Prime Numbers

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March 11, 2022

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Division

Definition

An integer number $d \in \mathbb{Z}$ divides another integer number $D \in \mathbb{Z}$ if there exists $k \in \mathbb{Z}$ such that $D = dk$.

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In this case we write $d|D$. We also refer to d as a divisor of D.

Prime numbers

Definition

A number $p \in \mathbb{Z}$ is called prime if $p \neq \pm 1$ and $\{\pm 1, \pm p\}$ are its only divisors.

Interesting Questions

- How many prime numbers are there?

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- Can you make money out of them?

Pascal Triangle

Theorem

If p is a prime number, then $\binom{p}{n}$ is a multiple of p.

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$$\binom{p}{n} = \frac{p(p-1)\dots(p-n+1)}{n(n-1)\dots 2}.$$

Pascal Triangle

Theorem

If p is a prime number, then $\binom{p}{n}$ *is a multiple of p.*

Proof.

$$\binom{p}{n} = \frac{p(p-1)\dots(p-n+1)}{n(n-1)\dots 2}.$$

The result follows since p is prime and hence not divisible by any factor in the denominator. □

The Euler Zeta Function

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Definition

The Euler Zeta Function is defined as follows

$$\theta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \dots$$

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$$\left(1 - \frac{1}{2^s}\right)\theta(s) = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{9^s} + \dots$$

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$$\prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right) \theta(s) = 1$$

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Euler Sieve Property

$$\theta(s) = \prod_{p \text{ prime}} \left(\frac{1}{1 - p^{-s}} \right)$$

There Is No Largest Prime Number

The proof uses *reductio ad absurdum*.

Theorem (_, 300 BC)

There are infinitely many prime numbers.

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There are infinitely many prime numbers.

Proof.

1. Suppose there were finitely many prime numbers.
2. Consider p the largest prime number.
3. Let $q = p!$ be the product of the first p numbers.
4. Then $q + 1$ is not divisible by any of them.
5. But $q + 1$ is greater than p, thus divisible by some prime number . . .

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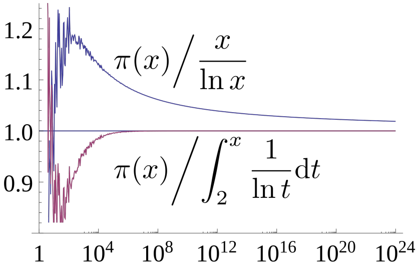

Sieve of Eratosthenes, 200 BC

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
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61	62	63	64	65	66	67	68	69	70
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$$11^2 = 121 > 100$$

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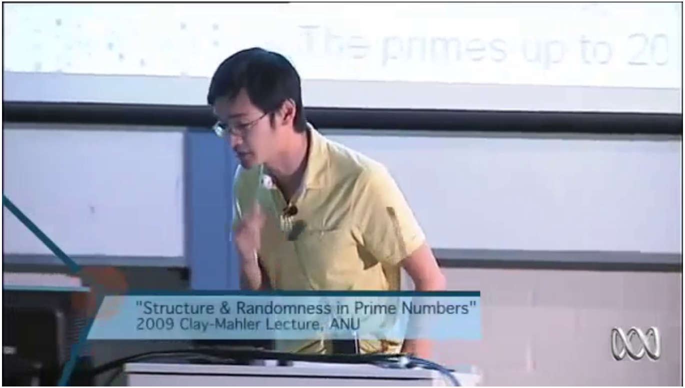
$$11^2 = 121 > 100$$

Goldbach Conjecture

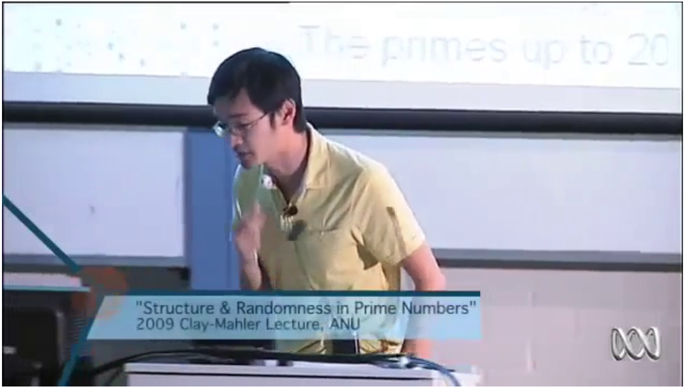
Goldbach, 1742

Every even integer greater than 2 can be expressed as the sum of two primes.

Local video file



External player



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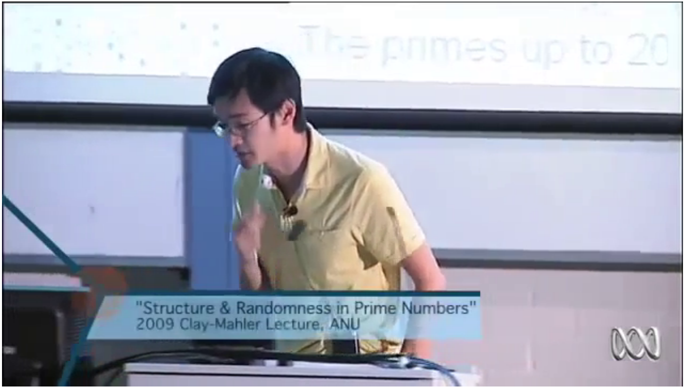

https://www.youtube.com/watch?v=lqKSXk5Xwg8

remote (YouTube player)

https://www.youtube.com/watch?v=1qKSXk5Xwg8