Prime Numbers

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Alexandria University

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- Distribution of Prime Numbers

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First Properties Main Result Other Results What are Prime Numbers Interesting Questions

Division

Definition

An integer number $d \in \mathbb{Z}$ divides another integer number $D \in \mathbb{Z}$ if there exists $k \in \mathbb{Z}$ such that D = dk.

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First Properties Main Result Other Results What are Prime Numbers Interesting Questions

Division

Definition

An integer number $d \in \mathbb{Z}$ divides another integer number $D \in \mathbb{Z}$ if there exists $k \in \mathbb{Z}$ such that D = dk.

In this case we write d|D. We also refer to d as a divisor of D.

First Properties Main Result Other Results What are Prime Numbers Interesting Questions

Prime numbers

Definition

A number $p \in \mathbb{Z}$ is called prime if $p \neq \pm 1$ and $\{\pm 1, \pm p\}$ are its only divisors.

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Interesting Questions

• How many prime numbers are there?

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Main Result

Other Results

What are Prime Numbers Interesting Questions

Interesting Questions

- How many prime numbers are there?
- Do they have interesting properties?

Image: A matrix and a matrix

Main Result

Other Results

What are Prime Numbers Interesting Questions

Interesting Questions

- How many prime numbers are there?
- Do they have interesting properties?
- How easy are they to detect?

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Main Result

Other Results

What are Prime Numbers Interesting Questions

Interesting Questions

- How many prime numbers are there?
- Do they have interesting properties?
- How easy are they to detect?
- Are they constructible?

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Interesting Questions

- How many prime numbers are there?
- Do they have interesting properties?
- How easy are they to detect?
- Are they constructible?
- Can you make money out of them?

Pascal Triangle The Euler Zeta Function

Pascal Triangle

Theorem

If p is a prime number, then $\binom{p}{n}$ is a multiple of p.

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$$\binom{p}{n} = \frac{p(p-1)\dots(p-n+1)}{n(n-1)\dots 2}.$$

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Proof.

$$\binom{p}{n} = \frac{p(p-1)\dots(p-n+1)}{n(n-1)\dots 2}.$$

The result follows since p is prime and hence not divisible by any factor in the denominator.

Image: A math a math

Pascal Triangle The Euler Zeta Function

The Euler Zeta Function

Pf. Euclid Prime Numbers

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Pascal Triangle The Euler Zeta Function

The Euler Zeta Function

Definition

The Euler Zeta Function is defined as follows

$$heta(s) = 1 + rac{1}{2^s} + rac{1}{3^s} + rac{1}{4^s} + rac{1}{5^s} + \dots$$

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Pascal Triangle The Euler Zeta Function

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$$\frac{1}{2^s}\theta(s) = \frac{1}{2^s} + \frac{1}{4^s} + \dots + \dots$$
$$\prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)\theta(s) = 1$$

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Euler Sieve Property

$$heta(s) = \prod_{p \text{ prime}} \left(rac{1}{1-p^{-s}}
ight)$$

Image: A mathematical states and a mathem

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There Is No Largest Prime Number Distribution of Prime Numbers

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There Is No Largest Prime Number

The proof uses reductio ad absurdum.

Theorem (_, 300 BC)

There are infinitely many prime numbers.

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- Suppose there were finitely many prime numbers.
- ② Consider p the largest prime number.
- Let q = p! be the product of the first p numbers.
- Then q + 1 is not divisible by any of them.
- But q + 1 is greater than p, thus divisible by some prime number . . .

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Prime Number Theorem

Let $\pi(n) := \#\{\text{prime numbers smaller than } n\}.$

There Is No Largest Prime Number Distribution of Prime Numbers

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Prime Number Theorem

Let $\pi(n) := \#\{\text{prime numbers smaller than } n\}$. For example $\pi(10) = 4$, $\pi(100) = 25$,...

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Theorem (Prime Number Theorem)

$$\pi(n) \sim \frac{n}{\log(n)}$$

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Sieve of Eratosthenes Goldbach Conjecture

Sieve of Eratosthenes, 200 BC

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

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X	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
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11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	,30
31	,32	33	,34	35	,36	37	,38	39	<i>4</i> 0
41	<i>4</i> 2	43	<i>4</i> 4	45	<i>4</i> 6	47	<i>4</i> 8	49	,50
51	, 5 2	53	,54	55	<i>,</i> 56	57	<i>5</i> 8	59	,60
61	<i>6</i> 2	63	,64	65	<i>,</i> 66	67	<i>6</i> 8	69	70
71	72	73	74	75	76	77	78	79	80
81	<i>,</i> 82	83	,84	85	,8 6	87	<u>88</u>	89	90
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61	<i>6</i> 2	<i>6</i> 3	,64	65	<i>,</i> 66	67	<i>6</i> 8	<i>,</i> 69	70
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61	,62	<i>6</i> 3	,64	<i>,</i> 65	<i>6</i> 6	67	68	,69	70
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 $11^2 = 121 > 100$

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Sieve of Eratosthenes Goldbach Conjecture

Goldbach Conjecture

Goldbach, 1742

Every even integer greater than 2 can be expressed as the sum of two primes.

Sieve of Eratosthenes Goldbach Conjecture

Local video file



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Sieve of Eratosthenes Goldbach Conjecture

External player



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Sieve of Eratosthenes Goldbach Conjecture

External player (href)



https://www.youtube.com/watch?v=lqKSXk5Xwg8

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Sieve of Eratosthenes Goldbach Conjecture

remote (YouTube player)

https://www.youtube.com/watch?v=lqKSXk5Xwg8

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