frac{k-\ell}{2}}U_{\frac{k+\ell}{2}-1}$ and $(T_k(x)-T_\ell(x))+x(U_{k-1}(x)+U_{\ell-1}(x))=(U_k(x)+U_{\ell-2}(x))$. Therefore:

$$\begin{aligned}\frac{d\varphi_1}{dt}&=-\frac{8k\ell ts^2}{N(s^2+t^2)^2}T_{\frac{k-\ell}{2}}U_{\frac{k+\ell}{2}-1},\\ 2ts\frac{d\varphi_2}{dt}-(s^2-t^2)\frac{d\varphi_1}{dt}&=\frac{8k\ell ts}{N(s^2+t^2)}T_{\frac{k-\ell}{2}+1}U_{\frac{k+\ell}{2}-1}.\end{aligned}\tag{6}$$

Thus the common zeroes to $\varphi_1':=\frac{d\varphi_1}{dt}$ and $\varphi_2':=\frac{d\varphi_2}{dt}$ are given by $tsU_{\frac{k+\ell}{2}-1}\left(\frac{s^2-t^2}{s^2+t^2}\right)=0$, that is,

$$\left\{(t,s)\mid t=0, s=0,\text{ or }\frac{s^2-t^2}{s^2+t^2}=\cos\left(\frac{2r\pi}{k+\ell}\right), r=1,\ldots,\frac{k+\ell}{2}-1\right\}.$$

This makes a total of $2\left(\frac{k+\ell}{2}-1\right)+2=N$ singularities. Using the previous equations one can check that the order of t in (φ_1',φ_2') is (1,2) and consequently such a singularity is an ordinary cusp of equation y^2-x^3 whose tangent is the line $L_0:=\{y=0\}$. By applying the symmetry (i), the remaining singularities are also cusps and their tangents are a rotation of L_0.

By the parity of N, the line L_0 intersects $\bar{C}_{k,\ell}$ at two cusps $\varphi([1:0])$ and $\varphi([0:1])$ with multiplicity 3 each, hence there are $2(k-3)$ extra points of intersection (counted with multiplicity). Let us denote by $\varphi([t_0:s_0])=P$ one such point. Due to the symmetry of $\bar{C}_{k,\ell}$ with respect to L_0, unless the tangent direction at P is vertical, the curve $\bar{C}_{k,\ell}$ possesses a node at P. In order to prove that the $2(k-3)$ extra points of intersection do in fact correspond to $(k-3)$ nodes, one just needs to check that φ_2 and φ_1' do not have any common zeroes. In order to do so, it is enough to note that, according to (6), all the roots of φ_1' that are not critical points of the parametrization are of the form $\frac{s^2-t^2}{s^2+t^2}=\cos\left(2\frac{2r-1}{k-\ell}\pi\right)$, which are not roots of φ_2. Applying the symmetry of order N and the fact that the orbit of L_0 by such symmetry has $\frac{N}{2}$ lines, one obtains the existence of $\frac{(k-3)N}{2}$ nodes. Let us denote this group of nodes by A_1.

Another group of nodes is placed on the line L_{2N} which is the rotation of L_0 by an angle of $\frac{\pi}{N}$ radians. In order to do so, let us reparametrize the hypocycloid so that L_{2N} is sent to the horizontal line. This corresponds to switching $\ell\theta$ by $\ell\theta+\pi$ in the inner circle. The real equations are transformed as follows (compare with (1)):

$$\tilde{X}_{k,\ell}(\theta) = \frac{1}{N}\left(\ell\cos k\theta - k\cos \ell\theta\right), \quad \tilde{Y}_{k,\ell}(\theta) = \frac{1}{N}\left(\ell\sin k\theta + k\sin \ell\theta\right) \tag{7}$$

which provide the rational parametrization:

$$\tilde{x}_{k,\ell}(t) := \frac{\tilde{P}_{k,\ell}\left(\frac{1-t^2}{1+t^2}\right)}{N}, \quad \tilde{y}_{k,\ell}(t) := \frac{2t\tilde{Q}_{k,\ell}\left(\frac{1-t^2}{1+t^2}\right)}{N(1+t^2)}, \tag{8}$$

where $\tilde{P}_{k,\ell}(x) := \ell T_k(x) - kT_\ell(x)$, and $\tilde{Q}_{k,\ell}(x) := \ell U_{k-1}(x) + kU_{\ell-1}(x)$. Note that again $\tilde{x}_{k,\ell}(-t) = \tilde{x}_{k,\ell}(t)$ and $\tilde{y}_{k,\ell}(-t) = -\tilde{y}_{k,\ell}(t)$, and that the rotation of $\frac{2\pi}{N}$ radians is a symmetry of the curve. Using the formula $U_{k-1} - U_{\ell-1} = 2T_{\frac{k+\ell}{2}}U_{\frac{k-\ell}{2}-1}$ one can check that

$$\tilde{\varphi}_1' := \frac{\tilde{\varphi}_1}{dt} = -\frac{8k\ell st^2}{N(s^2+t^2)^2}T_{\frac{k+\ell}{2}}U_{\frac{k-\ell}{2}-1},$$

and hence $\tilde{\varphi}_1'$ and $\tilde{\varphi}_2$ have common zeroes only at $t = 0$ and $s = 0$, which are two vertical tangents. This shows that there are $(k-1)$ remaining nodes on L_{2N} and, after applying the rotation, one finds $\frac{(k-1)N}{2}$ new nodes. Let us denote this group of nodes by A_2 and define $A = A_1 \cup A_2$.

- If N is odd, then proceeding as above, one can use the formula: $T_k(x) - T_\ell(x) = (1-x)W_{\frac{k+\ell-1}{2}}(x)W_{\frac{k-\ell-1}{2}}(x)$ and check that the critical points of the parametrization are given by $t = 0$ and by $W_{\frac{k+\ell-1}{2}}\left(\frac{s^2-t^2}{s^2+t^2}\right) = 0$, that is, $\{(t,s) \mid t = 0, \text{ or } \frac{s^2-t^2}{s^2+t^2} = \cos\left(\frac{2r\pi}{k+\ell}\right), r = 1, ..., \frac{k+\ell-1}{2}\}$.
  Analogously as in the previous case, the singularity at $t = 0$ is an ordinary cusp $y^2 - x^3$ whose tangent is L_0. There are $2k-3$ remaining points of intersection which are necessarily a vertical tangent and $k-2$ nodes, since φ_2 and $\frac{\varphi_1}{dt}$ do not have any common zeroes similarly as above. Again, after applying the rotation of order N one can find $(k-2)N$ nodes. Let us denote this group of nodes by A.

Summarizing, we have found $\#A = (k-2)N$ nodes and N ordinary cusps. Assuming (iii) $\bar{C}_{k,\ell}$ contains also 2 singular points of type $u^k - v^{k-\ell}$. Using the genus formula:

$$\frac{(2k-1)(2k-2)}{2} - (k-2)N - N - 2\frac{(k-\ell-1)(k-1)}{2} - \alpha = 0$$

where the first summand is the virtual genus of a curve of degree $2k$, the second summand comes from the nodes, the third summand comes from the cusps, the fourth one comes from the singularities at infinity, and the last one comes from any further singularities. Since the equation has to equal zero due to the fact that $\bar{C}_{k,\ell}$ is rational, this forces α to be equal to zero, and thus $\bar{C}_{k,\ell}$ contains no further singularities.

To finish the proof of part (ii) one needs to make sure that only $(\ell-1)N$ nodes are real. If N is odd this can be done by verifying that there are only $(\ell-1)$ real nodes on L_0. Note that the real nodes come from branches joining the cusps. The line L_0 contains one cusp and divides the set of remaining cusps into two groups of $\frac{N-1}{2}$ each. Since the real branches join a cusp with the ℓ-th consecutive cusp, there is a total of $2(\ell-1)$ branches crossing L_0 which lead to $(\ell-1)$ nodes. Again, applying the rotation one obtains the $(\ell-1)N$ real nodes. The case when N is even is analogous. Part (ii) will be proved if (iii) is checked.

Let P_1, P_2 be the two points in $\bar{C}_{k,\ell} \cap L_\infty$, where L_∞ is the line at infinity. Let us note that any reflection in $\mathbb{D}_{2N}$ fixes L_∞ and C and interchanges P_1 and P_2. By Bezout's Theorem, $L_\infty \cdot \bar{C}_{k,\ell} = 2k$; because of the symmetry, $(L_\infty \cdot \bar{C}_{k,\ell})_{P_i} = k$. On the other hand it is easily seen that the only intersection points of $\bar{C}$ and $\bar{C}_{k,\ell}$ are the N cusps and P_1, P_2 (compute $X(\theta)^2 + Y(\theta)^2$). Since $\bar{C}$ and $\bar{C}_{k,\ell}$ are transversal at the cusps, their intersection number at these points is 2. One obtains the following:

$$4k = \bar{C} \cdot \bar{C}_{k,\ell} = 2(k+\ell) + 2(\bar{C} \cdot \bar{C}_{k,\ell})_{P_1}$$

and $(\bar{C} \cdot \bar{C}_{k,\ell})_{P_1} = (\bar{C} \cdot \bar{C}_{k,\ell})_{P_2} = k - \ell$.

It is a standard fact in singularity theory that a locally irreducible curve germ intersecting two smooth transversal branches with coprime multiplicities p and q has the same topological type as $u^p - v^q = 0$. Therefore (iii) follows. Note that if $k - \ell = 1$ then the points at infinity are smooth, otherwise they are singular: this also proves (iv). □

3. Fundamental group of the complement of a curve

Let us consider an affine complex plane algebraic curve $C \subset \mathbb{C}^2$. Let us denote by $f(x, y) \in \mathbb{C}[x, y]$ a reduced equation of the curve C. For simplicity f will be assumed to be monic as a polynomial in y (this can be achieved by applying a generic change of coordinates and dividing by a non-zero constant). Let d be the degree of f in y (which may be smaller than the total degree of f).

The classical Zariski-van Kampen method works as follows. One considers a generic vertical line L in $\mathbb{C}^2$; the group $\pi_1(L \setminus C)$ is isomorphic to the free group $\mathbb{F}_d$, since $L \cap C$ consists of d points. A basis of loops in this group also generates $\pi_1(\mathbb{C}^2 \setminus C)$ and the relations are obtained via the monodromy of this fibration, which is basically *moving L around* the non-generic vertical lines. Let us state it more precisely (see [1] for a more complete version).

For $t \in \mathbb{C}$, L_t denotes the vertical line $x = t$. Let $NT := \{t \in \mathbb{C} \mid L_t \not\pitchfork C\}$; NT is a finite set and it is the zero locus of the discriminant of f with respect to y (which is a polynomial in x). If $t \notin NT$, then $C \cap L_t$ consists of d points, and by the continuity of roots one can see f as a holomorphic mapping $\tilde{f} : \mathbb{C} \setminus NT \to V \setminus \Delta$, where:

- V is the space of monic complex polynomials in one variable and degree d (naturally isomorphic to $\mathbb{C}^d$ via the coefficients);
- Δ is the discriminant of V, i.e., the set of polynomials in V with multiple roots (which is a hypersurface of V).

The space $V \setminus \Delta$ can be naturally identified with the configuration space of d different points in $\mathbb{C}$, whose fundamental group is the braid group $\mathbb{B}_d$ in d strings. Let us recall the Artin presentation of this group:

$$\mathbb{B}_d = \left\langle \sigma_1, \ldots, \sigma_{d-1} \middle| \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1},\ {\scriptstyle i=1,\ldots,d-2},\ [\sigma_i, \sigma_j] = 1,\ {\scriptstyle 1 \le i < j-1 < d-2} \right\rangle. \tag{9}$$

There is a free action of this group on the free group $\mathbb{F}_d$ with generators $a_1, \ldots, a_d$ defined as follows:

$$a_i^{\sigma_j} := \begin{cases} a_{i+1} & \text{if } i = j \\ a_{i+1} a_i a_{i+1}^{-1} & \text{if } i = j+1 \\ a_i & \text{if } i \neq j, j+1. \end{cases} \tag{10}$$

Note that for any $\tau \in \mathbb{B}_d$, $(a_d \cdot \ldots \cdot a_1)^\tau = a_d \cdot \ldots \cdot a_1$.

One can define the braid monodromy of C with respect to the coordinates x, y as follows. Let $t_0 \in \mathbb{C} \setminus NT$ and let $\mathbb{F} := \pi_1(\mathbb{C} \setminus NT; t_0)$, which is a free group. Then one defines $\nabla : \mathbb{F} \to \mathbb{B}_d$ as the morphism defined by $\tilde{f}$ at the level of fundamental groups (with a suitable identification of $\mathbb{B}_d$ with $\pi_1(V \setminus \Delta; \tilde{f}(t_0))$.

Zariski-van Kampen Theorem 3.1. *The fundamental group of $\mathbb{C}^2 \setminus C$ is the quotient of $\mathbb{F}_d$ by the subgroup normally generated by $w^{-1} w^{\nabla(\tau)}$, $w \in \mathbb{F}_d$, $\tau \in \mathbb{B}_d$. If $b_1, \ldots, b_r$ is a free generating system of $\mathbb{F}$, then it admits the following presentation:*

$$\left\langle a_1, \ldots, a_d \middle| \, a_i = a_i^{\nabla(b_j)}, \ i = 1, \ldots, d-1, \ j = 1, \ldots, r \right\rangle. \tag{11}$$

Remark 3.2. A natural interpretation of $\pi_1(V \setminus \Delta; \tilde{f}(t_0))$ can be given when $f \in \mathbb{R}[x, y]$, $t_0 \in \mathbb{R}$ and all the roots of $f(t_0, y)$ are real. Let $y_1 > \cdots > y_d$ be the roots of $f(t_0, y)$. Then σ_i is the homotopy class of the mapping $[0, 1] \to V \setminus D$,

$$t \mapsto \{y_1, \ldots, y_{i-1}, c_i + r_i e^{\pi t \sqrt{-1}}, c_i - r_i e^{\pi t \sqrt{-1}}, y_{i+2}, y_d\},$$

where c_i is the middle point between y_i and y_{i+1} and r_i is half their distance. A similar argument works without the *real* assumptions.

Remark 3.3. Let us assume again that $f \in \mathbb{R}[x, y]$, $t_0 \in \mathbb{R}$ and all the roots of $f(t_0, y)$ are real. For the free group $\pi_1(L_{t_0} \setminus C; y_0)$, $y_0 \gg y_1$, a basis can be chosen as follows. Fix a small radius $\varepsilon > 0$.

Consider a lasso $a_i := u_i \cdot \delta_i \cdot u_i^{-1}$ based at y_0 as follows. The path u_i runs along the real line from y_0 to $y_i - \varepsilon$ and avoids the points $y_1, \ldots, y_{i-1}$ by the upper semicircles of radius ε centered at these points; the lasso δ_i runs counterclockwise along the circle of radius ε centered at y_i.

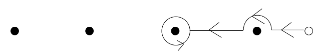

Figure 2: An element of a standard geometric basis

The *ordered* basis $(a_1, \ldots, a_d)$ is called a *standard geometric basis*. Note that a_i is a meridian of the point y_i (see [1] for a definition) and $(a_d \cdot \ldots \cdot a_1)^{-1}$ is a meridian of the point at infinity. These identifications give the geometric counterpart of the action (10). In the non-real case, *standard geometric bases* play the same role (see [1]).

In general *standard pseudogeometric bases* are preferred for the group $\mathbb{F}$ (if $NT \subset \mathbb{R}$); the only difference with geometric bases being that the condition on the position of the base points is weakened.

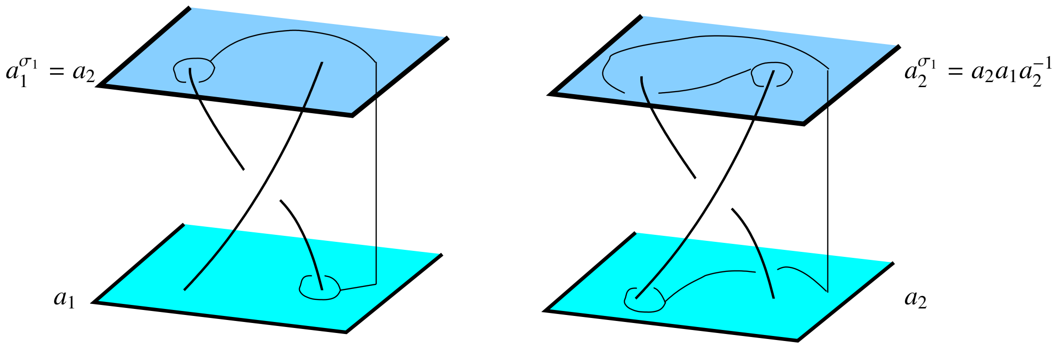

Figure 3: Geometric action of the braid group

Example 3.4. Let us assume that b_1 corresponds to a small loop surrounding a point t such that L_t is an ordinary tangent line. For suitable choices of loops and paths, $\nabla(b_1) = \sigma_1$ and the only non-trivial relation is given by $a_1 = a_2$. Analogously, for other non-transversal vertical lines L_t, one obtains the following braids and relations:

- If L_t passes transversally through a node, then $\nabla(b_1) = \sigma_1^2$ and the only non-trivial relation is given by $[a_1, a_2] = 1$.
- If L_t passes transversally through an ordinary cusp, then $\nabla(b_1) = \sigma_1^3$ and the only non-trivial relation is given by $a_1 a_2 a_1 = a_2 a_1 a_2$.
- If L_t is tangent to an ordinary cusp, then $\nabla(b_1) = (\sigma_2\sigma_1)^2$ and the only non-trivial relations are given by $a_1 = a_3$ and $a_2 = a_3 a_2 a_1 a_2^{-1} a_1^{-1}$.
- If L_t passes transversally through an m-tacnode (two smooth branches with intersection number m), then $\nabla(b_1) = \sigma_1^{2m}$ and the only non-trivial relation is given by $(a_1 a_2)^m = (a_2 a_1)^m$.

Remark 3.5. Two remarks about the relations explained in Example 3.4 should be made. First of all, such relations can be expressed in such a simple manner when: t_0 is close enough to t, b_1 is a small meridian around t, and a suitable choice for generators of $\mathbb{F}_d$ is considered (essentially a standard geometric basis). In such cases, $\nabla(b_1)$ produces r effective relations, where $r := d - \#(C \cap L_t)$.

In the general case, for instance when the base point is not close to t, the braid $\nabla(b_1)$ is written as $\tau^{-1}\sigma\tau$, where:

- The open braid τ^{-1} goes from L_{t_0} to a fiber $L_{t_0'}$ close to L_t.
- The braid σ is as in Example 3.4 (or a product of these braids involving disjoint subsets of strings).

When a (standard) geometric basis $a_1', \ldots, a_d'$ in $\pi_1(L_{t_0'} \setminus C)$ is considered, on which σ acts, only r non-trivial relations are produced. This choice of basis allows one to consider τ as an element of $\mathbb{B}_d$. Using this technique, one can see that $\pi_1(\mathbb{C}^2 \setminus C)$ can be described by a system of relations of type $a_i' = a_i^\tau$.

The relations will involve conjugates of the standard generators of $\mathbb{F}_d$. The number of relations and their *type* (equality, commutation, braid relations,...) depend only on the braids σ, which depend only on algebraic properties of C (degree, topological type of singularities), but the involved conjugations of the generators depend on the coefficients of f. In general, finding the braid τ explicitly is a very difficult task and unless the coefficients of f are rational or Gaussian integers, computational methods are far from being efficient.

The braid τ, or equivalently the relationship between the two geometric bases, can be obtained algorithmically from the real picture, when f has real equations (*real curve*), all (or almost all) of the non-transversal vertical lines are real and the branches around the critical points are also real. Such curves are called *totally real* curves.

4. Fundamental group of hypocycloids and Artin groups

In this section we compute fundamental groups for some hypocycloids. Hypocycloids are real curves, but not totally real curves. However, they are very symmetric and it is by quotienting the plane by these symmetries that one can obtain a curve that is closer to being totally real.

In these computations a special type of groups, called *Artin groups* will be obtained. Artin groups can be defined as follows. Let Γ be a finite graph (with no loops and no multiple edges between vertices); let $S := S(\Gamma)$ be the set of vertices and let $E := E(\Gamma)$ be the set of edges (considered as a subset of $\{A \subset S \mid \#A = 2\}$). The Artin group G_Γ associated with Γ is generated by the elements of S and has a system of relations given as follows for any $s \neq t$:

- If $\{s,t\} \in E$ then $sts = tst$.
- If $\{s,t\} \notin E$ then $[s,t] = 1$.

For example, $\mathbb{B}_d$ is the Artin group associated with the $\mathbb{A}_{d-1}$ graph (a connected linear graph with $d-1$ vertices).

4.1. The Deltoid

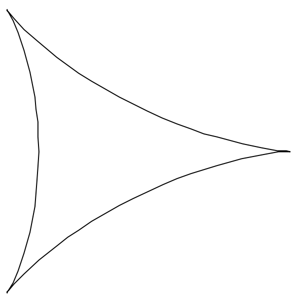

Figure 4: Deltoid

The deltoid corresponds to $\rho = \frac{1}{3}$, that is, $k = 2$, $\ell = 1$, and $N = 3$. In order to obtain an explicit equation for $C_{k,\ell}$, given by parametric equations $(x_{k,\ell}(t), y_{k,\ell}(t))$, one needs to compute the resultant of $x_{k,\ell}(t) - x$ and $y_{k,\ell}(t) - y$, with respect to t. In this case one obtains

$$C_{2,1} : 3(x^2 + y^2)^2 + 24x(x^2 + y^2) + 6(x^2 + y^2) - 32x^3 - 1 = 0. \tag{12}$$

Let $\pi : \mathbb{C}^2 \to \mathbb{C}^2$ be the 2-fold ramified covering given by $\pi(x, y) := (x, y^2)$. Let $D_{2,1} := \pi(C_{2,1})$; since $C_{2,1}$ is symmetric with respect to the involution $\sigma : \mathbb{C}^2 \to \mathbb{C}^2$, $\sigma(x, y) := (x, -y)$ (which generates the automorphism group of π), one can check that $C_{2,1} = \pi^{-1}(D_{2,1})$, where $D_{2,1}$ is given by

$$D_{2,1} : 3(x^2 + y)^2 + 24x(x^2 + y) + 6(x^2 + y) - 32x^3 - 1 = 0. \tag{13}$$

In order to compute $\pi_1(\mathbb{C}^2 \setminus (D_{2,1} \cup X))$ (where X is the x-axis), one has to compute the discriminant of the equation (13) with respect to y. Since all its roots are real, that is, $D_{2,1}$ is a totally real curve, Figure 5 contains all the topological information needed to compute the group $\pi_1(\mathbb{C}^2 \setminus (D_{2,1} \cup X))$.

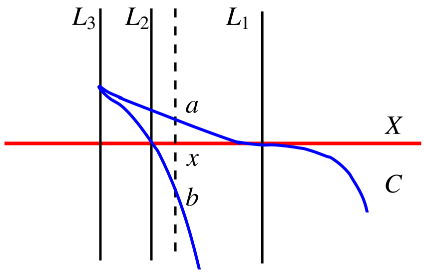

Figure 5: The curve $D_{2,1} \cup X$

Following §3, the dotted line L represents a generic vertical line (the other three lines in Figure 5 are the non-transversal vertical lines). After fixing a *big enough* real number as the base point, one can consider the natural free basis of $\pi_1(L \setminus (D_{2,1} \cup X))$ given by a, x, b (positive meridians around the intersections with the curve), such that $(bxa)^{-1}$ is a meridian of the point at infinity. Moving around the line L_1 one obtains the braid σ_1^6, which produces the relation:

$$(ax)^3 = (xa)^3.$$

The braid around L_2 is σ_2^2, and the relation is $[b, x] = 1$. In order to compute the relations provided by L_3 one can proceed as follows. Consider a vertical line L' between L_2 and L_3 and generators a', b', x' of $\pi_1(L' \setminus (D_{1,2} \cup X))$ as done with L. In order to see them as elements in $\pi_1(L \setminus (D_{2,1} \cup X))$ it is necessary to connect the base points in the same horizontal line. It is easily seen that $a = a'$, $b = b'$ and $x = x'$. The relation obtained around L_3 is $aba = bab$.

One obtains

$$\pi_1(\mathbb{C}^2 \setminus (D_{2,1} \cup X)) = \langle a, x, b \mid (ax)^3 = (xa)^3, [b, x] = 1, aba = bab \rangle.$$

Since $\pi_| : \mathbb{C}^2 \setminus (C_{2,1} \cup X) \to \mathbb{C}^2 \setminus (D_{2,1} \cup X)$ is a double unramified covering, one can check that the group $\pi_1(\mathbb{C}^2 \setminus (C_{2,1} \cup X))$ is the subgroup of index 2 normally generated by a, b, x^2. It is well-known that $\pi_1(\mathbb{C}^2 \setminus C_{2,1})$ can be obtained from $\pi_1(\mathbb{C}^2 \setminus (C_{2,1} \cup X))$ by adding the relation $x^2 = 1$.

Remark 4.1.1. In fact, the above operations, computing the index 2 subgroup and factoring by x^2, commute. Therefore, one $\pi_1(\mathbb{C}^2 \setminus C_{2,1})$ can also be considered as a subgroup of index 2 of $G_{2,1} := \pi_1(\mathbb{C}^2 \setminus (D_{2,1} \cup X))/\langle x^2 \rangle$.

Proposition 4.1.2. *The group $\pi_1(\mathbb{C}^2 \setminus C_{2,1})$ is the Artin group of the triangle.*

Proof. The group $\pi_1(\mathbb{C}^2 \setminus C_{2,1})$ is the kernel of the morphism $\rho : G_{2,1} \to \langle t \mid t^2 = 1 \rangle$, given by $\rho(a) := \rho(b) := t$, $\rho(x) := 1$.

Using the Reidemeister-Schreier method, $\pi_1(\mathbb{C}^2 \setminus C_{2,1})$ is generated by a, b, c where $c := xax$ (note that $b = xbx$, because of the second relation). The third relation gives $aba = bab$ and $cbc = bcb$. The first relation gives $aca = cac$. The presentation of the Artin group for the triangle results directly. □

Remark 4.1.3. According to Proposition 2.1(iii), the projective closure $\bar{C}_{2,1}$ of $C_{2,1}$ in $\mathbb{P}^2$ is such that the line at infinity is a bitangent and the contact points are the imaginary cyclic points of order 4. The fundamental group $\pi_1(\mathbb{P}^2 \setminus \bar{C}_{2,1})$ is a non-abelian finite group of size 12 which was first computed by Zariski [9] and it is the braid group of 3 strings in the 2-sphere. This is the curve of smallest degree with a non-abelian fundamental group. Its dual is a nodal cubic curve (as a real curve with a node with imaginary tangent lines).

4.2. Astroid

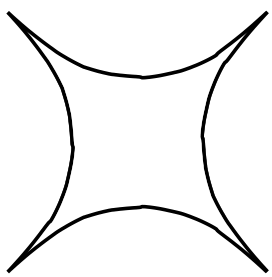

Figure 6: Astroid

The astroid corresponds to $\rho = \frac{1}{4}$, that is, $k = 3$, $\ell = 1$, and $N = 4$. In order to apply a more suitable symmetry, a rotation of the astroid should be performed to obtain a curve as in Figure 6. One obtains

$$C_{3,1} : 4(x^2 + y^2)^3 + 15(x^2 + y^2)^2 + 12(x^2 + y^2) - 108x^2y^2 - 4 = 0. \tag{14}$$

As we did for the deltoid in §4.1, let us consider $D_{3,1} := \pi(C_{3,1})$ the quotient of $C_{3,1}$ by the symmetry σ, which has equation

$$D_{3,1} : 4(x^2 + y)^3 + 15(x^2 + y)^2 + 12(x^2 + y) - 108x^2y - 4 = 0. \tag{15}$$

In order to compute $\pi_1(\mathbb{C}^2 \setminus (D_{3,1} \cup X))$ (where X is the horizontal axis), one computes the discriminant of the equation (13) with respect to y. In this case $D_{3,1}$ is not totally real, thus

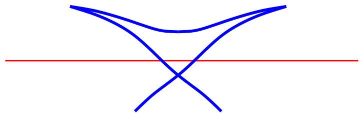

Figure 7: The curve $D_{3,1} \cup X$

Figure 7 is not enough to compute the group $\pi_1(\mathbb{C}^2 \setminus (D_{3,1} \cup X))$. According to Proposition 2.1(ii) the astroid has four ordinary (non-real) double points: two of them are sent to the *real* double point shown in Figure 7, the other two are on the symmetry axis and are sent to two (non-real) points where $D_{3,1}$ is tangent to X, hence $D_{3,1}$ is not a totally real curve. However, $D_{3,1}$ is symmetric with respect to the $Y := \{x = 0\}$ axis, and hence one can perform the quotient with respect to this symmetry and obtain the curve $X \cup Y \cup E_{3,1}$, where

$$E_{3,1} : 4(x + y)^3 + 15(x + y)^2 + 12(x + y) - 108xy - 4 = 0. \tag{16}$$

The curve is rotated in order to have a better projection.

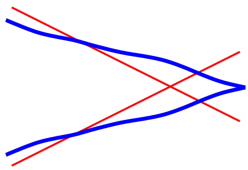

Figure 8: The curve $E_{3,1} \cup X \cup Y$

According to §3, $\pi_1(\mathbb{C}^2 \setminus (E_{3,1} \cup X \cup Y))$ is generated by x, a, b, y and a system of relations can be given as follows: $aba = bab$, $[a, x] = 1$, $[b, y] = 1$, $[x, y] = 1$, $(ay)^2 = (ya)^2$, and $(bx)^2 = (xb)^2$. Defining $G_{3,1}$ as before by adding the relations $x^2 = 1$, $y^2 = 1$, one needs to compute the appropriate index four subgroup. A straightforward application of the Reidemeister-Schreier method gives the following result.

Proposition 4.2.1. *The group $\pi_1(\mathbb{C}^2 \setminus C_{3,1})$ is the Artin group of the square.*

Remark 4.2.2. The curve $\bar{C}_{3,1}$ is a sextic with six cusps and four nodes, i.e., the dual of a nodal quartic. These curves were studied by O. Zariski [10]. The group $\pi_1(\mathbb{P}^2 \setminus \bar{C}_{3,1})$ is isomorphic to the braid group of four strings on the sphere.

4.3. Hypocycloid $\rho = \frac{2}{5}$

This is a particular case of the hypocycloids for $\rho = \frac{n}{2n+1}$ where all nodes are real. The curve $C_{3,2}$ has equation

$$80(x^2 + y^2)^3 + 165(x^2 + y^2)^2 - 30(x^2 + y^2) - 216x(x^4 - 10\,x^2y^2 + 5\,y^4) + 1 = 0. \tag{17}$$

The quotient $D_{3,2}$ of $C_{3,2}$ by the action of σ is a totally real curve, and thus the real picture contains all the topological information needed to compute the group $\pi_1(\mathbb{C}^2 \setminus (D_{2,1} \cup X))$.

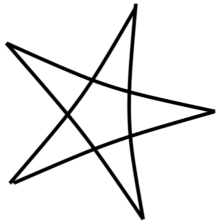

Figure 9: Hypocycloid $\rho = \frac{2}{5}$

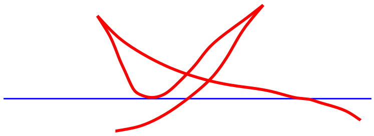

Figure 10: The curve $D_{3,2} \cup X$

Following §3, one needs to select a generic vertical line L_{t_0}, which will be chosen to sit between the two real double points of $D_{3,2}$. The group $G_{3,2}$, which is the quotient of $\pi_1(\mathbb{C}^2 \setminus (D_{3,2} \cup X))$ by the square of a meridian of X, is generated by a, b, x, c a has the following system of relations $x^2 = 1$, $[a, b] = 1$, $(ax)^2 = (xa)^2$, $(xax)b(xax) = b(xax)b$, $[c, x] = 1$, $[b, c] = 1$, $aca = cac$, and $(bx)^3 = (xb)^3$.

Proposition 4.3.1. *The group $\pi_1(\mathbb{C}^2 \setminus C_{3,2})$ is the Artin group of the pentagon.*

Proof. After using Reidemeister-Schreier again, one obtains the following set of generators: $a, b, c, u := xax, v := xbx$, a set of *cuspidal* relations among the adjacent list of generators u, b, v, a, c in a cyclic manner, and commutation relations amongst the non-adjacent generators. □

Remark 4.3.2. This result was already obtained in [3, 4]. The authors computed the correct presentation of the group, but then they stated that this group is isomorphic to $\mathbb{B}_5$; this is not true as can be checked either directly or using Artin group theory.

Remark 4.3.3. With some more computations, it is possible to show that the group for the hypocycloid $C_{4,3}$ (for $\rho = \frac{3}{7}$) is the Artin group of the polygon of 7 edges. Since the quotient of the curves $C_{n+1,n}$ is *totally real*, we will compute the general case in a forthcoming paper.

4.4. Hypocycloid $\rho = \frac{3}{8}$

We complete this preliminary study of the topology of the hypocycloid curves with $C_{5,3}$ corresponding to $\rho = \frac{3}{8}$, i.e., $k = 5$, $\ell = 3$, and $N = 8$. As with the astroid in 4.2, a rotation will be performed in order to use more suitable symmetries. One obtains:

$$\begin{aligned} C_{5,3} :&11664 \left(x^2 + y^2\right)^5 + 47655 \left(x^2 + y^2\right)^4 + 40240 \left(x^2 + y^2\right)^3 - \\ &- 17040 \left(x^2 + y^2\right)^2 + 1920 (x^2 + y^2) + 1350000\, x^2 y^2 \left(x^2 - y^2\right)^2 - 64 = 0. \end{aligned} \tag{18}$$

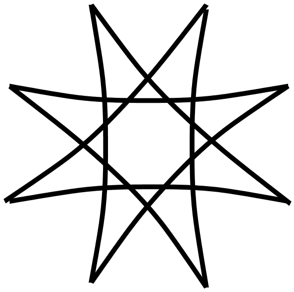

Figure 11: Hypocycloid $\rho = \frac{3}{8}$

Let $D_{5,3} := \pi(C_{5,3})$.

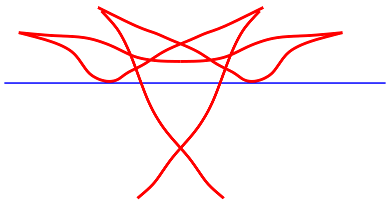

Figure 12: The curve $D_{5,3} \cup X$

Since $D_{5,3}$ is not totally real, Figure 12 is not enough to compute $\pi_1(\mathbb{C}^2 \setminus (D_{5,3} \cup X))$. However, one can use the symmetry along $Y := \{x = 0\}$ to obtain the quotient $E_{5,3}$ of $D_{5,3}$. The curve $E_{5,3} \cup X \cup Y$ is not totally real yet. According to Proposition 2.1(iii), $C_{5,3}$ has eight imaginary nodes. Since two of them are on X, they produce another three nodes for $D_{5,3}$, one of them on Y and real. After rotating the axes one obtains Figure 13. Note that $E_{5,3}$ has a

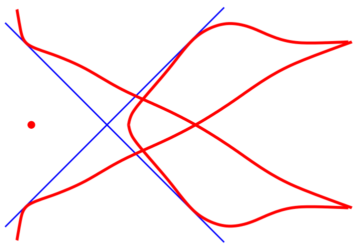

Figure 13: The curve $E_{5,3} \cup X \cup Y$

(real) node with imaginary tangent lines and has a symmetry along the line joining the node and the origin. By this symmetry, X and Y are symmetric to one another.

The resulting quotient is a totally real curve, whose fundamental group can be obtained from the information shown in Figure 14.

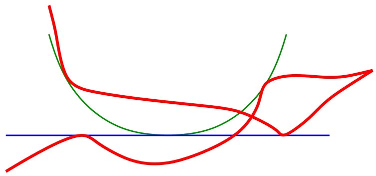

Figure 14: Final curve

Proposition 4.4.1. *The group $\pi_1(\mathbb{C}^2 \setminus C_{3,1})$ is the Artin group of the octagon.*

Proof. One has to apply the Reidemeister-Schereier method and use `GAP` [6] to obtain the desired presentation. □

5. Conclusions

Note that with this method one finds not only the expected groups, but also the expected presentations. This happens also in other computations of the fundamental groups of the complements of special curves, like in [2]. This seems to suggest that there is a deep geometrical connection between the symmetries and the fundamental group, which should be better understood.

It seems to be possible to generalize the method of §4 for curves $C_{n+1,n}$, since all nodes and vertical tangents are real. For the general case, in order to *make visible* all singular points we may consider the quotient of $\mathbb{C}^2$ by the action of $\mathbb{D}_{2n}$, which is not a smooth surface.

Acknowledgements

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