On Sextic Curves with Big Milnor Number

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1 Introduction

section-introduction

In the present paper we provide an exhaustive description, up to projective isomorphism, of all irreducible sextic curves in $\mathbb{P}^2$ having a singular point of type $\mathbb{A}_n$, $n \geq 15$, only rational singularities and global Milnor number at least 18. Moreover, we have developed a method for explicit construction of sextic curves with at least eight – possibly infinitely near – double points. This method allows to express such sextic curves in terms of arrangements of curves with lower degrees and it provides a geometric picture of possible deformations. Because of the large number of cases, we have developed only some which give an insight of the general situation.

A classification of sextic curves with only rational singularities has been given by Yang in [?], where all possibilities are listed. Nevertheless, for any given configuration of singular points Yang does not provide further information about the number or the dimension of the families ocurring with such combinatorics.

In most cases we will exhibit several families of curves satisfying the same configuration of singular points which are either deformation of different families or can be deformed into different configurations. In some cases, the generic members of theses families are topologically different. This fact is detected either by the Alexander polynomial (as in [?], [?] or [?]) or by the fundamental group. In other cases, we do not know whether generic members are topologically equivalent. The main difficulty, as it was already mentioned in [?], is that such families are associated with Galois transformations in some number fields. In such a situation where conjugated curves in a number field come into play, no one of the known invariants of curves has been shown to be effective.

Another asset of this method is allows one to compute the fundamental group of the sextic curves using fundamental groups of the arrangements of curves of lower degree involved in aforementioned the construction.

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2 General method

section-general

Let $\mathcal{C} \subset \mathbb{P}^2$ be an irreducible sextic curve with at least eight double points (possibly infinitely near). We recall that a point Q is said to be infinitely near to another point P if Q appears as a result of a sequence of blow-ups in a neighborhood of P. By convention, we shall say that P is infinitely near to itself. Let us choose eight such double points with the following convention: if Q is infinitely near to P and Q is chosen, then P has to be chosen as well.
Let us list these eight points as

$$\mathcal{L} := \bigcup_{j=1}^{r} \{P_0^j, P_1^j \ldots, P_{n_j}^j\}, \quad \sum_{j=1}^{r} n_j + 1 = 8, \tag{1}$$

eq-basepts

where for each $j = 1, \ldots, r$, the ordered sequence $\mathcal{L}^j := \{P_0^j, P_1^j, \ldots, P_{n_j}^j\}$ of n_j+1 distinct points satisfies that P_i^j is infinitely near to P_k^j if and only if $i \geq k$. One can consider the set $\mathcal{L}^j$ as a subset of points in the multiplicity tree of $\mathcal{C}$ at P_0^j. One can also consider the restriction of a meromorphic germ $\frac{f}{g} \in \mathcal{O}_{\mathbb{P}^2, P_1^j}$ to P_i^j, denoted by $\frac{f}{g}|_{P_i^j}$, as the difference of multiplicities of the strict transform of f and g at P_i^j. We will say that a divisor $\mathcal{D} \subset \mathbb{P}^2$ **passes through a point** $P_i^j \in \mathcal{L}$ if $D|_{P_i^j} > 0$, where D is a local equation for $\mathcal{D}$ at P_0^j. Thus, the following defines an ideal sheaf in $\mathbb{P}^2$ supported on $\mathcal{L}_0 := \{P_0^1, ..., P_0^r\}$:

$$(\mathcal{I}_{\mathcal{L}})_{P_i^j} := \{g \in \mathcal{O}_{\mathbb{P}^2, P_i^j} \mid g \text{ passes through } P_i^j\}.$$

Since $\dim H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}/\mathcal{I}_{\mathcal{L}}) = 8$ and

$$0 \to \mathcal{I}_{\mathcal{L}}(3) \to \mathcal{O}_{\mathbb{P}^2}(3) \to \mathcal{O}_{\mathbb{P}^2}/\mathcal{I}_{\mathcal{L}}(3) \to 0$$

is an exact sequence, then there is a pencil of cubics $\Gamma \in H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3))$. Since the sextic $\mathcal{C}$ is irreducible, then the set of base points of Γ is finite and, by an argument of multiplicities of intersection, the generic cubic in this pencil is smooth. Let $\sigma\, X \to \mathbb{P}^2$ be the resolution of the pencil obtained by blowing up the nine base points of Γ, that is, the eight points of $\mathcal{L}$ and an extra one, denoted by G (which may be infinitely near to some points in $\mathcal{L}$).

convention-blowup

Convention 2.1. After blowing up a point, we use the same notation for the exceptional divisor as we did for the point. The same convention will be followed for strict transforms. When necessary, we shall specify the surface where the divisor is being considered. For example, the multiplicity of intersection of $\mathcal{D}_1$ and $\mathcal{D}_2$ considered as divisors in $\mathbb{P}^2$ shall be denoted by $(\mathcal{D}_1 \cdot \mathcal{D}_2)^{\mathbb{P}^2}$, whereas the multiplicity of intersection of their strict transforms in X shall be denoted by $(\mathcal{D}_1 \cdot \mathcal{D}_2)^X$.

Note that the surface X is rational. Therefore the group of divisors algebraically equivalent to zero module linear equivalence is trivial, and hence the Neron-Severi group of X, denoted by $NS(X)$, coincides with its Picard group $\mathrm{Pic}(X)$. The linear system $\sigma^*\Gamma$ has a divisor of base components, say

$$B := \sum_{j=1}^{r} \sum_{k=0}^{n_j} p_k^j P_k^j.$$

Hence $\sigma^*\Gamma - B$ defines an elliptic fibration $\pi\ X \to \mathbb{P}^1$ of which G is a section. Let us consider the Mordell-Weil group $MW(X)$ of the sections of π with zero element G. According to [?] one can define an epimorphism

$$\varphi : NS(X) \to MW(X)$$

as follows. Let $\mathcal{D} \in \mathrm{Div}(X)$ be a divisor in X. On each smooth fiber X_ζ, $\mathcal{D}$ can be restricted to a divisor $\mathcal{D}_\zeta$. Using the group law on X_ζ, any divisor $\mathcal{D}_\zeta$ defines, by adding its points, a point $s_\zeta \in X_\zeta$. Since the closure of these elements gives rise to a section and degree zero divisors are sent to G (by Abel's theorem), this map factorizes through $NS(X) = \mathrm{Pic}(X)$.
It is well known that $\mathrm{Pic}(X)$ is freely generated by H (the strict transform of a generic line in $\mathbb{P}^2$), $P_0^j, P_1^j, \ldots, P_{n_j}^j$, $j = 1, \ldots, r$, and G. Moreover, the kernel of φ is freely generated by G, a generic fiber F and the irreducible components of the singular fibers of φ which do not intersect G. After fixing a singular fiber, one can replace F by the irreducible component of this fiber intersecting G. There are two (non-disjoint) kinds of reducible fibers of π: fibers containing exceptional components of σ and fibers coming from reducible elements of the pencil.
Following the construction above, let us denote by $\mu : X \to X$ the involution determined by taking the opposite on each fiber X_ζ. Then, one has the following

lemma-involution **Lemma 2.2.** *The curve $\mathcal{C}$ does not intersect G and $\mu(\mathcal{C}) = \mathcal{C}$.*

Proof. Note that the divisor $\mathcal{C}$ in X –see the convention above– is linearly equivalent to

$$6H - \sum_{j=1}^{r} \sum_{k=1}^{n_j} 2(k+1) P_k^j - \alpha G \tag{2}$$ eq-alpha

for a certain positive integer α. Analogously, the strict transform of a generic fiber F in X is linearly equivalent to

$$3H - \sum_{j=1}^{r} \sum_{k=0}^{n_j} (k+1) P_k^j.$$

Hence $\mathcal{C}$ is a linear combination of F and G. Since both F and G are in $\ker\varphi$, then $\mathcal{C} \in \ker\varphi$ also.

By Bezout theorem $(\mathcal{C}\cdot F)^{\mathbb{P}^2}=18$. Since the intersection multiplicity at the base points of the pencil Γ is equal to $16+\varepsilon$, where $\varepsilon=0,1$ or 2, then $(\mathcal{C}\cdot F)^X=2-\varepsilon$. By the previous paragraph $\mathcal{C}\in\ker\varphi$, then the points of $\mathcal{C}$ in any fiber add up to zero. Hence, if $\varepsilon=1$ then G has to be a component of F, but F is a strict transform and G is an exceptional divisor. If $\varepsilon=2$ then $\mathcal{C}$ intersects G, and hence the multiplicity of intersection of $\mathcal{C}$ with at least one fiber will be greater than 18, which is not possible. Hence, $\varepsilon=0$ and also $\alpha=0$ in eq-alpha (2). Therefore, the two points of intersection of $\mathcal{C}$ with F are opposite each other and hence $\mu(\mathcal{C})=\mathcal{C}$. $\square$

lemma-Yrational

Lemma 2.3. *Let $\tau: X\to Y=X/\mu$ be the quotient induced by μ from the above construction. Then the resulting surface Y is smooth and rational.*

Proof. Y is rational since X is and τ is onto. Moreover, since the fixed points of μ are a disjoint union of smooth irreducible curves – see figures 1-10 – then Y is also smooth. $\square$

corollary-XYZ

Corollary 2.4. *Let $\tilde{\mathcal{C}}\subset X$ be a reducible curve such that:*

- *$\mathcal{C}\subset\tilde{\mathcal{C}}$;*
- *$\mu(\tilde{\mathcal{C}})=\tilde{\mathcal{C}}$;*
- *$\tau_|: X\setminus\tilde{\mathcal{C}}\to Y\setminus\tau(\tilde{\mathcal{C}})$ is an unramified 2-fold covering and*
- *there exists a sequence of blowing-downs $\kappa: Y\to Z\cong\mathbb{P}^2$ which is an isomorphism on $Y\setminus\tau(\tilde{\mathcal{C}})$.*

Then, there exists a curve $\mathcal{C}$ with the chosen eight double points (and may be some added conditions) if and only if the (reducible) curve $\kappa(\tau(\tilde{\mathcal{C}}))\subset Z$ exists.

Because of the large number of cases, in this paper we restrict our attention to the case where $\mathcal{L}=\{P_0^1,\ldots,P_7^1\}$ in eq-basepts (1), i.e., $\mathcal{C}$ has a singular point of type $\mathbb{A}_n$ with $n\geq 15$. We will also ask $\mathcal{C}$ to have some other singular points of type $\mathbb{A}_k$ so that its global Milnor number is greater than or equal to 18.

3 Settings and preliminaries

section-settings

Given $\mathcal{D}_1,\mathcal{D}_2\in\mathrm{Div}(X)$ two divisors on the rational surface X, we will use the following notation:

- $\mathcal{D}_1=\mathcal{D}_2$ if they are equal as divisors,
- $\mathcal{D}_1\sim\mathcal{D}_2$ if they are linearly equivalent,
- $\mathcal{D}_1\overset{MW}{=}\mathcal{D}_2$ if they are equal in the Mordell-Weil group.

Let us recall some definitions of transforms of curves under rational morphisms. Let X and X' be smooth surfaces and $f : X \to X'$ a birational morphism. Let $\mathcal{D} = \sum_{i=1}^r m_i \mathcal{D}_i$ be an effective divisor in $f(X)$, where $\mathcal{D}_1, \ldots, \mathcal{D}_r$ are irreducible curves; the **total transform** $f^*\mathcal{D} := \sum_{i=1}^r m_i f^*\mathcal{D}_i$ is obtained as the pull-back of $\mathcal{D}$. Let $E_1, \ldots, E_s$ be the exceptional components of f on X. It is well known that there exist $n_1, \ldots, n_s \in \mathbb{Z}_{\geq 0}$ positive integers such that

$$f^*\mathcal{D} = \sum_{i=1}^r m_i \mathcal{D}_i + \sum_{j=1}^s n_j E_j.$$

The **strict transform** of $\mathcal{D}$ is obtained as

$$f^*\mathcal{D} - \sum_{j=1}^s n_j E_j.$$

Let us consider a pencil Γ in X' without base components, whose resolution is given by f. Let us denote by $f^*\Gamma$ its pull-back. Then the base locus of the linear system $f^*\Gamma$ is an effective divisor given by a linear combination of the exceptional components $E_1, \ldots, E_s$, say $B = \sum_{j=1}^s b_j E_j$. The **virtual transform** of $\mathcal{D}$ with respect to Γ can be defined as

$$\check{\mathcal{D}} := \sum_{i=1}^r m_i \mathcal{D}_i + \sum_{j=1}^k (n_j - b_j) E_j = f^*\mathcal{D} - B.$$

One has the following

proposition-miembrohaz **Proposition 3.1.** *Let $\mathcal{C} \in \operatorname{Div}(\mathbb{P}^2)$ be an effective divisor and let F be a general member of a pencil Γ as above. Suppose that $|F - \mathcal{C}|$ is non-empty and that $\check{\mathcal{C}}$ is effective. Then $\mathcal{C}$ is a member of Γ.*

Proof. The result is a consequence of the following facts:

i) $\check{\mathcal{C}}^2 = \mathcal{C}^2 - F^2$,

ii) $F^2 \geq \mathcal{C}^2$,

iii) $\check{F} \cdot \check{\mathcal{C}} \geq 0$, since $\check{\mathcal{C}}$ is effective.

Using *iii)* and the fact that $\check{F}^2 = 0$ one has

$$0 \leq (F - \mathcal{C})^2 = (\check{F} - B - \check{\mathcal{C}} + B)^2 = \check{\mathcal{C}}^2 - 2\check{F} \cdot \check{\mathcal{C}}.$$

Hence *i)* and *ii)* imply that $\check{\mathcal{C}}^2 = 0$ and therefore $\check{\mathcal{C}} \sim k\check{F}$. Note, however, that $F^2 = \mathcal{C}^2 = k^2 F^2$ and that $\mathcal{C}$ is effective, hence $k = 1$. □

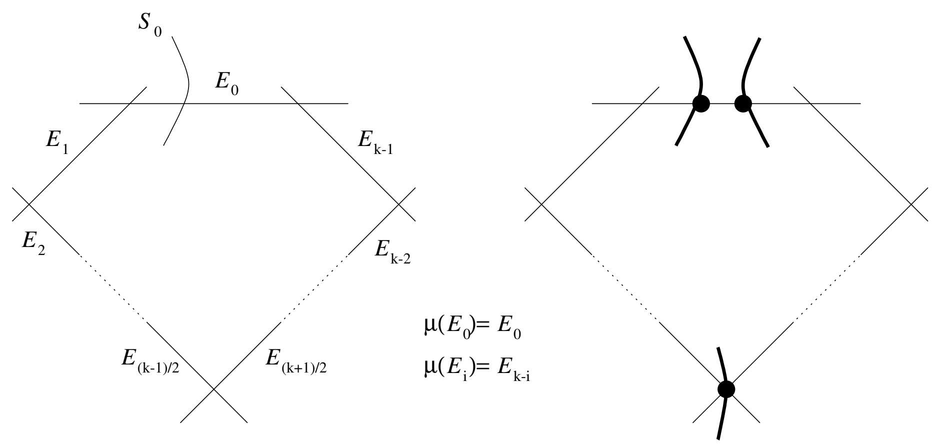

Figure 1: fiber I_k for k odd. fig-ik

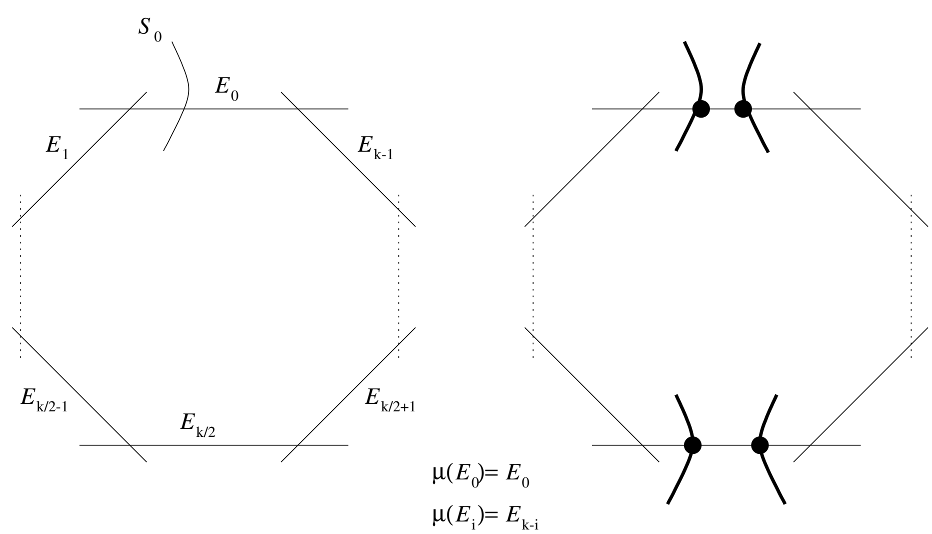

Figure 2: fiber I_k for k even. fig-i2k

Suppose the linear system Γ^* considered above defines an elliptic fibration $X \to \mathbb{P}^1$, that is, we assume the general fiber F is a smooth elliptic curve. Such a fibration can only have a certain type of special fibers, classified by Kodaira in [?]. The following table describes the possible fibers and the action of the involution μ –see Lemma 2.2– on such fibers. Thick curves or dots mean fixed points by μ. The divisor S_0 represents the section and the added divisors that do not belong to the fiber represent the torsion 2 curves and the 0-section.

As mentioned at the end of the previous section we will only deal with the case where $\mathcal{L} = \mathcal{L}^1$, that is, where the eight points chosen are infinitely near to the first one. In order to simplify notation we denote by $E = E_1$ the chosen point in $\mathbb{P}^2$ and by $E_2, \ldots, E_8$ the remaining points in $\mathcal{L}$. The ninth base point in the pencil of cubics passing through $\mathcal{L}$ is denoted by G. The following two divisors will be

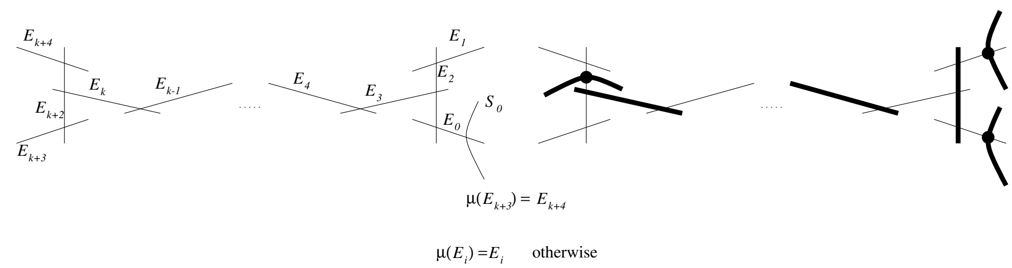

Figure 3: fiber I_k^* for k odd. `fig-isk`

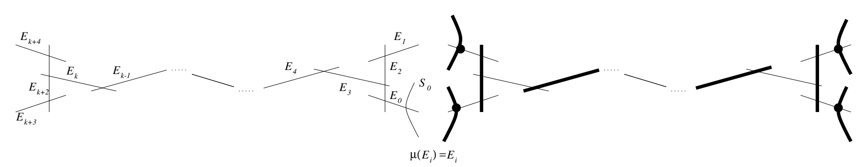

Figure 4: fiber I_k^* for k even. `fig-is2k`

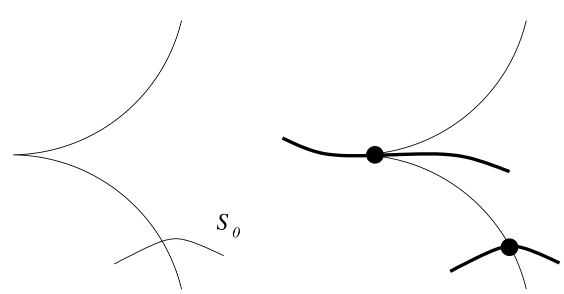

Figure 5: fiber II. `fig-ii`

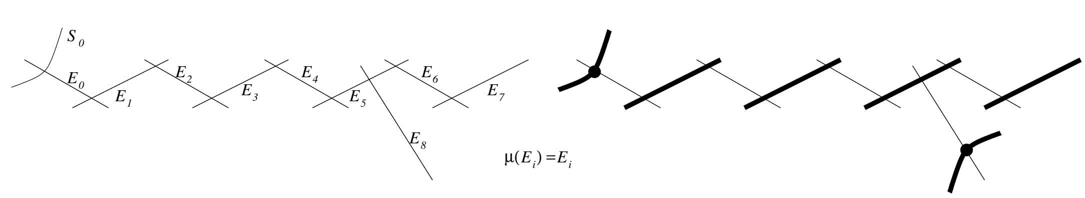

Figure 6: fiber II^*. `fig-iis`

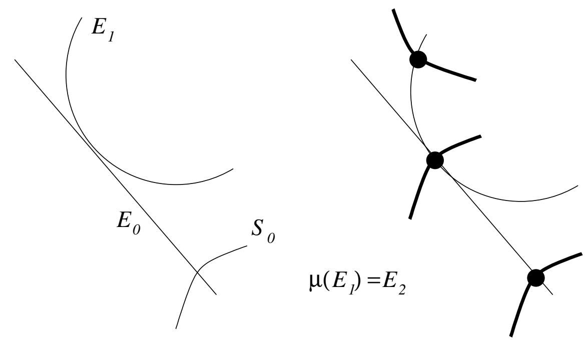

Figure 7: fiber III. fig-iii

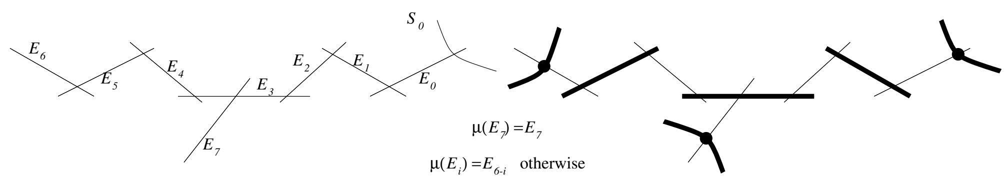

Figure 8: fiber III^*. fig-iiis

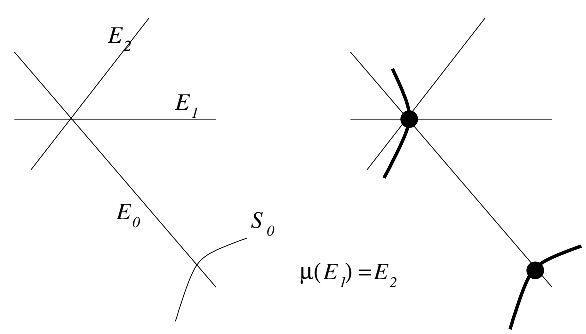

Figure 9: fiber IV. fig-iv

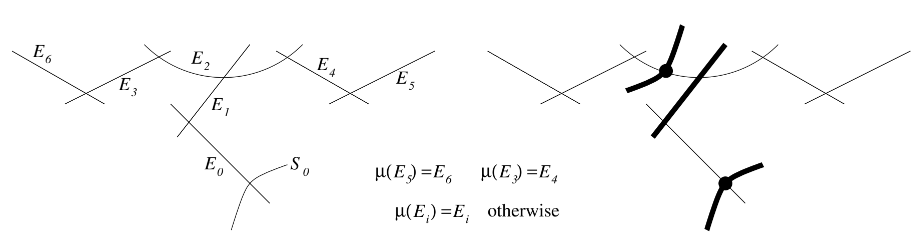

Figure 10: fiber IV^*. fig-ivs

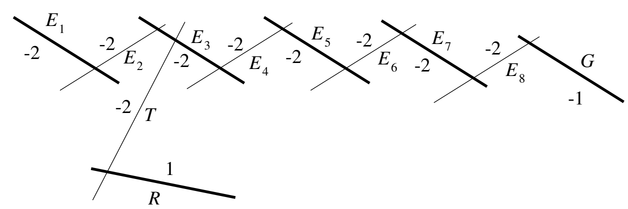

Figure 11: curves in X. fig-1x

frequently used: the tangent line to $\mathcal{C}$ at E, denoted by T and the conic of highest contact with $\mathcal{C}$ at E, denoted by Q. We will distinguish several cases, depending on the values of $(\mathcal{C}\cdot T)_E$ and $(\mathcal{C}\cdot Q)_E$ and on the position of G. In these cases the starting point will always be the pencil Γ of cubics passing through $\mathcal{L}$ as defined in section 1. Also, once and for all, the map $\sigma : X \to \mathbb{P}^2$ denotes the resolution of Γ; the map $\pi : X \to \mathbb{P}^1$ denotes the resulting elliptic fibration; μ denotes the involution considered above; Y the quotient of X by μ; τ the projection from X onto Y; Z the projective plane obtained after a sequence of blowing-downs from Y and R the divisor of X containing the points of order 2, that is, points not in G, fixed by μ.

4 Case $(\mathcal{C}\cdot T)_E = 6$

section-casect6

This is the most special and simple case and it will help to understand the general case. It is easily seen that $Q = 2T$. The virtual transform of $3T$ with respect to Γ is not only an effective divisor, but in fact passes through $\mathcal{L}$. In particular, by Proposition (proposition-miembrohaz 3.1), $3T$ is a member of Γ. Let F be a generic member of Γ, then $(3T\cdot F)_E = 3(T\cdot F)_E = 9$ and hence, the ninth base point of Γ, G, is also infinitely near to E.

The fiber F_0 of π containing T is the virtual transform of $3T$ by σ with respect to Γ. One has

$$F_0 \sim 3T + 2\sum_{j=1}^{3} jE_j + \sum_{j=4}^{8}(9-j)E_j$$

and F_0 is a singular fiber of π of type II^*. Note that F_0 is the only reducible fiber of π since $E_1,\ldots,E_8$ are contained in F_0 and $3T$ is the only reducible cubic of the pencil. Let us consider the group $MW(X)$ of sections of π with zero element G. The kernel of the epimorphism φ is generated by G, $E_1,\ldots,E_8$ and T, where $T \sim H - E_1 - 2E_2 - 3\sum_{j=1}^{9} E_j$. Thus, $MW(X)$ is trivial.

The irreducible component of F_0 intersecting G is E_8. According to Figure fig-iis 6, R intersects F_0 only at T (and transversally). Note also that R intersects the generic fiber in X at three points. From these data one can deduce that R is the strict transform of a line not containing E.

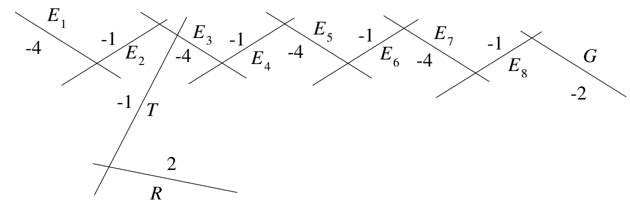

Figure 12: curves in Y. fig-1y

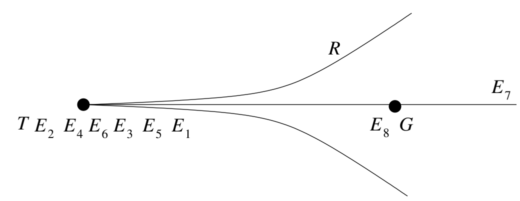

Figure 13: curves in Z. fig-1z

Note that $\tilde{\mathcal{C}} := E_1 \cup \cdots \cup E_8 \cup G \cup T \cup R \cup \mathcal{C}$ verifies the conditions in Corollary (2.4). Figure 12 shows the images of $\tilde{\mathcal{C}} \setminus \mathcal{C}$ in Y.
Y has Euler characteristic equal to 12. Hence, blowing down the following 9 curves $T, E_2, E_4, E_6, E_8, G, E_3, E_5$ and E_1 will produce a surface Z isomorphic to $\mathbb{P}^2$. The image of R is a cuspidal cubic and the image of E_7 is the tangent line to R at the cusp. The contracted components are sent to two points.
In order to decide whether such a configuration exists, we need to study the curve $\mathcal{C}$ is transformed into. Note from the sequence of blowing-ups, that $\mathcal{C}$ should have intersection number 2 with E_8 (and 0 with $E_1, \ldots, E_7$). By Lemma (2.2) $\mathcal{C}$ cannot pass through the last base point G. According to Figure 6, the action of μ on E_8 fixes only two points which are $E_8 \cap E_7$ and $E_8 \cap G$. Since $\mathcal{C}$ is invariant and does not pass through these points, $\mathcal{C}$ must intersect E_8 in X at *generic* points and then $\mathcal{C}$ has an $\mathbb{A}_{15}$-singularity at E. By Bezout theorem, it has to intersect R with multiplicity 6.
The image of $\mathcal{C}$ in Z (denoted again by $\mathcal{C}$) is a conic tangent to E_7 (not at the cusp of R).
Let $Q_1, \ldots, Q_k$ be the points in $\mathcal{C} \cap R \subset Z$, with contact orders $m_0 \geq \cdots \geq m_k$, where $m_0 + \cdots + m_k = 6$. Such a situation will be denoted by $[m_0, ..., m_k]$. Note that the curve $\mathcal{C}$ is irreducible if and only if there are at least two odd m_j's and that the inverse image of Q_j is a singular point on $\mathcal{C}$ of type $\mathbb{A}_{m_j-1}$.
These are the possible configurations of singular points for a curve with total Milnor number 18 or greater:

1. $[5,1]$, $(\mathbb{A}_{15} + \mathbb{A}_4)$;
2. $[3,3]$, $(\mathbb{A}_{15} + 2\mathbb{A}_2)$;
3. $[4,1,1]$, $(\mathbb{A}_{15} + \mathbb{A}_3)$;

4. $[3,2,1]$, $(\mathbb{A}_{15}+\mathbb{A}_2+\mathbb{A}_1)$.

The next step consists of giving suitable equations to the components of $\tilde{\mathcal{C}}$ in Z and decide which configuration of singular points does exist. After a change of coordinates in Z, one can assume that R has equation $x^2z-y^3=0$ and that the line E_7 has equation $x=0$. Note that the parametrization $\mathbb{C}\to\operatorname{Reg}(R)$, $t\mapsto[1:ut:(ut)^3]$ is an isomorphism of groups for any $u\in\mathbb{C}^*$, where $\mathbb{C}$ has the additive group structure and $\operatorname{Reg}(R)$ has the geometric group structure with zero element the inflection point of R (the sum of three elements is equal to zero if and only if they lie in the same line).
Let t_0 be the parameter of the point of highest contact (at least three) between R and $\mathcal{C}$. Then, there exist $a,b,c\in\mathbb{C}$ such that the parameters for the other three intersection points are the roots of $t^3+at^2+bt+c=0$. Since $\mathcal{C}$ does not pass through the cusp and is tangent to E_7 one can assume that its equation is:

$$a_{2,0}x^2+a_{1,1}xy+\frac{1}{4}a_{0,1}^2y^2+a_{1,0}xz+a_{0,1}yz+z^2$$

Replacing (x,y,z) by $(1,t,t^3)$ in this equation one should obtain the polynomial $p(t)=(t-t_0)^3(t^3+at^2+bt+c)$. This way one can verify that $t_0=0$ produces only $[3,1,1,1]$ or $[6]$. Hence t_0 can be assumed to be 1 by changing u properly. Then, the equation of $\mathcal{C}$ becomes

$$\frac{1}{12}\left((-3b^2+60b-144)xy+(b^2-24b+48)x^2+\right.$$
$$\left.(3b^2-36b+108)y^2+12z^2+(12b-72)yz+(-b^2-12b+48)xz\right)$$

and the polynomial $p(t)$ becomes

$$p(t)=(t-1)^3p_1(t)=(t-1)^3(12t^3+36t^2+12bt-b^2+24b-48)$$

Case 1. One should ask $p_1(t)$ to have 1 as a double root. One can check that there is no such solution.
Case 2. There is no solution $p_1(t)=(t-t_1)^3$ with $t_1\neq 1$.
Case 3. The only solution of $p_1(t)=(t-1)p_2(t)$ is $b=36$. In this case

$$p(t)=(t-1)^4(t^2+4t+40)$$

and

$$C(x,y,z)=225\,y^2-156\,yx+30\,yz+40\,x^2-140\,xz+z^2.$$

Case 4. The only solution of $p_1(t)=(t-t_1)^2(t-t_2)$ is $b=\frac{8}{3}$. In this case

$$p(t)=\frac{1}{27}\,(t-1)^3(3t+2)^2(3t+5)$$

and

$$C(x,y,z)=\frac{1}{27}\,(-12\,yx+20\,xz+75\,y^2-20\,x^2+27\,z^2-90\,yz).$$

remark-ratmap1

Remark 4.1. One can also calculate the rational map $\omega = \kappa \circ \tau \circ \sigma^{-1} : \mathbb{P}^2 \dashrightarrow Z$ as follows: the set $\{\omega_1(x,y,z) = 0\}$ is the inverse image of the curve $\{x = 0\}$, that is, E_7. Following the sequence of blowing-ups and blowing-downs of ω, one can check that $\omega^* E_7 = 3T$. Analogously, $\{\omega_2(x,y,z) = 0\}$ is the inverse of $L_2 = \{y = 0\}$, which is a line intersecting R at the cusp and an inflexion point. One can check that $\omega^* L_2 = 2T + \tilde{L}_2$, where $\tilde{L}_2$ is a line. Finally, $\{\omega_3(x,y,z) = 0\}$ is the inverse of $L_3 = \{z = 0\}$, which is the line tangent to R at the inflexion point $L_2 \cap R$. One has that $\omega^* L_3$ is a cubic with a cusp at $R \cap \tilde{L}_2$ and an inflexion point at $T \cap \tilde{L}_2$ having T as tangent line. Using suitable coordinates on $\mathbb{P}^2$ such that $T = \{x = 0\}, \tilde{L}_2 = \{y = 0\}$ and $R = \{z = 0\}$ one has

$$\omega(x,y,z) = [x^3 : x^2 y : xz^2 + y^3].$$

The equation of the curve $\mathbb{A}_{15} + \mathbb{A}_3$ (case 3) is

$$-156\, x^5\, y - 140\, x^4 z^2 - 140\, x^3 y^3 + 225\, x^4 y^2 +$$

$$+40\, x^6 + x^2 z^4 + 2\, xz^2 y^3 + y^6 + 30\, x^3 y z^2 + 30\, x^2 y^4$$

where $\mathbb{A}_{15} = [0 : 0 : 1]$ and $\mathbb{A}_3 = [1 : 1 : 0]$. Finally, the equation of the curve $\mathbb{A}_{15} + \mathbb{A}_2 + \mathbb{A}_1$ (case 4) is

$$-12\, x^5 y + 20\, x^4 z^2 + 20\, x^3 y^3 + 75\, x^4 y^2 - 20\, x^6 +$$

$$+27\, x^2 z^4 + 54\, xz^2 y^3 + 27\, y^6 - 90\, x^3 y z^2 - 90\, x^2 y^4$$

where $\mathbb{A}_{15} = [0 : 0 : 1]$, $\mathbb{A}_2 = [1 : 1 : 0]$ and $\mathbb{A}_1 = [5 : -3 : 0]$.

5 Equations of the General Case

section-equations

The purpose of this section is to provide equations – up to projective change of coordinates – for both the divisor of two-torsion R and the birational map $\omega = \kappa \circ \tau \circ \sigma^{-1} : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ in the most general situation.

We will describe geometrically the birational map ω. Let us denote by $\mathcal{L} := \{P = P_0, P_1, \dots, P_7\}$ the list of base points of the linear system Γ satisfying the conditions of (1). eq-basepts We assume such infinitely close points to be generic, that is, no line passes through the first three, no conic through the first six and the ninth base point is not infinitely close to P. Consider S a nodal cubic passing through $\mathcal{L}$. This can be achieved if the node is at E, one branch is transversal and the other has order of contact seven with F, a general member of Γ. There is a special fiber

$$F_0 = \check{S} \sim S + \sum_{j=1}^{7} E_j$$

of type I_8. After the resolution of the pencil, G and E_8 are sections and $MW(X)$ (group of sections of π with with zero element G) is freely generated by $E_1, \ldots, E_8, G, H$. The kernel of φ is generated by $G, S, E_1, \ldots, E_7$, where

$$S = 3H - \sum_{j=1}^{7}(j+1)E_j - 8E_8 - G.$$

Thus $MW(X)$ is generated by H and E_8 with the relation $3H \overset{MW}{=} 8E_8$, that is, $MW(X)$ is isomorphic to $\mathbb{Z}$. By assumption, the conic Q has five infinitely near points in common with $\mathcal{C}$ at E. Therefore $(Q \cdot F)_E = 5$, where F is a general member of the pencil Γ, and hence, it is a section of π and $Q \overset{MW}{=} 2H - 5E_8$. Analogously, since $(T \cdot F)_E = 2$, T is a section and $T \overset{MW}{=} H - 2E_8$. Calculating determinants one can see that both Q and T are generators of $MW(X)$, in fact

$$E_8 \overset{MW}{=} 3Q, \quad H \overset{MW}{=} 8Q, \quad \text{and} \quad T \overset{MW}{=} 2Q.$$

According to Figure 2, $\mu(E_j) = E_{8-j}$, $j = 1, \ldots, 7$ and $\mu(S) = S$, that is, S and E_4 are globally fixed. Let $E_8' = \mu(E_8)$ be the image of E_8 by μ. Since $E_8 \cap S = E_8 \cap G = E_8 \cap E_2 = \cdots = E_8 \cap E_7 = \emptyset$ and E_8 intersects E_1 transversally, then so behaves E_8' with respect to their images by μ. Note also that since E_8 is a section and rational, then so is E_8'.

notation-multseq **Notation 5.1.** We say that a curve of degree d is of type $E(m_1, \ldots, m_8)$ if $(m_1, \ldots, m_8)$ is the multiplicity sequence of the strict transforms of the curve at the base points over E.

Let us denote by a the multiplicity of intersection of E_8' with E_8 and by d the degree of $E_8' \subset \mathbb{P}^2$. With this notation $E_8' \subset \mathbb{P}^2$ is of type $E(a+1, a, \ldots, a)$. Since $E_8' \subset X$ is a section and $E_8' \cap G = \emptyset$, then

$$(E_8' \cdot F)_E = (a+1) + 7a = 3d - 1. \tag{3}$$ eq1

Moreover, since $E_8' \subset X$ is rational and smooth, then the genus formula on $\mathbb{P}^2$ gives:

$$(d-1)(d-2) = (a+1)a + 7a(a-1). \tag{4}$$ eq2

The only possible solution of (3) and (4) is $d = 6$, $a = 2$.
From Figure 2, the curve R intersects E_4 transversally at two points and S at one point. Note that $E_8' \cap E_8$ consists of two distinct points fixed by μ. Hence R passes through $E_8' \cap E_8$ transversally to E_8' and E_8. Therefore R is of type $E(4,4,4,4,2,2,2,2)$ and hence has degree 9. Let us denote by R_0 a component of R passing through E_4, then R_0 is of type $E(2+\ell_1, 2+\ell_1, 2+\ell_1, 2+\ell_1, \ell_2, \ell_2, \ell_2, \ell_2)$ where $\ell_1, \ell_2 \in \{0,1,2\}$. Moreover,

$$9 \cdot 3 = (R_0 \cdot S)^{\mathbb{P}^2} = 5\,(2+\ell_1) + 3\,\ell_2 + (R_0 \cdot S)^{Y}.$$

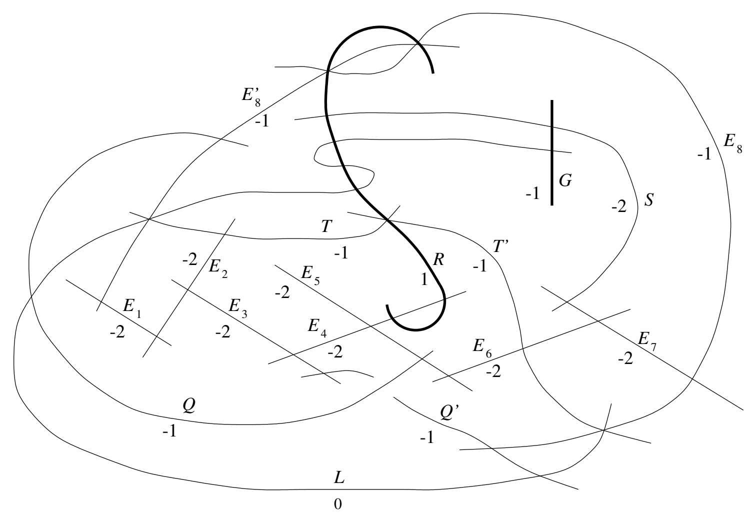

Figure 14: curves in X. fig-4x

Hence $\ell_1 = \ell_2 = 2$ and $(R_0 \cdot S)^Y = 1$. that is, $R_0 = R$ is irreducible. Note also that $(E_8' \cdot Q)^{\mathbb{P}^2} = 12 = (E_8' \cdot Q)_E^{\mathbb{P}^2} + (E_8' \cdot Q)^X = 11 + \ell$, hence $(E_8' \cdot Q)^X = 1$ and therefore $(E_8 \cdot \mu(Q))^X = 1$. Let us denote by $Q' = \mu(Q)$ the image of Q under μ; then Q' intersects $\mu(E_5) = E_3$ and $\mu(E_8') = E_8$. Therefore Q' is of type $E(2, 2, 2, 1, \ldots, 1)$ and its self-intersection is -1. Thus $Q' \subset \mathbb{P}^2$ is a rational quartic. Finally, note that $\pi_{|R}$ is three to one.
Let us consider

$$\tilde{\mathcal{C}} := \mathcal{C} \cup S \cup \bigcup_{j=1}^{8} E_j \cup E_8' \cup R \cup G \cup Q \cup Q'$$

satisfying the conditions of Corollary (2.4). The curves in $\tilde{\mathcal{C}} \setminus \mathcal{C}$ are shown in Figure 14 and their images in Y appear in Figure 15.
By Lemma (2.3), Y is a smooth rational surface and its Euler characteristic is 8. So, after blowing down the following curves: S, Q, E_1, E_2 and E_4 one obtains a surface Z isomorphic to $\mathbb{P}^2$. The image of R is a nodal cubic, where E_4 is its singular point and S an inflexion point. The curve G on Z is the tangent line to R at S. The image of E_3 is the line joining S and E_4; this line contains also the point Q. Finally, the image of E_8 is a smooth conic tangent to G at S; passing through Q and tangent to R at two points.
Up to projective change of coordinates, one can fix the first seven points on the list $\mathcal{L}$. This can be achieved by fixing the following data:

1. the point P, as $[0 : 0 : 1]$,

2. the point P_1 by fixing, for instance, the line $\ell = \{x = 0\}$ as the line passing through P_1, that is, the common tangent direction of Γ at P.

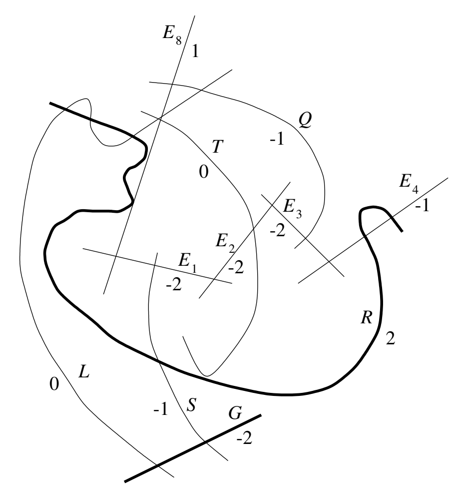

Figure 15: curves in Y. fig-4y

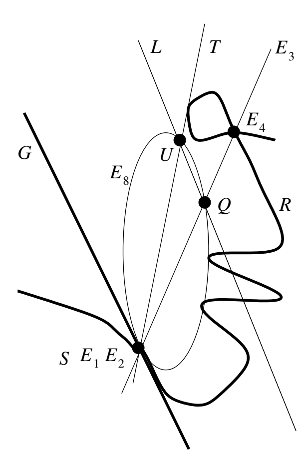

Figure 16: curves in Z. fig-4z

3. the points P_2, P_3 and P_4 by fixing the conic Q of maximal contact with Γ. Note that such conic has to pass through P and P_1, for instance $Q = \{xz - y^2 = 0\}$.

4. the remaining points P_5 and P_6.

Note that the ideal $\mathcal{I} \subset \mathbb{C}\{x, y\}$ defining the germs that pass throught the points $P, P_1, ..., P_6$ is generated by $(x-y^2-y^5, y^7)$. Therefore, the complex number $u \in \mathbb{C}$ in $\varphi_u \equiv x - y^2 - y^5 - uy^7$ parametrizes the last infinitely close point P_7 by means of the intersection of the strict transform of φ_u and E_7.
On the other hand, by a suitable selection of coordinates on Z, one can define R as the set of zeroes of the nodal cubic $xyz + x^3 - y^3 = 0$, where $[0 : 0 : 1]$ is the nodal point and, for instance, $[1 : 1 : 0]$ is the inflection point S. Then

$$G = \{3(x - y) + z = 0\}, \quad \text{and} \quad E_3 = \{x - y = 0\}.$$

Note that the parametrization

$$\begin{array}{rccc} \varphi: & \mathbb{C} & \to & R \\ & t & \mapsto & [t : t^2 : t^3 - 1]. \end{array}$$

induces a group isomorphism, by restriction to $\mathbb{C}^*$, between the multiplicative group $\mathbb{C}^*$ and the set of non-singular points of R endowed with the geometric group structure of zero element S. We recall that G corresponds to the tangent line of R at S and E_3 corresponds to the line joining the points S and the singular point of R. The equation of E_8 is given by a homogeneous polynomial of degree 2, say $f_{E_8}(x, y, z)$. Since the nodal point is not in E_8 one can assume that the coefficient of z^2 in f is 1. The composition $p(t) := (f \circ \varphi)(t)$ is, hence, a one-variable polynomial of degree 6. Moreover, p has the following factorization

$$p(t) = (t - 1)^2(t - t_1)^2(t - t_2)^2$$

where the values 1, t_1 and t_2 correspond to the three points of contact two. According to the group structure of $\operatorname{Reg}(R)$ one has $(t_1t_2)^2 = 1$. The case $t_1t_2 = 1$ forces E_8 to be a double line; thus, one can suppose $t_1t_2 = -1$. Then:

$$p(t) = (t - 1)^2(t - t_1)^2(t + \frac{1}{t_1})^2, \quad t_1 \neq 1,$$

or

$$p(t) = (t - 1)^2(t^2 - wt - 1)^2, \quad w := t_1 - \frac{1}{t_1} \neq 0.$$

This condition determines f_{E_8} as

$$z^2+(2-2w)zx+(-2-2w)zy+(-6w+w^2-3)x^2+(-2w^2+6)xy+(-3+6w+w^2)y^2.$$

We will describe two more curves in Z that will be required to calculate the morphism φ. There are two lines in the pencil of lines through S that intersect R non transversaly, namely, G and a T. Using this fact one can easily find that

$$T = \{x - y - z = 0\}.$$

Note that $T \cap E_8 = \{S, U\}$ as denoted in figure 15. We will denote by L the line in Z passing through U and Q. One can easily check that

$$L = \{(w^2 - 8w - 4)x - (w^2 + 8w - 4)y + 4z = 0\}.$$

Using the lines T, E_3 and L, the fact that

$$\begin{array}{rcl} \varphi^{-1}T & = & T + T' + S \\ \varphi^{-1}E_3 & = & Q + Q' + S \\ \varphi^{-1}L & = & Q + Q' + L \\ \varphi^{-1}E_8 & = & Q + Q' + E_8' + 2S \\ \varphi^{-1}R & = & 2R + 3S \end{array}$$

and the equations of $T, T', Q, Q', E_8', S, L, R$ in X one can calculate φ and also deduce that the parameters u and w are linked by the equation $16u^3 + w^2 = 0$. Using a new parameter v such that $u = -v^2$ and $w = 4v^3$ one can give all the equations in terms of v. For instance in Z one has

$$\begin{array}{rcl} E_8 & = & \{z^2 + (16v^6 - 24v^3 - 3)x^2 + (16v^6 + 24v^3 - 3)y^2 - \\ & & -(32v^6 - 6)xy - (8v^3 - 2)zx - (8v^3 + 2)zy = 0\} \end{array}$$

and in X one has

$$\begin{array}{rcl} Q' & = & \{v^2z^2x^2 + zx^2y - 2v^2zxy^2 - x^4 - xy^3 + v^2y^4 = 0\} \\ & & \\ T' & = & \{v^2z^3x^2 + z^2x^2y - 2v^2z^2xy^2 - zx^4 - \\ & & 2v^2zx^3y - 2zxy^3 + v^2zy^4 + v^2x^5 + x^3y^2 + 2v^2x^2y^3 + y^5 = 0\} \\ & & \\ R & = & \{v^2z^5x^4 - v^4z^4x^5 + z^4x^4y - 4v^2z^4x^3y^2 - \\ & & -z^3x^6 - 4v^2z^3x^5y + 4v^4z^3x^4y^2 - 4z^3x^3y^3 + 6v^2z^3x^2y^4 + \\ & & +3v^2z^2x^7 + 2z^2x^5y^2 + 12v^2z^2x^4y^3 - 6v^4z^2x^3y^4 + 6z^2x^2y^5 - \\ & & -4v^2z^2xy^6 + 2zx^7y - 6v^2zx^6y^2 - zx^4y^4 - 12v^2zx^3y^5 + 4v^4zx^2y^6 - \\ & & -4zxy^7 + v^2zy^8 - x^9 - 2x^6y^3 + 3v^2x^5y^4 + 4v^2x^2y^7 - v^4xy^8 + y^9 = 0\} \\ & & \\ L & = & \{-v^2z^2x - v^4zx^2 + (-1 + v^6)zxy + v^2zy^2 + \\ & & +(1 - v^6)x^3 + v^2x^2y + v^4xy^2 + (1 - v^6)y^3 = 0\} \end{array}$$

The curve $\mathcal{C} \subset Z$ must be an irreducible cubic with the following two features:

1. $\mathcal{C}$ has a double point at $Q = E_8 \cap E_3 \subset Z$.

2. $S \in Z$ is an inflection point of $\mathcal{C}$ and G is the tangent line to $\mathcal{C}$ at S.

Therefore one has the following data:

1. $(\mathcal{C} \cdot E_8)^Z_S = 2$, $(\mathcal{C} \cdot E_8)^Z_Q \geq 2$ and $(\mathcal{C} \cdot E_8)^Z = 6$. Hence, in general there will be two other points of intersection outside $\{Q, S\}$: this situation produces a singular point of type $\mathbb{A}_{15}$ (note that this also includes the possibility $(\mathcal{C} \cdot E_8)^Z_Q = 3$ and only one intersection point outside $\{Q, S\}$). If there is only one additional intersection point or $(\mathcal{C} \cdot E_8)^Z_Q = 4$, and it is not in R, the induced singular point is of type $\mathbb{A}_{16}$.

2. – $(\mathcal{C} \cdot R)^Z_S \varepsilon 3$ and $\mathcal{C} \cdot E_8 = 9$. In the general case, we have six other points of intersection. If at one *generic* point we have an intersection order equal to r, it produces a singular point of type $\mathbb{A}_{r-1}$.

3. – It may happen that $\mathcal{C}$, E_8 and R have common points. If S is such a common point, and $(\mathcal{C} \cdot R)_S = 3 + r$, it produces a singular point of type $\mathbb{A}_r$. If it is one of the other points of intersection of E_8 and R, say A, and $(R \cdot E_8)_A = 2$, $(\mathcal{C} \cdot R)^Z_A = s$, then it produces a singular point of type $\mathbb{A}_{15+s}$.

Let us describe the strategy as in the previous section. One starts by considering the pencil of cubics generated by $\mathcal{C}$ and R. It is easily seen that G is a component of a fiber $G + B$, where B is a conic which intersects R in the same manner as $\mathcal{C}$ does, except that $(R \cdot B)_S = (R \cdot \mathcal{C})_S - 3$. Namely, let $q(x, y, z) = 0$ be the equation of $\mathcal{C}$ unitary in z^3 and let $b(x, y, z) = 0$ be the equation of B unitary in z^2. If $g(t) := b(t, t^2, t^3 - 1)$, then:

$$q(t, t^2, t^3 - 1) = g(t)(t - 1)^3.$$

This property defines B. One can consider $\mathcal{C}$ as the element in the pencil generated by R and $G + B$ passing through Q. There is an additional condition in this construction: $\mathcal{C}$ must have a double point at Q.

We list the possibilities for the singularities of the sextic curve with global Milnor number at least 18. We classify them by means of the decomposition of $g(t)$. We take into account that the independent term of g is equal to 1. We note also that one should have roots of odd multiplicity in order to avoid reducible curves.

Case $g(t) = (t - s)^4(t^2 - ut + s^{-4})$, $(2us^2 - 1)(2us^2 + 1) \neq 0$, $s \in \mathbb{C}^*$. *One obtains $\mathbb{A}_{15} + \mathbb{A}_3$ for $s \in \mathbb{C}^* \setminus \{(s^8 - 1)(s^3 - 1)(s^4 - 2s^3 + 4s^2 - 2s + 1)(s^4 - 2s^3 - 2s^2 - 2s + 1) \neq 0\}$. Equations are very long and we will write down them only for $\mu = 19$.*

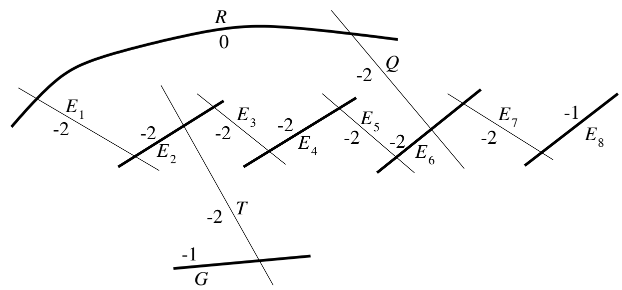

Figure 17: curves in X. fig-2x

6 Case $(\mathcal{C} \cdot T)_E = 4$, $(\mathcal{C} \cdot Q)_E = 12$

section-casect4cq12

Note that in this case the cubic $Q+T$ passes exactly through eight base points of the pencil, hence $Q+T$ is a member of Γ and the ninth base point G belongs to T but is not infinitely near to E. The virtual transform F_0 of $Q+T$ is

$$(Q+T)\check{} = F_0 = Q + T + E_1 + 2\sum_{j=2}^{6} E_j + E_7,$$

hence F_0 is of type I_4^*. Note that F_0 is the only reducible fiber of π. The Picard group of X is freely generated by $H, E_1, \ldots, E_8, G$ and one has:

$$T \overset{MW}{=} H - G - E_1 - 2\sum_{j=1}^{8} E_j, \quad Q \overset{MW}{=} 2H - \sum_{j=1}^{6} jE_j + 6E_7 + 6E_8.$$

Then $MW(X)$ is generated by $\mathrm{Pic}(X)$ with relations

$$G \overset{MW}{=} E_1 \overset{MW}{=} E_2 \overset{MW}{=} \ldots \overset{MW}{=} E_7 \overset{MW}{=} T \overset{MW}{=} Q \overset{MW}{=} 0,$$

that is,

$$G \overset{MW}{=} E_1 \overset{MW}{=} E_2 \overset{MW}{=} \ldots \overset{MW}{=} E_7 \overset{MW}{=} H \overset{MW}{=} 0, \quad 2E_8 \overset{MW}{=} 0.$$

Then $MW(X)$ is cyclic of order 2 with elements G, E_8. According to Figure fig-is2k 4, all the irreducible components of F_0 are globally preserved by μ and E_6, E_4, E_2 are pointwise fixed. Also, the 2-torsion points of π form at least two irreducible components, one of them being E_8. Let us denote by R the remaining components, then R intersects the smooth part of F_0 at E_1 and Q. Checking multiplicities of intersection of the component of R passing through E_1 with Q, one concludes that R is in fact a line passing through E and hence, it verifies that $\pi_{|R}$ is $2:1$.

Note that $\tilde{\mathcal{C}} := E_1 \cup \cdots \cup E_8 \cup G \cup Q \cup T \cup R \cup \mathcal{C}$ verifies the conditions in Corollary (lemma-involution 2.2). Figure fig-i2k 2 shows the images of $\tilde{\mathcal{C}} \setminus \mathcal{C}$ in Y.

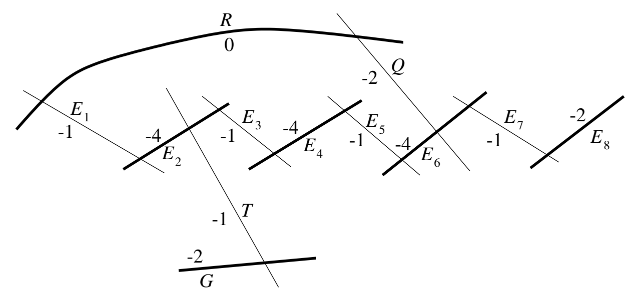

Figure 18: curves in Y. fig-2y

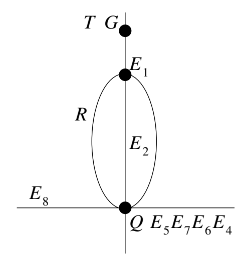

Figure 19: curves in Z. fig-2z

By Lemma (2.3), Y is a smooth rational surface with Euler characteristic equal to 12. So, blowing-down the following 9 curves: $T, Q, E_1, E_3, E_5, E_7, G, E_6$ and E_4 will produce a surface Z isomorphic to $\mathbb{P}^2$. The image of R in Z is a smooth conic, the image of E_8 is a tangent line to R at a point and E_2 becomes a line through the point of tangency. The contracted components are sent to three points.

In order to decide whether such a configuration exists, one needs to determine the image of $\mathcal{C}$ on Z. Note that $(\mathcal{C} \cdot E_8)^Y = 2$, $(\mathcal{C} \cdot E_i)^Y = 0$ for $i = 1, \ldots, 7$ and $(\mathcal{C} \cdot F)^X_{E_8} = 4$ where F is a general cubic of the pencil. By Lemma (2.2), $\mathcal{C}$ does not intersect G. According to Figure 6, E_8 is pointwise fixed and since $\mathcal{C} \overset{MW}{=} 0$ then $\mathcal{C}$ is globally preserved, hence $\mathcal{C}$ must intersect E_8 in X at either two distinct points ($\mathbb{A}_{15}$) or at a node ($\mathbb{A}_{17}$), (the cases of tangency ($\mathbb{A}_{19}$) or cusp are this way excluded). Note also that $(\mathcal{C} \cdot R)^Y = (\mathcal{C} \cdot R)^Z = 4$ and $(\mathcal{C} \cdot T)^Y = 2$ (the latter are opposite each other). The result in Z is very simple: $\mathcal{C}$ must be a conic tangent to E_2 at the image of G (or T). It could happen that $\mathcal{C}$ is tangent to E_8 (producing $\mathbb{A}_{17}$) or not (producing $\mathbb{A}_{15}$). The first case will be denoted by $\epsilon = 2$ whereas the second by $\varepsilon = 1$. The other singular points of $\mathcal{C} \subset \mathbb{P}^2$ should come from the intersection points of $\mathcal{C}$ with R in Z. Let us denote such points by $P_1, \ldots, P_k$, where $(\mathcal{C} \cap R)^Z_{P_j} = m_j$, $m_1 \geq \ldots, \geq m_k$ and $\sum_{j=1}^k m_j = 4$. Such a configuration for $\mathcal{C}$ will be denoted by $[\varepsilon, m_1, \ldots, m_k]$. Note that the inverse image of P_j is a singular point of $\mathcal{C}$ of type $\mathbb{A}_{m_j-1}$, $j = 1, \ldots, k$, where $\mathbb{A}_0$ means the type of a smooth point. These are the possible configurations of singular points for a curve with total Milnor number 18 or greater:

(a) $[2,2,1,1]$, $[2,3,1]$, $(\mathbb{A}_{17}+\mathbb{A}_n)$, $n=1,2$.
(b) $[1,4]$, $(\mathbb{A}_{15}+\mathbb{A}_3)$;

Next, we give equations for the arrangement $\tilde{\mathcal{C}}$ in Z. By fixing three points in the projective plane Z, one can assume that

$$E_8=\{x=0\},\quad E_2=\{y=0\},\quad R=\{xz-y^2\}=0.$$

Fixing a fourth point in Z, one can also assume that $[1:1:1]\in\mathcal{C}\cap R$ has the highest contact order.

Case 2. We look for a conic tangent to E_2 and having a contact of order 4 with R at $[1:1:1]$. A straightforward computation gives the following equation for $\mathcal{C}$:

$$8\,y^2-4\,yx-4\,yz+x^2-2\,xz+z^2=0$$

as the only irreducible curve satisfying 2.

Case 1. We look for a conic tangent to E_2 and E_8 and having a contact of order $n+1$ with R at $[1:1:1]$. The case $n=2$ is obtained by degeneration of $n=1$. One has a one parametric family of equations for $\mathcal{C}$ satisfying $n=1$:

$$(s+3)^2\,y^2-2\left(s^2+s+2\right)yx-2\,(s+3)\,yz+s^2x^2-2sxz+z^2=0,$$

where $s\in\mathbb{C}\setminus\{-1,3\}$. The value $s=-1$ gives a reducible conic and $s=3$ results in the case $n=2$, $(\mathbb{A}_{17}+\mathbb{A}_2)$:

$$9\,x^2-28\,xy-6\,xz+36\,y^2-12\,yz+z^2=0.$$

remark-ratmap2

Remark 6.1. In order to calculate the rational map $\omega=\kappa\circ\tau\circ\sigma^{-1}:\mathbb{P}^2\dashrightarrow\mathbb{P}^2$ one can proceed as in the previous section. Note that $\omega^*E_8=2Q$, $\omega^*E_2=Q+2T$ and $\omega^*L_3=\tilde{L}_3$ where $\tilde{L}_3$ is a quartic satisfying $(\tilde{L}_3\cdot T)_E^{\mathbb{P}^2}=(\tilde{L}_3\cdot R)_E^{\mathbb{P}^2}=4$ and $(\tilde{L}_3\cdot Q)_E^{\mathbb{P}^2}=8$. One can fix three points on $\mathbb{P}^2$ so that $T=\{x=0\}$, $R=\{y=0\}$ and $Q=\{xz-y^2=0\}$. Hence

$$\omega(x,y,z)=[\alpha_1(xz-y^2)^2:\alpha_2x^2(xz-y^2):\alpha_3x^4-\alpha_4y^2(xz-y^2)].$$

Fixing a fourth point one can impose, for instance, $\omega(1,1,0)=[1:-1:0]$ and therefore $\alpha_1=\alpha_2=\alpha_3=\alpha_4$. Thus

$$\omega(x,y,z)=[(xz-y^2)^2:x^2(xz-y^2):x^4-y^2(xz-y^2)].$$

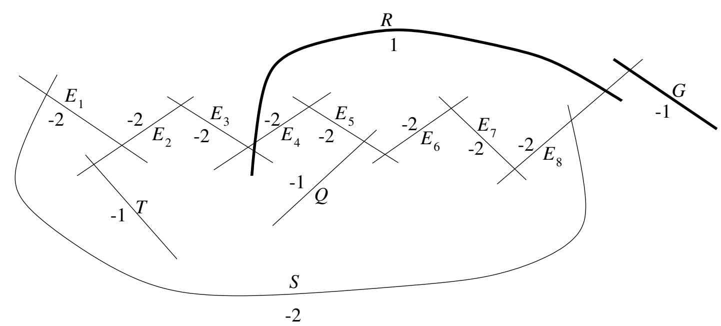

Figure 20: curves in X. fig-3x

7 Case $(\mathcal{C}\cdot T)_E=4$, $(\mathcal{C}\cdot Q)_E=10$ and G infinitely near to E

section-casect4cq10

This case may be considered as a particular case of the next section. We present it in a separate form because computations are different.

Let S be a nodal cubic with order of contact 9 at E with a general cubic F of the pencil. This can be achieved if the node is at E, one branch is transversal and the other has order of contact eight with F. Then

$$\check{S}=S+\sum_{j=1}^{8}E_j$$

and hence $\check{S}$ is the only reducible fiber of π. Its type is I_9. The group $MW(X)$ is in this case generated by $\operatorname{Pic}(X)=\langle E_1,\ldots,E_8,G,H\rangle$. The kernel of φ is generated by $E_1,\ldots,E_8,G$ and S, where

$$S\overset{MW}{=}3H-\sum_{j=1}^{8}(j+1)E_j-9G.$$

Thus, $MW(X)$ is cyclic of order three and its elements are G,T,Q satisfying $G\overset{MW}{=}0$ and $Q\overset{MW}{=}2T\overset{MW}{=}-T$.

According to Figure fig-ik 1, one has $\mu(E_8)=E_8$, $\mu(S)=E_7$ and $\mu(E_j)=\mu(E_{7-j})$, $j=1,2,3$. These components are globally fixed (not pointwise fixed). Let R be the curve of the 2-torsion points. If R was reducible, then the irreducible component passing through $E_3\cap E_4$ would have the same multiplicity of intersection with Q than with S at E. This is a contradiction, since $R\cap S=\{E\}\subset\mathbb{P}^2$. Hence R is irreducible. The sequence of multiplicities of R at E is $(3_3,2,1_4,0)$, where M_n represents an n-tuple of M's. Thus R is the strict transform of a sextic curve. Also note that $(R\cdot F)^X=3$.

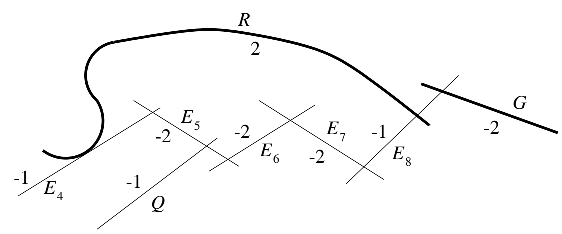

Figure 21: curves in Y. fig-3y

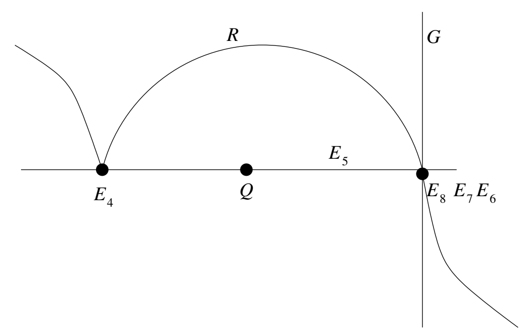

Figure 22: curves in Z. fig-3z

The Euler characteristic of the quotient Y is 8. Hence, after blowing-down the following 5 curves: $Q(=T), E_4, E_8, E_7, E_6$, a surface Z isomorphic to $\mathbb{P}^2$ is produced. The image of R in Z is a cuspidal cubic whose singular point is at E_4; G is the tangent line to the inflection point E_8 and E_5 is the line joining E_8 and E_4.

Let us study what the curve $\mathcal{C}$ is transformed into. Keeping track of its self-intersection accordingly to the sequence of blowing-ups and blowing-downs one can see that $\mathcal{C}$ becomes a cubic curve with a double point at Q. Also one can check that its intersection number with R at E_8 (an inflection of R) is at least 3 (thus E_8 is also an inflection point of $\mathcal{C}$). Let $E_8 = P_0, P_1, \ldots, P_k$ be the points in $\mathcal{C} \cap R \subset Z$ with contact orders m_0, $m_1 \geq \cdots \geq m_k$. Note that $\sum_{i=0}^k m_i = 9$ and that $m_0 = 3+\ell$ where $\ell \geq 0$. Hence, $\ell + \sum_{j=1}^k m_j = 6$. We will denote this situation by $[\ell, m_1, \ldots, m_k]$. Note that the inverse image of P_0 is a singular point of $\mathcal{C} \subset \mathbb{P}^2$ of type $\mathbb{A}_{15+\ell}$, while the inverse image of P_j, $j \geq 1$ is of type $\mathbb{A}_{m_j-1}$.

Recall we are only considering those configurations in Z that result from a curve $\mathcal{C} \subset \mathbb{P}^2$ with global Milnor number at least 18. Before listing such cases, let us explain a general procedure to obtain equations in Z. Consider the pencil of cubics generated by R and $\mathcal{C}$ in the projective plane Z. Let G' be a conic intersecting $\mathcal{C}$ at P_j with multiplicity m_j, $j = 0, \ldots, k$. By proposition-miembrohaz Proposition (3.1) $G + G'$ is an element of the pencil. In other words, $\mathcal{C}$ is an element of the pencil generated by R and $G + G'$ satisfying that $\mathcal{C}$ is singular at Q. Fixing three points in Z one can choose coordinates so that

$$R = \{x^2 z - y^3 = 0\}, \quad E_5 = \{y = 0\}, \quad G = \{z = 0\}.$$

Since $[1:1:1]$ is neither on E_5 nor on G, one can assume that $P_1 = [1:1:1]$. Let $\varphi : t \mapsto [1:t:t^3]$ be the parametrization of $\operatorname{Reg}(R)$ given in section 4 and $t_j := \varphi^{-1}(P_j)$, $j = 0, \ldots, k$. Note that $t_0 = 0$. Hence the equation of G' is:

$$a_{0,2}\, y^2 + a_{1,1}\, xy + a_{0,1}\, yz + a_{2,0}\, x^2 + a_{1,0}\, xz + z^2 = 0,$$

satisfying that

$$\varphi^{-1}(G') = \prod_{j=0}^{k} (t - t_j)^{m_j} \quad \text{and} \quad \sum_{j=1}^{k} m_j t_j = 0.$$

The following is a list of all existing configurations and equations for $\mathcal{C} \subset Z$:

(a) $[0,4,1,1]$, $(\mathbb{A}_{15} + \mathbb{A}_3)$:

$$63\, zy^2 - 48\, zxy + 3\, z^2 y + 256\, x^2 z - 32\, z^2 x + z^3 - 243\, y^3 = 0.$$

(b) $[1,3,1,1]$, $(\mathbb{A}_{16} + \mathbb{A}_2)$:

$$(22 + 6\, s)\, x^2 z + (-6\, s - 22)\, y^3 - \left(\frac{14 + 2\, s}{3}\right) xyz +$$

$$+ z^3 + (11 + 2\, s)\, y^2 z - \left(\frac{4 - 2\, s}{3}\right) yz^2 + (-6 - 2\, s)\, xz^2 = 0,$$

where $s^2 = 13$.

(c) $[1,2,2,1]$, $(\mathbb{A}_{16} + \mathbb{A}_1 + \mathbb{A}_1)$:

$$21\, y^2 z - 9\, xyz - 36\, yz^2 + 8\, xz^2 + z^3 + x^2 z - y^3 = 0$$

(for symmetry reasons in this case we considered $t_3 = 1$ as opposed to $t_1 = 1$).

(d) $[2,2,1,1]$, $(\mathbb{A}_{17} + \mathbb{A}_1)$:

$$3\, zy^2 - 4\, z^2 x + z^3 + 4\, x^2 z - 4\, y^3 = 0.$$

(e) $[3,1,1,1]$, $(\mathbb{A}_{18})$:

$$4\, z^3 - 4\, xz^2 + x^2 z - y^3 = 0.$$

For this construction there is no configuration with Milnor number 19.

Remark 7.1. Using the procedure explained in the previous sections one can give suitable coordinates in $\mathbb{P}^2$ so that the rational map $\omega = \kappa \circ \tau \circ \sigma^{-1} : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ is defined by

$$\omega(x, y, z) = [4q(x, y, z) : 4vx(xz - y^2)(xyz + x^3 - y^3) : -v^3(xyz + x^3 - y^3)^2],$$

where $Q = [4 : 0 : v^3]$ and

$$q(x, y, z) := -y^6 + 3\, y^4 xz + y^3 x^3 - 3\, y^2 x^2 z^2 - yx^4 z - x^6 + x^3 z^3.$$

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