# CHARACTERISTIC CLASSES

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In this text $X$ is a topological space which is Hausdorff and paracompact, and in most cases admits a countable basis. We will deal mostly with differentiable (or complex analytic) manifolds, and with $CW$-complexes. In the previous lectures, the concept of vector bundle has been defined. The goal of these notes is to define Euler, Chern and Pontrjagin characteristic classes in different ways, in particular from classifying spaces, and their relationship with $K$-theory.

The readers should already have notions on vector bundles, $K$-theory, singular homology and cohomology (including cup-products).

## 1. About characteristic classes

We can see a characteristic class of vector bundles as a map from the set of $\mathbb{K}$-vector bundles ($\mathbb{K} = \mathbb{R}, \mathbb{C}$) over a *nice* topological space $X$ (mostly a connected manifold, differentiable or analytic) to the cohomology ring of $X$ over a ring (usually $\mathbb{Z}$ or a field). This map must be compatible with pull-backs. If $\mathbf{p}$ is a characteristic class, $\pi : E \to X$ is a vector bundle and $f : Y \to X$ is continuous, then $\mathbf{p}(f^*E) = f^*(\mathbf{p}(E))$.

Such a characteristic class is said to be *stable* if for $E_1, E_2$ vector bundles, $\mathbf{p}(E_1 \oplus E_2) = \mathbf{p}(E_1) \smile \mathbf{p}(E_2)$. Note that in that case, if we replace $H^*(X; R)$ by its completion $\hat{H}^*(X; R)$ (i.e., replace the direct sum of homogeneous cohomology groups by the cartesian product) and we assume that $\mathbf{p}(E)$ is invertible in $H^*(X; R)$ (basically its zero-part is invertible in $R$) the characteristic class descends to the $K$-theory of $X$ if we replace its cohomology ring by its completion in order to define $\mathbf{p}(-E)$.

We will follow several approaches for the definitions which should converge to compatible ones. Let $X$ be a paracompact space. Let us recall that the set of isomorphisms classes of $n$-vector bundles is in bijection with the set $[X, \mathrm{Gr}(n, \infty)]$ of homotopy classes of maps from $X$ to the infinite Grasmannian.

**Theorem 1.1.** *The cohomology ring of* $\mathrm{Gr}(n, \infty)$ *is isomorphic to* $\mathbb{Z}[c_1, \dots, c_n]$*, where* $\deg c_j = 2j$.

This provides a *simple* way to define characteristic classes, after a precise choice of signs for the generators of the ring. The $j^{\text{th}}$-*Chern characteristic class* $c_j(E) \in H^{2j}(X; \mathbb{Z})$ of a complex vector bundle $\pi : E \to X$ (of rank $n$) defined by a map $f : X \to \mathrm{Gr}(n, \infty)$ is defined as $f^*(c_j)$ (if $j \leq n$, and $0$ if $j > n$). This definition obviously satisfies the first requirement and if we define the *total Chern class* as $1(= c_0(E)) + \sum_{j=1}^{n} c_j(E)$ we will prove later that it descends to the $K$-theory.

This definition is not so much useful neither for computations nor for guessing their properties. This is why it may be useful to get other insights.

The second family of characteristic classes are Pontrjagin ones. Let $\pi : E \to X$ be an $\mathbb{R}$-vector bundle of rank $n$. It determines a complex vector bundle $E_{\mathbb{C}}$ such that its fibers are $E_x \otimes_{\mathbb{R}} \mathbb{C}$; from the structural group point of view, we extend $\mathrm{GL}(n;\mathbb{R}) \hookrightarrow \mathrm{GL}(n;\mathbb{C})$.

Recall that given a complex vector bundle $E_1$, we can consider its conjugate $\bar{E}_1$ (where multiplication by $\pm\sqrt{-1}$ is exchanged). Using a hermitian metric, we have that $\bar{E}_1 \cong E_1^*$; we will see later that $c_j(\bar{E}_1) = (-1)^j c_j(E_1)$.

Since $E_{\mathbb{C}} \cong \bar{E}_{\mathbb{C}}$, we deduce that $2c_{2j+1}(E_{\mathbb{C}}) = 0$ (they will vanish if there is no 2). The $j^{\text{th}}$-Pontrjagin class of $E$ is defined as $p_j(E) := (-1)^j c_{2j}(E_{\mathbb{C}})$.

We finish this part with some ideas about the Euler class the construction of the Euler class. Let $\pi : E \to X$ be a rank $n$ oriented $\mathbb{R}$-vector bundle. In this introduction we consider only the case $n = 2$. Let us start with the case $n = 2$. Let us assume that $X$ is a $CW$-complex. Since $\pi$ is orientable we can fix a trivialization in the 1-skeleton $X_1$. Let $e$ be a 2-cell, and let $f_e : \bar{\mathbb{B}}^2 \to X$ be the *attaching map*. Since $f_e^*(E)$ is a trivial bundle ($\bar{\mathbb{B}}^2$ is contractible) we can take a nowhere vanishing section of $f_e^*(E)$; if we denote by $E_e$ the restriction of this bundle to $\mathbb{S}^1$, the restriction of the section defines $g_e : \mathbb{S}^1 \to E_e$. On the other since $E$ is trivial over $X_1$ we have another trivialization $h_e : f_e^*(E)_{\mathbb{S}^1} \to \mathbb{S}^1 \times \mathbb{R}^2$. Normalizing the image of $\pi_{\mathbb{R}^2} \circ h_e \circ g_e$, we obtain a map $\rho_e : \mathbb{S}^1 \to \mathbb{S}^1$.

We define the cochain $c : C_2(X) \to \mathbb{Z}$ by $c(e) := -\deg \rho_e$; it is a cocycle and a different choice of trivialization over $X_1$ provides a cobordant cocycle, so its cohomology class is well-defined and it is called the *Euler class of* $E$, denoted by $e(E) \in H^2(X;\mathbb{Z})$.

*Remark* 1.2. If $E$ is a real oriented vector bundle of rank $n > 2$ over a $CW$-complex $X$ and it admits a trivialization over the 2-skeleton, then it admits also a trivialization over the $(n-1)$-skeleton and the same kind of arguments hold to provide the Euler class $e(E) \in H^n(X;\mathbb{Z})$. We will use the Thom isomorphism to define the Euler class without this restriction.

**Example 1.3.** Let us consider the tautological line bundle over $\mathbb{P}^1$, i.e., $E := \{(p,\ell) \in \mathbb{C}^2 \times \mathbb{P}^1 \mid p \in \ell\}$, with the restriction of the second projection. Let $\mathbb{C}_x := \{[x:y] \in \mathbb{P}^1 \mid y \neq 0\} = \{[x:1] \in \mathbb{P}^1 \mid x \in \mathbb{C}\}$; we define in the same way $\mathbb{C}_y$ for $x \neq 0$. We have the charts,

$$([x:y],t) \in \mathbb{C}_x \times \mathbb{C} \to \left(\left(\frac{x}{y}t, t\right), [x:y]\right), \qquad ([x:y],t) \in \mathbb{C}_y \times \mathbb{C} \to \left(\left(t, \frac{y}{x}t\right), [x:y]\right).$$

The change of charts is given by

$$([x:y],t) \xmapsto{\Phi_x} \left(\left(\frac{xt}{y},t\right),[x:y]\right) \xleftarrow{\Phi_y} \left([x:y],\frac{xt}{y}\right)$$

and the transition function $\mathbb{C}_x \cap \mathbb{C}_y \to \mathrm{GL}(1;\mathbb{C}) = \mathbb{C}^*$ is given by $[x:1] \mapsto x$. In order to simplify the computation of the Euler class we fix the following $CW$-complex decomposition of $\mathbb{P}^1$: one cell $e^0$ of dimension 0, the point $[-1:1]$, one cell $e^1$ of dimension 1, the arc $\{[\exp(\sqrt{-1}\pi u):1] \mid u \in (-1,1)\}$, and two cells of dimension 2, $e^2_x := \{[x:1] \mid |x| \leq 1\}$ and $e^2_y := \{[1:y] \mid |y| \leq 1\}$. If we identify $\bar{\mathbb{B}}^2$ with $\{x \in \mathbb{C} \mid |x| \leq 1\}$, the attaching maps are given by:

$$u \in [-1,1] \xmapsto{f_{e^1}} [\exp(\sqrt{-1}\pi u):1], \quad x \in \bar{\mathbb{B}}^2 \xmapsto{f_{e^2_x}} [x:1], \quad y \in \bar{\mathbb{B}}^2 \xmapsto{f_{e^2_y}} [1:y],$$

The bundle is trivialized over the 1-skeleton:

$$([x:1],t) \mapsto ((tx,t),[x:1])$$

For the cell $e^2_x$ the pull-back is

$$\{(x,(s,t),[x:1]) \in \mathbb{S}^1 \times \mathbb{C} \times \mathbb{P}^1 \mid s = tx\}$$

and as a nonvanishing section we can consider $x \mapsto (x,(x,1),[x:1])$. The map $\rho_{e^2_x}$ is given by $x \mapsto 1$, of degree 0.

For the cell $e^2_y$ the pull-back (restricted to the boundary) is

$$\{(y,(s,t),[1:y]) \in \mathbb{S}^1 \times \mathbb{C} \times \mathbb{P}^1 \mid t = sy\}$$

and the restriction of the nonvanishing section is given by $y \mapsto (y,(1,y),[1:y])$. The map $\rho_{e^2_y}$ is given by

$$y \mapsto (y,(1,y),[1:y]) \mapsto ([y^{-1}:1],y) \mapsto y.$$

The cocycle is defined by $c(e^2_x) = 0$, $c(e^2_y) = -1$, i.e., the image of fundamental (positive class) of $\mathbb{P}^1$ is $-1$.

**1.4. More on oriented vector bundles of rank** 2. Let $\pi : E \to X$ such a bundle, where $X$ is a $CW$-complex. Let us restrict our attention to the 2-skeleton; for simplicity, we assume $X = X_2$. Recall that we can trivialize $E$ over the 1-skeleton $X_1$, in particular, this fiber bundle will come as the pull-back of a fiber bundle over $X_2/X_1$ which is a *bouquet* of a number of $\mathbb{S}^2$. So, we are interested in understanding rank 2 oriented vector bundles over $\mathbb{S}^2$; since we can assume these bundles to be riemannian, they are identified with vector bundles. If we identify $\mathbb{S}^2 \equiv \mathbb{P}^1(\mathbb{C}) \equiv \mathbb{C} \cup \{\infty\}$, the Euler class can be interpreted as follows. Let $s_x : \bar{\mathbb{B}}^2 \to E$, $s_y : (\mathbb{C} \setminus \mathbb{B}^2) \cup \{\infty\} \to E$ be nowhere vanishing sections. We orient $\mathbb{S}^1 \subset \mathbb{C}$ as the boundary of $\bar{\mathbb{B}}^2$. Let $x \in \mathbb{S}^1$; note that $\frac{s_y(x)}{s_x(x)}$ can be interpreted as a complex number $\rho(x)$; let $m := \deg \rho$. Then, the Euler class is $m\check{\mathbf{1}}$, where $\check{\mathbf{1}}$ is the positively oriented generator of $H^2(\mathbb{S}^2, \mathbb{Z})$.

**1.5. Chern classes from Euler class of line bundles.** Let $\pi : E \to X$ be a rank $n$ $\mathbb{C}$-vector bundle. It is not hard to define the projectivized bundle $\pi_{\mathbb{P}} : \mathbb{P}(E) \to X$. As a set, it is $\coprod_{x\in X} \mathbb{P}(E_x)$, i.e., its fibers are complex projective spaces of dimension $n-1$. Its *transition functions* are modelled on $\mathrm{PGL}(n; \mathbb{C})$. We can consider the pull-back $\mathbb{P}^*_\pi(E)$. As a vector subbundle of this one we have the tautological bundle

$$\tau(E) := \coprod_{x\in X} \{(v, \ell) \in E_x \times \mathbb{P}(E_x) \mid v \in \ell\}.$$

The natural map $\pi_\tau : \tau(E) \to \mathbb{P}(E)$ is a line bundle and let $x := -e(\tau(E)) \in H^2(\mathbb{P}(E); \mathbb{Z})$ the Euler class.

**Theorem 1.6.** *The cohomology ring $H^*(\mathbb{P}(E); \mathbb{Z})$ is a free $H^*(X; \mathbb{Z})$-module with basis $1, x, \ldots, x^{n-1}$.*

This theorem provides an alternative definition of Chern classes. Since $x^n$ is an element of $H^*(\mathbb{P}(E); \mathbb{Z})$, we have a relation

$$x^n + c_1(E)x^{n-1} + \cdots + c_{n-1}(E)x + c_n(E) = 0,$$

where the coefficients are the Chern classes.

## 2. Infinite grasmannians

We have already defined the infinite Grasmannians $\mathrm{Gr}_{\mathbb{K}}(r, \infty)$ ($\mathbb{K} = \mathbb{R}, \mathbb{C}$) as an inductive limit $\varinjlim \mathrm{Gr}_{\mathbb{K}}(r, n+r)$. We are going to present them as $CW$-complex in order to compute their cohomology. We borrow to J. Milnor the following definitions.

**Definition 2.1.** An *open $n$-cell* (or *$n$-cell* for short) is a topological space homeomorphic to an open ball $\mathbb{B}^n$ in $\mathbb{R}^n$. A *closed $n$-cell* is a topological space homeomorphic to an closed ball $\bar{\mathbb{B}}^n$ in $\mathbb{R}^n$.

**Definition 2.2.** A *finite CW-complex* consists of a Hausdorff topological space $X$ with a *finite* partition of $X$ as union of $n$-cells such that

(CW1) For each cell $e$ (of dimension $n_e$) there exists a continuous map $f_e : \bar{\mathbb{B}}^{n_e} \to X$ which carries $\mathbb{B}^{n_e}$ homeomorphically onto $e$ (the characteristic map of $e$).

(CW2) For each cell $e$ its closure $\bar{e} = f_e(\bar{\mathbb{B}}^{n_e})$ satisfies that $\bar{e} \setminus e$ is a disjoint union of cells of dimension $< n_e$.

**Definition 2.3.** A *CW-complex* consists of a Hausdorff topological space $X$ with a partition of $X$ as union of $n$-cells satisfying (CW1), (CW2) and

(CW3) Each point of $X$ is contained in a finite union of cells which form a closed finite $CW$-complex (such finite unions will be called *finite $CW$-subcomplex*).

(CW4) The space $X$ is homeomorphic to the inductive union of its finite $CW$-complex, i.e. $U \subset X$ is open if and only if $U \cap K$ is open in $K$ for any finite $CW$-subcomplex $K$.

The $k$-skeleton of a $CW$-complex is the subcomplex obtained from the union of all cells of dimension $\leq k$. Let us recall the construction of the cellular chain complex. We orient (arbitrarily) all the cells of $X$. Let us fix first a cell $e$ of dimension $n$. The property (CW2) provides a map $f_{e|} : \mathbb{S}^{n-1} \to K_{n-1}$. Let $e'$ be a cell of dimension $n-1$. For this cell we construct the space $\mathbb{S}^{n-1}_{e'}$ which is obtained from $K_{n-1}$ by collapsing to a point the closures of the cells of $K_{n-1}$ distinct from $e'$; the notation comes from the fact that it is homeomorphic to $\mathbb{S}^{n-1}$ (with a cell decomposition with two cells, $e'$ and the collapsed point). The previous map induced a new one $g_{e,e'} : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}_{e'}$. Since both spheres are oriented, the degree of the map is well-defined and we define $\deg(e, e') := \deg g_{e,e'}$. The cellular chain complex $C_*(X, \mathbb{Z})$ of $X$ is the free abelian group with basis the cells of the decomposition. This group is graduated by the dimensions of the cells. The boundary is defined by

$$\begin{array}{ccc} C_n(X;\mathbb{Z}) & \xrightarrow{\quad\partial\quad} & C_{n-1}(X;\mathbb{Z}) \\ & & \\ e & \longmapsto & \displaystyle\sum_{e'} \deg(e,e')e' \end{array}$$

*Remark* 2.4. Note that $C_n(X;\mathbb{Z})$ is naturally isomorphic to $H_n(K_n, K_{n-1};\mathbb{Z})$ ($K_{n-1} = \emptyset$). Moreover, $\delta : C_n(X;\mathbb{Z}) \to C_{n-1}(X;\mathbb{Z})$ is the connexion morphism of the long exact sequence of the triple $(K_n, K_{n-1}, K_{n-2})$. The following diagram explains why we get a complex

$$\begin{tikzcd}[column sep=small]
& H_{n-1}(K_{n-1};\mathbb{Z}) \arrow[dr, "i"] & & & \\
H_n(K_n, K_{n-1};\mathbb{Z}) \arrow[ur, "\partial"] \arrow[rr, "\partial"] & & H_{n-1}(K_{n-1}, K_{n-2};\mathbb{Z}) \arrow[rr, "\partial"] \arrow[dr, "\partial"'] & & H_{n-2}(K_{n-2}, K_{n-3};\mathbb{Z}) \\
& & & H_{n-2}(K_{n-2};\mathbb{Z}) \arrow[ur, "i"'] &
\end{tikzcd}$$

The cochain complex is the dual one.

Let us construct a $CW$-complex structure on the infinite Grasmannian. Let us fix in $\mathbb{K}^\infty$ an infinite *flag*

$$0 \subset \mathbb{K} \subset \mathbb{K}^2 \subset \cdots \subset \mathbb{K}^n \subset \dots$$

Given $H \in \mathrm{Gr}_{\mathbb{K}}(r, \infty)$, there is a sequence of integers

$$\dim_{\mathbb{K}}(H \cap \{0\}) \leq \dim_{\mathbb{K}}(H \cap \mathbb{K}) \leq \dim_{\mathbb{K}}(H \cap \mathbb{K}^2) \leq \cdots \leq \dim_{\mathbb{K}}(H \cap \mathbb{K}^n) \leq \dots$$

This sequence starts from 0 and ends in $r$, while the gap between two terms is either 1 or 0. The *Schubert symbol* of $H$ isa sequence $\sigma := (\sigma_1, \dots, \sigma_r)$ of integers, such that $0 < \sigma_1 < \cdots < \sigma_r$, representing the $r$ places where $\dim_{\mathbb{K}}(H \cap \mathbb{K}^n)$ jumps. Sometimes it is better to consider the sequence $\eta = (\eta_i := \sigma_i - i)_{i=1}^r$, where $0 \leq \eta_1 \leq \cdots \leq \eta_r$. If we see the elements of $\mathbb{K}^\infty$ as sequences (with only a finite number of non-zero entries), the Schubert symbol of $H$ determines a unique basis $(v_1, \dots, v_r)$ as follows:

- $v_1 = (\underbrace{*, \dots, *, 1}_{\sigma_1 \text{ entries}}, 0, \dots)$.
- $v_2 = (\underbrace{\underbrace{*, \dots, *, 0}_{\sigma_1 \text{ entries}}, *, \dots, *, 1}_{\sigma_2 \text{ entries}}, 0, \dots)$.
- We proceed in the same way. Each vector has $\eta_i$ free entries.

Hence the set $\Sigma_\sigma$ of subspaces with fixed Schubert symbol $\sigma$ is parametrized by $\mathbb{K}^{|\eta|}$, where $|\eta| = \eta_1 + \cdots + \eta_r$.

**Exercise 1.** *Show that $\Sigma_\sigma$ is homeomorphic to $\mathbb{K}^{|\eta|}$.*

**Lemma 2.5.** *Let $H \in \mathrm{Gr}_{\mathbb{K}}(r, \infty)$ and let $\tilde{\sigma}$ be its Schubert symbol. Then, $H \in \bar{\Sigma}$ if and only if $\tilde{\eta} \leq \eta$, i.e., $\tilde{\eta}_i \leq \eta_i$.*

*Proof.* Since the finite Grasmannians $\mathrm{Gr}_{\mathbb{K}}(r, N)$, $r \leq N$, are closed in $\mathrm{Gr}_{\mathbb{K}}(r, \infty)$, then we can work in a finite Grasmannian, for $N \geq \max(\sigma \cup \tilde{\sigma})$. Recall that $\mathrm{Gr}_{\mathbb{K}}(r, N)$ is the quotient by right action of $\mathrm{GL}(r; \mathbb{K})$ of

$$\{A \in \mathrm{Mat}(N \times r; \mathbb{K}) \mid \operatorname{Rank} A = r\}.$$

Each $H \in \mathrm{Gr}_{\mathbb{K}}(r, N)$ admits a unique echelon representative. The Schubert symbol represents the *steps* of the echelon. An open neighborhood of $H$ is given by the subspaces for which the minor associated to $\tilde{\sigma}$ is non-zero. This open set is in bijection with the matrices which are the identity on these rows. As a necessary condition for $H \in \bar{\Sigma}_\sigma$, there must be subspaces with this Schubert symbol in this open set. It is easily seen that the symbols present in this open set are those greater or equal than $\tilde{\sigma}$ and that all of them have elements close to $H$. ☐

*Sketch of the proof of Theorem* 1.1. The *CW*-complex structure has cells only in even dimension, in particular the homology is isomorphic to the cell chain complex, and the cohomology is dual to the cell cochain complex.

Let us sketch first the proof for the projective space, i.e. $\mathbb{P}(\mathbb{C}^\infty) = \mathrm{Gr}(1, \infty)$. We know that $H^{2j}(\mathbb{P}(\mathbb{C}^\infty); \mathbb{Z}) \cong \mathbb{Z}$ and the odd terms vanish. Let us denote by $t \in H^2(\mathbb{P}(\mathbb{C}^\infty); \mathbb{Z})$ such that $t(\mathbb{P}^1) = 1$.

We proceed as follows. If $X$ is an oriented compact manifold of real dimension $n$ with no torsion in its homology, given $\alpha \in H^k(X; \mathbb{Z}) = \mathrm{Hom}(H_k(X; \mathbb{Z}), \mathbb{Z})$, we consider its Poincaré dual $a := \mathrm{PD}(\alpha) \in H_{n-k}(X; \mathbb{Z})$, defined by the property $\alpha(b) = a \cdot b$. The relationship of Poincaré duality with cup product is defined as follows:

$$\alpha \smile \beta = \mathrm{PD}^{-1}(\mathrm{PD}(\alpha) \cdot \mathrm{PD}(\beta))$$

For the previous case, it is enough to compute $t^2 = t \smile t$ in $H^2(\mathbb{P}^N(\mathbb{C}); \mathbb{Z})$ for $N \gg 1$. Note that $\mathrm{PD}(t) = H$, the class of a hyperplane (of dimension $N-1$). Choosing two generic such hyperplanes, we see that $H_1 \cdot H_2 = S$ a projective subspace of codimension 2 which is the Poincaré dual of $t^2$. We check in this way that $H^*(\mathbb{P}(\mathbb{C}^\infty); \mathbb{Z})$ is isomorphic to $\mathbb{Z}[t]$, where $t^n$ is homogeneous of degree $2n$ and $t^n(\mathbb{P}^n(\mathbb{C})) = 1$.

Let us go to the general case. For a Schubert symbol $\sigma$, let $c_\sigma$ be the dual cochain and $a_\sigma$ the corresponding cycle (it is the closure of the cell). For two such symbols $\sigma', \sigma''$, let us compute its cup product $c_{\sigma'} \smile c_{\sigma''}$. Let us choose $N \gg 1$ such that the cells of $\sigma', \sigma''$ are contained in $\mathrm{Gr}_{\mathbb{C}}(n, N)$. Let us consider the Poincaré dual for the cochains $c_{\sigma'}, c_{\sigma''}$.

The cell decomposition is associated to the choice of a basis $(v_1, \ldots, v_N)$ of $\mathbb{C}^N$. We can fix a hermitian product for which it is an orthonormal basis. Let us consider the symbol

$$\sigma^\perp := (N + 1 - \sigma_{r-j+1})_{j=1}^r,$$

and construct the cell associated to the basis $(v_N, \ldots, v_1)$. From the matrix point of view, the cells of $\sigma$ and $\sigma^\perp$ consist of upper echelon matrices for $\sigma$ and lower ones for $\sigma^\perp$ with common boundary. We claim that the cells intersect at only one point. Note that the sum of the dimensions of the cell is

$$\sum_{j=1} \sigma_j + \sum_{j=1}^{r} (N + 1 - \sigma_{r-j+1}) - r(r+1) = r(N+1) - r(r+1) = r(N-r),$$

the dimension of the Grasmannian. Now, we want to compute the intersection of the closures of the cells for $(\sigma')^\perp$ and $(\sigma'')^\perp$, using again reversed bases for each one of them. Let us study the dimension of the common vectors in each position. In the $j^{\text{th}}$-position, the first cell provides dimension $N + 1 - \sigma'_{r-j+1} - j$, the second one $N + 1 - \sigma''_{r-j+1} - j$, and the total space has dimension $N$ ($N \gg r$). Then, the dimension of the intersection is

$$(N + 2 + r - j - \sigma'_{r-j+1} - \sigma''_{r-j+1})_{j=1}^r.$$

We claim now that this intersection can be seen as the closure of a cell with this symbol. The dual of this cell is given by

$$(\sigma'_j + \sigma''_j - j)_{j=1}^r.$$

If $\sigma$ is the corresponding symbol, note that $\eta = \eta' + \eta''$.

For a Schubert symbol $\sigma$, we denote $c_\sigma = \tilde{c}_\eta$. We have proved that $\tilde{c}_{\eta'} \smile \tilde{c}_{\eta''} = \tilde{c}_{\eta'+\eta''}$. It implies that $H^*(\mathrm{Gr}_\mathbb{C}(r,\infty);\mathbb{Z})$ is a polynomial ring in the variables $\tilde{c}_j := \tilde{c}_{\eta_j}$ for $j = 1,\dots,r$ where

$$\eta_j := (\underbrace{0,\dots,0}_{r-j \text{ times}}, \underbrace{1,\dots,1}_{j \text{ times}}).$$

The closure $\mathbb{P}_j$ for the cell for $\eta_j$ is isomorphic to the Grasmannian of hyperplanes in $\mathbb{C}^(r+1)/\mathbb{C}^{r-j}$, isomorphic to a projective space of dimension $j$. For the orientations, we impose that $\tilde{c}_j(\mathbb{P}_j) = (-1)^j$. □

## 3. Principal bundles

**Definition 3.1.** Let $G$ be a Lie group. A $G$-principal bundle is a map $\pi : P \to X$, locally trivial bundle, such that there is a right free action of $G$ on $P$ such that

$$\begin{tikzcd} & P \arrow[dl, "\pi"'] \arrow[dr, "\pi_G"] & \\ X \arrow[rr, "\cong"] & & P/G \end{tikzcd}$$

Note that in the case of vector bundles the fibers are naturally vector spaces while the fibers of a principal bundle are homeomorphic to $G$ but they are *not* Lie groups, since there is no neutral element, they are *$G$-torsor spaces.*

*Remark* 3.2. If a $G$-principal bundle $\pi : P \to X$ admits a section $s : X \to P$, then $\pi$ is a trivial bundle. Note that $X \times G \to P$, $(x,g) \mapsto s(x)\cdot g$ is a trivialization.

**Example 3.3.** Let $\pi : E \to X$ be a vector bundle of rank $n$. For each $x \in X$, let $F_x$ be the set of ordered bases (*frames*) of $E_x$. There is a natural right action $F_x \times \mathrm{GL}(n,\mathbb{R}) \to F_x$ such that

$$\left(\begin{pmatrix} v_1 & \dots & v_n \end{pmatrix}, A\right) \mapsto \begin{pmatrix} v_1 & \dots & v_n \end{pmatrix} \cdot A,$$

where the frames are intepreted as row matrices with vector entries.

We can induce a topological structure on $\mathcal{F} := \coprod_{x\in X} F_x$ as usual. Let $\pi_\mathcal{F} : \mathcal{F} \to X$ the natural projection. Let $U$ be a trivializing open set for $\pi$, i.e., we have a homeomorphism $\Phi_U : U \times \mathbb{R}^n \to \pi^{-1}(U)$. Let $v_i : U \to \pi^{-1}(U)$, the local section given by $v_i(x) = \Phi(x, \mathbf{e}_i)$. Then, $v : U \to \pi_\mathcal{F}^{-1}(U)$, given by $v(x) = (v_1(x)\dots v_n(x))$ is a local section of $\mathcal{F}$ for which $\Phi_U(x,\mathbf{t}) = v(x)\cdot\mathbf{t}$ (where $\mathbf{t} \in \mathbb{R}^n$ is seen as a column vector). Moreover

$$\tilde{\Phi}_U : U \times \mathrm{GL}(n,\mathbb{R}) \to \pi_\mathcal{F}^{-1}(U), \qquad (x,A) \mapsto v(x)\cdot A.$$

If $\Phi_V : V \times \mathbb{R}^n \to \pi^{-1}(V)$ is another trivialization we can define in the same a local section $w : V \to \pi_\mathcal{F}^{-1}(V)$. Recall that the transition functions $\Phi_{V,U} : U \cap V \to \mathrm{GL}(n;\mathbb{R})$ are determined by the following property (for any $x \in U \cap V$):

$$(x, \mathbf{t}) \xrightarrow{\quad \Phi_U \quad} v(x) \cdot \mathbf{t} = \underbrace{w(x) \cdot \Phi_{V,U}(x)}_{v(x)} \cdot \mathbf{t} \xleftarrow{\quad \Phi_V \quad} (x, \Phi_{V,U}(x) \cdot \mathbf{t}).$$

In the same way:

$$(x, A) \xrightarrow{\quad \tilde{\Phi}_U \quad} v(x) \cdot A = \underbrace{w(x) \cdot \Phi_{V,U}(x)}_{v(x)} \cdot A \xleftarrow{\quad \Phi_V \quad} (x, \Phi_{V,U}(x) \cdot A)$$

The frame bundle of $E$ is a $\mathrm{GL}(n; \mathbb{K})$-principal bundle.

**3.4. Reduction of the structural group.** Note that the frame bundle of $E$ contains essentially all the information contained in $E$. But in most cases this information can be encoded using smaller groups.

Assume that $\pi : E \to X$ is a real vector bundle of rank $n$ endowed with a riemannian metric, i.e., with a scalar product $\langle \bullet, \bullet \rangle_x$ of $E_x$, such that for any pair of sections $s, t : U \to E$ (over an open set $U \subset X$), the map $x \in U \mapsto \langle s(x), t(x) \rangle_x$ is continuous (or differentiable, or real analytic,...).

Then, we can work with *orthonormal frames*, i.e., orthonormal bases of $E_x$, $x \in X$. The above discussion can be rephrased if $\mathrm{GL}(n; \mathbb{K})$ is replaced by $\mathrm{O}(n; \mathbb{R})$. Not only, the group is simpler, but it captures the extra structure.

If we consider a rank $n$ complex bundle with a hermitian metric, the reduction is done to $U(n)$.

**3.5. Oriented bundles.** Recall that an orientation of a finite dimensional $\mathbb{R}$-vector space stands for the choice of an ordered basis (up to basis change of positive determinant). A real vector bundle is orientable if it is possible to choose an orientation for each $E_x$ such that the bundle can be locally trivialized by charts determining local sections of *positive* frames. In that case a rank $n$ real bundle admits a $\mathrm{GL}^+(n; \mathbb{R})$-reduction. If it is riemannian, the reduction can be done to $\mathrm{SO}(n; \mathbb{R})$.

Other functorial properties admit an interpretation in terms of reduction, i.e., the existence of a volume form on each fiber (compatible with trivializations) is an $\mathrm{SL}(n; \mathbb{K})$-reduction. The *projectivization* of a bundle can be understood in terms of *reduction* to $\mathrm{PGL}(n; \mathbb{K})$.

On the other side we can also extend the group. For example, any $\mathrm{O}(n; \mathbb{R})$-principal bundle naturally induces a $\mathrm{GL}(n; \mathbb{R})$-principal bundle (for the associated riemannian vector bundle it amounts to *forget* the metric). In the same way a $\mathrm{GL}(n; \mathbb{C})$-principal bundle induces a $\mathrm{GL}(2n; \mathbb{R})$-principal bundle (for the associated riemannian vector bundle it amounts to *forget* the multiplication by $\sqrt{-1}$).

**3.6. Lifting of a principal bundle.** Let $H, G$ be two Lie groups, with an epimorphism $\rho : H \to G$. A $G$-principal bundle $\pi : P_G \to X$ can be lifted to an $H$-principal bundle (via $\rho$) if there is a bundle morphism (over $1_X : X \to X$) equivariant with respect to $\rho$.

It can be done if the transition functions $\Phi_{V,U} : U \cap V \to G$ can be lifted to $\Psi_{V,U} : U \cap V \to H$, $\rho \circ \Psi_{V,U} = \Phi_{V,U}$, such that the cocycle condition is preserved. For example, recall that for $\mathrm{SP}(n; \mathbb{R})$ is the two-fold cover of $\mathrm{SO}(n; \mathbb{R})$. A *spin* structure of a riemannian vector bundle is related with the existencie of lifting of the orthogonal principal bundle to the spin group.

**3.7. Universal principal bundles.** Let us fix $G = \mathrm{GL}(r; \mathbb{R})$. For $N \geq r$, we consider the Stiefel manifold $V(r, N)$ defined as $\{V = (v_1, \ldots, v_r) \in (\mathbb{R}^N)^r \equiv \mathrm{Mat}(N \times r; \mathbb{R}) \mid \mathrm{Rank}\, V = r\}$. We can also define $V(r, \infty) := \bigcup_{N \geq r} V(r, N)$ with the limit topology. There is a natural map $\pi : V(r, \infty) \to \mathrm{Gr}(r, \infty)$, where $\pi(V)$ is the subspace generated by $v_1, \ldots, v_r$. We interpret the elements of $V(r, \infty)$ as matrices with $r$ columns and rows in bijection with $\mathbb{N}$ such the number of nonvanishing rows is finite. We denote by $\tau_+ : V(r, \infty) \to V(r, \infty)$ the map that shifts downward (one place) all the rows, adding a zero row in the first place.

**Lemma 3.8.** *Let $H : V(r, \infty) \times [0, 1]$ given by $H(V, t) := (1 - t)V + t\tau_+(V)$. It is a continuous homotopy relating $1_{V(r;\infty)}$ and $\tau_+$.*

*Proof.* We need to check that $H$ is well-defined, i.e., if $V \in V(r; \infty)$, then $\mathrm{Rank}\, H(V, t) = r$, for $t \in (0, 1)$. Let us take $N$ such that all the rows of $V$ vanish starting from $N$. Then, we can see $V$ and $H(V, t)$ as $N \times r$ matrices. Let $w_1, \ldots, w_N$ be the row vectors. We know that $\dim \mathbb{R}\langle w_1, \ldots, w_{N-1}\rangle = r$. The rows of $H(V, t)$ are

$$(1 - t)w_1, (1 - t)w_2 + tw_1, \ldots, (1 - t)w_{N-1} + tw_{N-2}, tw_{N-1}$$

which generate the same subspace of dimension $r$. □

Let $V_0$ be the matrix whose first $r$ rows are the identity and the next ones vanish.

**Lemma 3.9.** *Let $K : V(r, \infty) \times [0, 1]$ given by $H(V, t) := (1 - t)\tau_+^r(V) + tV_0$. It is a continuous homotopy relating $\tau_+$ and a constant map. In particular, $V(r, \infty)$ is contractible.*

The fact that $V(r, \infty)$ is important. Let us check some hidden consequence.

**Proposition 3.10.** *Let $\pi : P \to X$ be a $G$-principal bundle. Then there is a continuous map $f : X \to \mathrm{Gr}(r, \infty)$ such that $P = f^*(V(r, \infty))$. In particular the isomorphism types of $G$-principal bundles over $X$ is in bijection with $[X, \mathrm{Gr}(r, \infty)]$*

*Proof.* It is enough to consider the map for the associated vector bundle constructed as $E := P \times_G \mathbb{R}^n$ where

$$P \times_G \mathbb{R}^n := (P \times G)/(p \cdot g, v) \equiv (p, g \cdot v). \qquad \square$$

We are going to construct universal principal bundles for arbitrary Lie groups. Fix a Lie group $G$ and consider the cone $\mathrm{Cone}(G) := (G \times [0,1])/G \times \{0\}$. Its elements will be written as $t \cdot g$ (if $t = 0$, the value of $g$ is irrelevant). We define

$$EG_n := \{\langle t_1 \cdot g_1, \dots, t_n \cdot g_n \rangle \in \mathrm{Cone}(G)^n \mid \sum_{i=1}^{n} t_i = 1\}.$$

There is a natural right $G$-action $\langle t_1 \cdot g_1, \dots, t_n \cdot g_n \rangle \cdot g := \langle t_1 \cdot g_1 g, \dots, t_n \cdot g_n g \rangle$ and let $BG_n := EG_n/G$, whose elements (equivalence classes) will be denoted by $[t_1 \cdot g_1 : \dots : t_n \cdot g_n]$. Let us check that $\pi : EG_n \to BG_n$ is a principal bundle. If $U_i \subset BG_n$ is the open set for which $t_i > 0$, note that an element of $U_i$ can be written as $[t_1 \cdot g_1 : \dots : t_i \cdot \mathbf{1} : \dots : t_n \cdot g_n]$, and there is an obvious trivialization.

As usual we consider the union of these objects with inductive limit topology to construct $EG$, $BG$. Using the same kind of ideas as before, it is easily seen that $EG$ is contractible.

**Theorem 3.11.** *Let $\pi_P : P \to X$ be a $G$-principal bundle. Then there is map $f : X \to BG$ such that $P \cong f^*(EG)$.*

*Proof.* Let $\{U_\alpha\}_{\alpha \in A}$ be a locally finite open covering of $X$ for which $P$ is trivial over any $U_\alpha$. We assume also the existence of a partition of unity $\{\rho_\alpha\}_{\alpha \in A}$ supported by the covering. Given any finite subset $I \subset A$, we associate the open set

$$W_I := \{x \in X \mid \rho_i(x) > \rho_j(y) \text{ whenever } i \in I, j \in J\}.$$

Let $W_m$ the union of the open sets $W_I$ for $I$ such that $\#I = m$. From the properties of the partitions of a unity each $W_I$ is contained in any $U_i$ (for $i \in I$), $W_m$ is a disjoint union of the sets $W_I$, and $\{W_m\}_{m \in \mathbb{N}}$. The family $W_m$ is locally finite and a new partition of unity $\{\rho_m\}_{m \in \mathbb{N}}$ can be defined. Note that $P$ trivializes over each $W_m$.

Let $\Phi_m : W_m \times G \to \pi_P^{-1}(W_m)$ be a trivialization. For $p \in \pi_P^{-1}(W_m)$, let $\varphi_m(p) := \pi_G(\Phi_m^{-1}(p))$. Then, we define

$$F(p) := \langle \rho_m(\pi_P(p)) \cdot \Phi_m(p) \rangle_{m \in \mathbb{N}};$$

note that we need to have a value of $\Phi_m(p)$ if $p \notin W_m$. The map $F$ is clearly equivariant for the actions of $G$ and the result follows. □

*Remark* 3.12. With standard techniques we can prove that two principal bundles are isomorphic if and only they come from isotopic maps. As a consequence the bases of two universal $G$-principal bundles are homotopic. Since $EG$ is contractible this is the case for any one. There is a shift between the homotopy groups of $G$ and those of $BG$. Any contractible principal bundle is universal.

## 4. Thom isomorphism theorem and Euler class

In this section $\pi : E \to X$ is an oriented $\mathbb{R}$-vector bundle of rank $r$. If $V$ is an $\mathbb{R}$-vector space of dimension $r$, we recall that an orientation is the choice of an ordered basis up to a change of basis of positive determinant. Let us interpret it from a cohomology point of view. Let $V_0 := V \setminus \{0\}$. From the long exact cohomology sequence of pairs, we have that $H^n(V, V_0; \mathbb{Z})$ is isomorphic to $H^{n-1}(V_0; \mathbb{Z})$; if we set in $V$ a scalar product this is isomorphic to $\tilde{H}^{n-1}(\mathbb{S}^{n-1}; \mathbb{Z}) \cong \mathbb{Z}$. A choice of a generator is related with the choice of an orientation. For example, if we choose the orientation of a basis $(v_1, \dots, v_n)$, it is given by a cocyle determined by its value equal to 1 on the simplex generated by $(0, v_1, \dots, v_n)$ translated by its barycenter. Let $e^n$ be the chosen generator for $V = \mathbb{R}^n$ with the positive orientation. We denote by $E_0 := E \setminus \{0_x \mid x \in X\}$.

**Lemma 4.1.** *For $n \geq 1$, the map $H^j(B; \mathbb{Z}) \to H^{n+j}(B \times \mathbb{R}^n, B \times \mathbb{R}^n_0; \mathbb{Z})$ given by $c \mapsto c \times e^n$ is an isomorphism.*

*Proof.* Let us start with $n = 1$, and the cohomology sequence of triples for $(B \times \mathbb{R}, B \times \mathbb{R}_0, B \times (0, \infty))$. Note that the middle terms $H^n(B \times \mathbb{R}, B \times (0, \infty))$ clearly vanish and he have isomorphisms

$$H^j(B \times \mathbb{R}_0, B \times (0, \infty)) \to H^{j+1}(B \times \mathbb{R}, B \times \mathbb{R}_0)$$

Working with the first pair we have that in this sequence the last map is surjective, and then it is short

$$H^j(B \times \mathbb{R}_0, B \times (0, \infty)) \to H^j(B \times \mathbb{R}_0) \to H^j(B),$$

implying that the first space is isomorphic to $H^j(B)$. It is not hard to see that the isomorphism can be expressed as in the statement. If $U$ is an open set of $B$, we prove that

$$H^j(B, U) \to H^{j+1}(B \times \mathbb{R}, U \times \mathbb{R} \cup B \times \mathbb{R}_0), \quad c \mapsto c \times e^1,$$

is an isomorphism, following Künneth formula. Let us prove the case $n > 1$. Note that $e^n = e^1 \times \dots \times e^1$. From induction hypothesis the map $H^j(B) \to H^{j+n-1}(B \times \mathbb{R}^{n-1}, B \times \mathbb{R}^{n-1}_0)$, $c \mapsto c \times e^{n-1}$, is an isomorphism. Replacing $B \mapsto B \times \mathbb{R}^{n-1}$, $U \mapsto B \times \mathbb{R}^{n-1}_0$, we have an isomorphism $H^{j+n-1}(B \times \mathbb{R}^{n-1}, B \times \mathbb{R}^{n-1}_0) \to H^{j+n}(B \times \mathbb{R}^n, B \times \mathbb{R}^{n-1}_0 \times \mathbb{R} \cup B \times \mathbb{R}^{n-1} \times \mathbb{R}_0)$ and the result follows. □

**Theorem 4.2** (Thom isomorphism Theorem)**.** *There is exactly one cohomology class $u \in H^n(E, E_0; \mathbb{Z})$ which restricts to the orientation class on each fiber. Moreover, $H^j(E; \mathbb{Z})$ is isomorphic to $H^{j+n}(E, E_0; \mathbb{Z})$ via $c \mapsto c \cup u$.*

This theorem implies in particular that $H^j(E, E_0; \mathbb{Z}) = 0$ if $j < n$.

*Proof.* We proof first for $E$ trivial. The isomorphism result has been stated. We need to prove the existence and uniqueness of $u$. Note that $1 \times e^n$ satisfies the property and is clearly unique.

Let us assume that $X = B' \cup B''$ open sets such that the statement is true for the bundles over $B', B''$ and $B' \cap B''$, denoted by $E', E''$ and $E^\cap$. Let $u', u'', u^\cap$ be the classes of the statement. Note that

$$0 = H^{n-1}(E^\cap, E_0^\cap) \to H^n(E, E_0) \to H^n(E', E_0') \oplus H^n(E'', E_0'') \to H^n(E^\cap, E_0^\cap)$$

By uniqueness the images of $u'$ and $u''$ on $H^n(E^\cap, E_0^\cap)$ are $u^\cap$, and $u$ exists ad is unique. The isomorphism statement comes from Mayer-Viétoris and the five lemma. In particular, we can prove the statement for compact $X$. Since it works also for arbitrary disjoint unions, it can be extended to paracompact spaces. □

**Definition 4.3.** The Thom class is the unique class $u \in H^n(E, E_0; \mathbb{Z})$ of the Theorem. The Euler class $e(E) \in H^n(X; \mathbb{Z})$ is the unique class such that $i^*u = \pi^* e(E)$

We state the Gysin sequence Theorem. It is a direct consequence of Thom isomorphism Theorem.

**Theorem 4.4.** *There exists a long exact sequence of cohomology*

$$\cdots \to H^j(X) \xrightarrow{\smile e(E)} H^{j+n}(X) \xrightarrow{\pi_0^*} H^{j+n}(E_0) \to \ldots$$

**Example 4.5.** Let us consider the tautological line bundle studied in Example 1.3. It is not hard to see that $E_0$ has the homotopy type of $\mathbb{S}^3$. Hence the Gysin sequence gives

$$\underset{=0}{H^1(\mathbb{S}^3)} \to \underset{\cong\mathbb{Z}}{H^0(\mathbb{S}^2)} \xrightarrow{\smile e(E)} \underset{\cong\mathbb{Z}}{H^2(\mathbb{S}^2)} \xrightarrow{\pi_0^*} \underset{=0}{H^2(\mathbb{S}^3)}$$

We need only to decide if the Euler class represents the positive or the negative generator of $H^2(\mathbb{S}^2; \mathbb{Z})$. Recall that $e(E) = i^*(u) \in H^2(E; \mathbb{Z})$, where $u$ is the Thom class, whose restriction to any fiber corresponds to its complex orientation; by Poincaré duality it corresponds to the dual of the fiber, i.e., with the image of the zero section, identified with $\mathbb{S}^2$. Let $c$ be the positive generator of $H^2(\mathbb{S}^2; \mathbb{Z}) \equiv H^2(E; \mathbb{Z})$; we know that $e(E) = \varepsilon c$, $\varepsilon = \pm 1$ and that $c(\mathbb{S}^2) = F \cdot \mathbb{S}^2 = 1$, if $F$ is a fiber of the bundle. Then

$$\varepsilon = e(E)(\mathbb{S}^2) = \mathbb{S}^2 \cdot \mathbb{S}^2 = -1.$$

In particular, both definitions of Euler class match.

**Proposition 4.6.** *If $f : Y \to X$ is continuous then $e(f^*(E)) = f^*(e(E))$. In particular, if $E$ is trivial the $e(E) = 0$.*

**Proposition 4.7.** *Let $-E$ be the same bundle with the opposite orientation. Then, $e(-E) = -e(E)$. In particular, if $n$ is odd, then $2e(E) = 0$.*

The last statement comes from the fact that multiplying by $-1$ is an isomorphism from $E$ to $-E$ if $n$ is odd.

**Proposition 4.8.** *Let $E', E''$ two oriented vector bundles. Then $e(E' \times E'') = e(E') \times e(E'')$. If the base space coincides then $e(E' \oplus E'') = e(E') \smile e(E'')$.*

In terms of Thom classes the equality involve a sign for the product of ranks. It does not affect since this Euler class are annihilated by 2 if the ranks are odd.

**Proposition 4.9.** *If $E$ admits a nowhere vanishing section, then $e(E) = 0$.*

*Proof.* Let $E_1$ be the subbundle defined by the section; using a riemannian metric, we can consider its orthogonal $E_2$, i.e., $E = E_1 \oplus E_2$, where $E_1$ is trivial. Then $e(E) = e(E_1) \smile e(E_2) = 0$. □

**Example 4.10.** Let us consider the tautological universal bundle $\pi : \tau_{\mathbb{C}}(1, \infty) \to \mathbb{P}_{\mathbb{C}}(\infty)$; denote $\tau_{\mathbb{C}}(1, \infty)$ as $\tau$ for short. We are going to see that its Euler class coincides with $-c_1$. It is enough to study the tautological bundle over $\mathbb{P}^1(\mathbb{C})$. We need more knowledge to identify the Euler class for the tautological universal bundle over a Grasmannian.

**Proposition 4.11.** *Let $E_1, E_2$ be two complex line bundles over $X$. Then, $c_1(L_1 \otimes L_2) = c_1(L_1) + c_2(L_2)$.*

*Proof.* It is straightforward for line bundles over $\mathbb{P}^1$ and it extends to arbitrary $CW$-complex. □

## 5. Chern classes à la Bott-Tu

We are going to prove Theorem 1.6 which is a particular case of Leray-Hirsch Theorem.

*Proof of Theorem* 1.6. Let $U \in X$ be an open set and consider the map

$$\Psi_U : H^*(U)\langle 1, x, \dots, x^{n-1}\rangle \to H^*(\mathbb{P}(E)_{|U}, \mathbb{Z})$$

such that $\Psi_U(cx^j) := \pi_{\mathbb{P}}^*(c)x^j$. We follow the same ideas as in the proof of Thom isomorphism Theorem: prove it first for trivializing open sets, second for decomposition in two open sets such that the theorem holds for them and their intersection, third for disjoint unions of trivializing open sets. □

**Example 5.1.** From this definition the naturality goes easily. If $f : Y \to X$ is a map, then from the naturality of the Euler class, if $y$ is the Euler class of the tautological bundle over $\mathbb{P}(f^*(E))$ then $y = f^*x$. Then, $0 = f^*(\sum_{j=0}^n c_j(E)x^{n-j}) = \sum_{j=0}^n f^*(c_j(E))y^{n-j}$.

**Example 5.2.** If $L$ is a line bundle, then $X = \mathbb{P}(L)$. By the definition of the class $x$ we have $x + c_1(L) = 0$, $c_1(L) = e(L)$.

**Theorem 5.3.** *Let $E = E_1 \oplus E_2$. Then $c(E) = c(E_1) \smile c(E_2)$.*

*Proof.* Denote by $n_i$ the ranks of the bundles. Let us consider the projectivized bundles $\mathbb{P}(E) \supset \mathbb{P}(E_1), \mathbb{P}(E_2)$; note that $\mathbb{P}(E_1) \cap \mathbb{P}(E_2) = \emptyset$. Moreover, by standard projective geometry $U_1 := \mathbb{P}(E) \setminus \mathbb{P}(E_1) \simeq \mathbb{P}(E_2)$ and $U_2 := \mathbb{P}(E) \setminus \mathbb{P}(E_2) \simeq \mathbb{P}(E_1)$.

Let $x, x_1, x_2$ be the opposite of the Euler classes of the corresponding tautological bundles. By naturality, if $i_j : \mathbb{P}(E_j) \hookrightarrow \mathbb{P}(E)$ are the inclusions, then $i_j^*(x) = x_j$. Let us consider the classes

$$\mathbf{C}_i := \sum_{j=0}^{n} c_j(E_i) x^{n-j} \in H^{n_i}(\mathbb{P}(E); \mathbb{Z})$$

By definition $i_j^*(\mathbf{C}_j) = 0$; hence it lifts to a class

$$\mathbf{D}_j \in H^{n_j}(\mathbb{P}(E), \mathbb{P}(E_j); \mathbb{Z}) = H^{n_j}(\mathbb{P}(E), U_{3-j}; \mathbb{Z}).$$

The following exact sequence

$$\begin{array}{ccc} H^{n_1}(\mathbb{P}(E), U_2; \mathbb{Z}) \times H^{n_2}(\mathbb{P}(E), U_1; \mathbb{Z}) & \xrightarrow{\ \smile\ } & H^{n_1+n_2}(\mathbb{P}(E), U_1 \cup U_2; \mathbb{Z}) \equiv 0 \\ \downarrow & & \downarrow \\ H^{n_1}(\mathbb{P}(E); \mathbb{Z}) \times H^{n_2}(\mathbb{P}(E); \mathbb{Z}) & \xrightarrow{\ \smile\ } & H^{n_1+n_2}(\mathbb{P}(E); \mathbb{Z}) \end{array}$$

implies that $\mathbf{C}_1 \smile \mathbf{C}_2 = 0$ and the result follows. $\square$

**Corollary 5.4.** *Let $E = L_1 \oplus \cdots \oplus L_n$, splitting of $E$ as Whitney sum of line bundles. Then*

$$c(E) = \prod_{j=1}^{n} c(L_j) = \prod_{j=1}^{n} (1 + c_1(L_j)).$$

*In particular, the top Chern class is the Euler class.*

**Definition 5.5.** Let $E$ be complex vector bundle over $X$. We say that $f : Y \to X$ is a splitting map if $f^*(E)$ decomposes as Withney sum of line bundles and $f^* : H^*(X; \mathbb{Z}) \to H^*(Y; \mathbb{Z})$ is injective.

**Proposition 5.6.** *Any complex vector bundle has a splitting map (e.g, the* flag associated *bundle).*

*Proof.* We proceed by induction of the rank $n$. If $n = 1$, the result is obvious. Assume $n > 1$. We are going to use 1.6. Consider $\pi_{\mathbb{P}} : \mathbb{P}(E) \to X$; recall that $\pi_{\mathbb{P}}^*$ defines an injection of $H^*(X; \mathbb{Z})$ into $H^*(\mathbb{P}(E); \mathbb{Z})$. Let us consider the pull-back $\tilde{\pi} : \pi_{\mathbb{P}}^*(E) \to \mathbb{P}(E)$. We have considered that tautological line subbundle $\tau(E) \subset \pi_{\mathbb{P}}^*(E)$; using a hermitian metric we can construct the orthogonal subbundle $Q$ such that $\pi_{\mathbb{P}}^*(E) = \tau(E) \oplus Q$, where $Q$ is a complex bundle of rank $n-1$ over $\mathbb{P}(E)$. By induction, there is a map $f : Y \to \mathbb{P}(E)$ such that $f^*(Q)$ splits and $f^* : H^*(\mathbb{P}(E); \mathbb{Z}) \to H^*(Y; \mathbb{Z})$ is injective. Then, $\pi_{\mathbb{P}} \circ f : Y \to X$ is a split map. $\square$

*Remark* 5.7. We can be more explicit on the space $Y$. Given a (complex) vector space of dimension $n$, a *flag* of $V$ is a chain of subspaces $0 = H_0 \subset H_1 \subset \dots H_{n-1} \subset H_n = V$, $\dim_{\mathbb{C}} H_j = j$. The space $F(V)$ of all the flags of $V$ is called the *flag space* of $V$ and it is a complex variety of (complex) dimension $\frac{n(n-1)}{2}$. Under the presence of a hermitian scalar product we can identify a flag with an ordered decomposition of $V$ as direct sum of 1-dimensional subspaces.

Since $\mathrm{GL}(n;\mathbb{C})$ acts on $F(\mathbb{C}^n)$, we can define for a $\mathbb{C}$-vector bundle $\pi : E \to X$ of rank $n$ the associated flag bundle $F(E)$. Since it is constructed as a tower of projective bundles, $H^*(X;\mathbb{Z})$ is identified as a subring of $H^*(F(E);\mathbb{Z})$ and $E$ splits over $F(E)$.

**5.8. Splitting principle.** If a polynomial identity for Chern classes of complex vector bundles holds for Withney sums of line vector bundles, then it holds for any vector bundle.

Let us illustrate this principle for the result that top Chern class is Euler class, already proved for decomposable bundles. Let $\pi : E \to X$ be a complex bundle of rank $n$, and let $f : Y \to X$ be a splitting map. Then,

$$f^* e(E) = e(f^* E) = c_n(f^*(E)) = f^*(c_n(E)) \Longrightarrow e(E) = c_n(E).$$

**Proposition 5.9.** $c_1(E) = c_1(\det E)$.

*Proof.* Let us prove it for a splitted bundle $E = L_1 \oplus \dots \oplus L_r$. Note that $\det E = \bigwedge^r E = L_1 \oplus \dots \oplus L_r$. Then,

$$c_1(\det E) = \sum_{j=1}^{r} c_1(L_j) = c_1(E). \qquad \square$$

**Proposition 5.10.** *Let $E$ be a complex vector bundle. Then $c_j(\bar{E}) = (-1)^j c_j(E)$.*

*Proof.* It is enough to prove it for decomposable bundles. If $L$ is a line bundle, using a hermitian metric we have that $\bar{L} \cong L*$; since $L \otimes L^*$ is a trivial bundle, we have that $c(\bar{L}) = 1 - c_1(L)$. The result follows easily. $\square$

**5.11. Todd class.** Consider the analytic function

$$Td(x) := \frac{x}{1 - e^{-x}} = 1 + \frac{1}{2}x + \frac{1}{12}x^2 - \frac{1}{720}x^4 + \dots$$

The total *Todd class* for a line bundle $L$ is defined as $Td(L)$. If $E = L_1 \oplus \dots \oplus L_r$ then the Todd class of $E$ is $TD(E) := \prod_{j=1}^{r} Td(L_j)$:

$$Td(E) = 1 + \frac{1}{2}\sum_{j=1}^{r} c_1(L_j) + \frac{1}{12}\sum_{j=1}^{r} c_1(L_j)^2 + \frac{1}{4}\sum_{1 \leq i < j \leq r} c_1(L_i)c_1(L_j) + \dots =$$
$$1 + \frac{1}{2}c_1(E) + \frac{1}{12}c_1(E)^2 - \frac{1}{12}c_2(E) + \dots$$

The splitting principle allows to extend the definition to any bundle. The Todd class appears in a natural way in Riemann-Roch formula.

**5.12. Chern character.** We apply the ideas in the construction of the Todd class. For a decomposable character $E = L_1 \oplus \dots \oplus L_r$, we define

$$ch(E) := \sum_{j=1}^{r} \exp(c_1(L_j)) = r + c_1(E) + \frac{1}{2}(c_1(E)^2 - 2c_2(E)) + \frac{1}{3!}(c_1(E)^3 - 3c_1(E)c_2(E)) + \dots$$

The extension comes from the Splitting Principle. Note that $ch(E \oplus F) = ch(E) + ch(F)$ and $ch(E \otimes F) = ch(E) \smile ch(F)$. Note that the Chern character defines a ring homomorphism $K(X) \to H^*(X; \mathbb{Q})$.

We finish this section with the characterization of the cohomology ring $H^*(\mathrm{Gr}_{\mathbb{C}}(r, \infty))$. Let $F_r$ be the flag bundle of the tautological bundle. For simplicity a hermitian form is used in $\mathbb{C}^\infty$ in order to deal with $n$-tuples of subspaces of dimension 1 instead of sequence of subspaces. We can see $F_r$ as a subspace of $\mathbb{P}(\mathbb{C}^\infty)^r$, namely the set of $r$-tuples whose sum is of dimension $r$. A combination of $r$ downard shifts, the top-addition of the identity, allows to see that the inclusion is a homotopy equivalence.

Let $\pi_j : \mathbb{P}(\mathbb{C}^\infty)^r \to \mathbb{P}(\mathbb{C}^\infty)$ the $j^{\text{th}}$ projection. Let us denote by $L_j := \pi_j^*(\tau_1)$, and let $t_j := -c_j(\tau_1)$. Then,

$$H^*(F_r, \mathbb{Z}) = H^*(\mathbb{P}(\mathbb{C}^\infty)^r; \mathbb{Z}) = \mathbb{Z}[t_1, \dots, t_r], \quad \deg t_j = 2.$$

The map $\pi_F : F_r \to \mathrm{Gr}_{\mathbb{C}}(r, \infty)$ is splitting for $\tau_r$, and $\pi_F^*(\tau_r) = L_1 \oplus \dots \oplus L_r$ so $\pi^*(c(\tau_r)) = \prod_{j=1}^{r} = (1 - t_j)$, i.e., $c_j(\tau_r)$ is identified with the $j^{\text{th}}$-symmetric polynomial (up to $(-1)^j$).

Since any permutation map in the variables keeps everything, we have that the ring $H^*(\mathrm{Gr}_{\mathbb{C}}(r, \infty))$ is inside the symmetric polynomials by the permutation of the variables, generated by $c_j(\tau_r)$, then

$$H^j(\mathrm{Gr}_{\mathbb{C}}(r, \infty); \mathbb{Z}) = \mathbb{Z}[c_1(\tau_r), \dots, c_r(\tau_r)].$$

Let us explain an alternative proof from Milnor's book. We fix in $\mathbb{C}^\infty$ a hermitian scalar product; let us consider Gysin sequence:

$$\dots \to H^{j-2r}(\mathrm{Gr}_{\mathbb{C}}(r, \infty)) \xrightarrow{\smile c_r(\tau_r)} H^j(\mathrm{Gr}_{\mathbb{C}}(r, \infty)) \longrightarrow H_j((\tau_r)_0) \to \dots$$

Let us study the space $(\tau_r)_0$, its elements are of the form $(S, v)$, $S$ subspace with $\dim S = r$, $v \in S$, $v \neq 0$. From such an element we can construct the subspace $S_v^\perp$, the orthogonal to $v$ in $S$, $\dim S_v^\perp = r - 1$. We construct a map $F : (\tau_r)_0 \to \mathrm{Gr}_{\mathbb{C}}(r-1, \infty))$. Let us see what a fiber is. We fix $T$ subspace, $\dim T = r - 1$; its preimages are of the form $(S, v)$, where $v$ is a non-zero vector orthogonal to $T$ and $S = T \oplus \mathbb{C}\langle v \rangle$. Hence the fiber has the homotopy type of $\mathbb{S}^\infty$, which is contractible. Since $F$ is a fiber bundle, we rewrite the exact sequence

$$\dots \to H^{j-2r}(\mathrm{Gr}_{\mathbb{C}}(r, \infty)) \xrightarrow{\smile c_r(\tau_r)} H^j(\mathrm{Gr}_{\mathbb{C}}(r, \infty)) \longrightarrow H_j(\mathrm{Gr}_{\mathbb{C}}(r-1, \infty)) \to \dots$$

Let us study the last map. We start from the the tautological bundle $\tau_r$; if $\pi_0$ is the projection defined over $(\tau_r)_0$, then $\pi_0^*(\tau_r)$ is given by $(S, v, w)$, $\dim S = r$, $v, w \in S$, $v \neq 0$. This is equivalent to the data $(S_v^\perp, v, w - \frac{\langle v,w\rangle}{\|v\|^2} v)$, $\langle v, w\rangle$, i.e., $\pi_0^*(\tau_r)$ is isomorphic to $F^*$ of the Whitney sum of $\tau_{r-1}$ and the trivial line bundle. As a consequence, the last map satisifies $c_j(\tau_r) \mapsto c_j(\tau_{r-1})$ for $j < r$. For $j < 2r$ the map $H^j(\mathrm{Gr}_\mathbb{C}(r, \infty)) \longrightarrow H_j(\mathrm{Gr}_\mathbb{C}(r-1, \infty))$ is an isomorphism.

## 6. Pontrjagin characteristic classes

In this section, we deal with $\mathbb{R}$-vector bundles. We recall two concepts from linear algebra. Let $V$ be a $\mathbb{C}$-vector space of dimension $n$. We can define another vector space $\bar{V}$, where the underlying $\mathbb{R}$-vector space is the same as the one of $V$ but for $v \in \bar{V}$, we have $\sqrt{-1} \cdot v = -\sqrt{-1} v$ (the last product as element in $V$). Of course, since it is a $\mathbb{C}$-vector space of the same dimension, $V, \bar{V}$ are isomorphic, but they are not naturally isomorphic (if we perform a basis change in $V$ with matrix $A$, the change in $\bar{V}$ is given by $\bar{A}$). Nevertheless, if we fix a hermitian metric in $V$, then $\bar{V}$ and $V^* = \mathrm{Hom}_\mathbb{C}(V, \mathbb{C})$ are naturally isomorphic.

Hence, if $\pi : E \to X$ is a complex vector bundle, we can define a its conjugated as $\bar{\pi} : \bar{E} \to X$, where the real bundle structures coincide but for each $x \in X$, $\bar{E}_x = \overline{E_x}$. From the point of view of transition functions, we take the conjugate matrices. Note that $\bar{E}$ and $E^*$ are isomorphic if the base is paracompact, since a hermitian metric on $E$ exists.

The other operation is the complexification. Let $V$ an $\mathbb{R}$-vector space of dimension $n$. Then $V_\mathbb{C} := V \otimes_\mathbb{R} \mathbb{C}$ is a complex vector space of dimension $n$. Its elements are of the form $v + \sqrt{-1} w$ for $v, w \in V$. It is not hard to see that the map $v + \sqrt{-1} w \mapsto v - \sqrt{-1} w$ is an isomorphism of $V$ and $\bar{V}$.

Hence, if $\pi : E \to X$ is a real vector bundle, we can define a its complexified as $\pi_\mathbb{C} : E_\mathbb{C} \to X$, $(E_\mathbb{C})_x = (E_x)_\mathbb{C}$. We have performed a *reduction* of structural group, from $\mathrm{GL}(n; \mathbb{R})$ to $\mathrm{GL}(n; \mathbb{C})$. In this case $E$, $\bar{E}$ and $E^*$ are isomorphic.

Note that $c_j(E_\mathbb{C}) = c_j(\bar{E}_\mathbb{C}) = (-1)^j c_j(E_\mathbb{C})$. We have then $2c_{2k+1}(E) = 0$, in particular, it will vanish if there is no 2-torsion.

**Definition 6.1.** The $j^{\text{th}}$-*Pontrjagin characteristic class* of $E$ is $p_j(E) := (-1)^j c_{2j}(E_\mathbb{C})$. The *total Pontrjagin class* is $p(E) := \sum_{j \geq 0} p_j(E)$.

It is obvious that Pontrjagin classes behave nicely for pull-backs. For Whitney sums the equality $p(E_1 \oplus E_2) = p(E_1) \smile p(E_2)$ holds only up to 2-torsion. It is an equality if $E_2$ is a trivial bundle (the total Pontrjagin class of a trivial bundle is 1).

**Example 6.2.** The total Pontrjagin class of the tangent bundle to a sphere $\mathbb{S}^N$ is 1. Recall that the Whitney sum of the tangent bundle and the normal bundle (trivial) is trivial.

Let $V$ be a $\mathbb{C}$-vector space of dimension $n$. Let $V_{\mathbb{R}}$ its *realification* (we forget the complex structure), an $\mathbb{R}$-vector space of dimension $2n$. Consider now its complexification $(V_{\mathbb{R}})_{\mathbb{C}}$. The original multiplication by $\sqrt{-1}$ defines a morphism $J : (V_{\mathbb{R}})_{\mathbb{C}} \to (V_{\mathbb{R}})_{\mathbb{C}}$, such that $J^2$ is minus the identity. Let $V_{\pm}$ be the eigenspace for $\pm\sqrt{-1}$; note that $(V_{\mathbb{R}})_{\mathbb{C}} = V_+ \oplus V_-$, where

$$V_+ = \{v - \sqrt{-1}J(v) \mid v \in V\} \cong V, \qquad V_+ = \{v + \sqrt{-1}J(v) \mid v \in V\} \cong \bar{V}.$$

Hence $(V_{\mathbb{R}})_{\mathbb{C}} \cong V \oplus \bar{V}$.

Let us consider a complex vector bundle $E$. Then

$$c_{2j+1}((E_{\mathbb{R}})_{\mathbb{C}}) = c_{2j+1}(E \oplus \bar{E}) = \sum_{k=0}^{2j+1}(-1)^k c_k(E) \cdot c_{2j+1-k}(E) = 0,$$

and

$$c_{2j}((E_{\mathbb{R}})_{\mathbb{C}}) = c_{2j}(E \oplus \bar{E}) = \sum_{k=0}^{2j}(-1)^k c_k(E) \cdot c_{2j-k}(E) = (-1)^j c_j(E)^2 + 2\sum_{k=1}^{j}(-1)^{j+k} c_{j-k}(E) \cdot c_{j+k}(E),$$

i.e.,

$$p_j(E_{\mathbb{R}}) = c_j(E)^2 + 2\sum_{k=0}^{j-1}(-1)^{(k)} c_{j-k}(E) \cdot c_{j+k}(E).$$

## 7. Chern-Weil definition of Chern classes

In this section $X$ is a $\mathcal{C}^\infty$ manifold and $\pi : E \to X$ a $\mathcal{C}^\infty$ $\mathbb{C}$-vector bundle of rank $n$. Given this bundle we denote $\mathcal{E}^j(E)$ the space of $j$-forms with values in $E$. In particular, $\mathcal{E}^0(E) = \mathcal{C}^\infty(E)$, the space of sections of the bundle. Note that all these definitions keep their meaning for open sets of $X$. In particular, if $U \subset X$ is an open set where $E$ trivializes, this means that there exists $h_i \in \mathcal{C}^\infty(E|U)$ such that $\forall x \in U$, $h_x := {}^t(h_1(x), \ldots, h_n(x))$ is a basis of $E_x$. In this way, a section or a form on $U$ can be expressed as

$$s = \sum_{j=1}^{r} f_j h_j \in \mathcal{C}^\infty(E|U), \quad h_j \in \mathcal{C}^\infty(U), \qquad \omega = \sum_{j=1}^{r} \omega_j h_j \in \mathcal{E}^j(E_{|U}), \quad \omega_j \in \mathcal{E}^j(U).$$

**Definition 7.1.** A *connexion* on $X$ is $\mathbb{C}$-linear map $\nabla : \mathcal{C}^\infty(E) \to \mathcal{E}^1(E)$ such that if $f \in \mathcal{C}^\infty(X)$ and $s \in \mathcal{C}^\infty(E)$, then $\nabla(fs) = df \otimes s + f\nabla s$.

Let us compute the local form of a connexion. Let $U$ be a trivializing open set with $h$ (a column matrix of sections) a local basis of sections. Then, the connexion is determined by

$$\nabla h = \omega_h h, \quad \omega_h \in \mathrm{Mat}(n; \mathcal{E}^1(U))$$

and the product means matrix multiplication. Let $k$ be another basis such that $k = gh$. A similar formula with $\omega_k$ exists, let us see the relationship:

$$\omega_k k = \nabla k = \nabla gh = dg \otimes h + g\omega_h \otimes h = (dg \cdot g^{-1} + g\omega_h g^{-1})h.$$

As a consequence, $\omega_k = dg \cdot g^{-1} + g\omega_h g^{-1}$.

The definition of the connexion can be extended to $\mathbb{C}$-linear maps $\nabla : \mathcal{E}^j(E) \to \mathcal{E}^{j+1}(E)$, where for $\alpha \in \mathcal{E}^j(X)$ and $s \in \mathcal{C}^\infty(E)$, we have

$$\nabla(\alpha \otimes f) = d\alpha \otimes f + (-1)^j \alpha \nabla(f).$$

More generally, if $\alpha \in \mathcal{E}^j(X)$, and $\eta \in \mathcal{E}^i(E)$, then

$$\nabla(\alpha \otimes \eta) = d\alpha \wedge \eta + (-1)^j \alpha \nabla \eta \in \mathcal{E}^{i+j}(E).$$

We computate the *curvature* of the connexion as $\nabla^2$. Let us express it in local equations:

$$\nabla^2 h = \nabla(\omega_h \otimes h) = (d\omega_h - \omega_h \wedge \omega_h) \otimes h = \Omega_h \otimes h,$$

where $\omega \wedge \omega$ stands for matrix and wedge product combined. Note that $0 = d(g \cdot g^{-1}) = dg \cdot g^{-1} + gd(g^{-1})$. Then,

$$\begin{gathered} d\omega_k - \omega_k \wedge \omega_k = d(dg \cdot g^{-1} + g\omega_h g^{-1}) - (dg \cdot g^{-1} + g\omega_h g^{-1}) \wedge (dg \cdot g^{-1} + g\omega_h g^{-1}) = \\ dg(g^{-1})dg(g^{-1}) + dg \wedge \omega_h g^{-1} + g \wedge d\omega_h g^{-1} + \\ g \wedge \omega_h (g^{-1})dg(g^{-1}) - dg(g^{-1})dg(g^{-1}) - dg \wedge \omega_h g^{-1} - g\omega_h g^{-1} dg \cdot g^{-1} - g\omega_h \wedge \omega_h g^{-1} = \\ g(d\omega_h - \omega_h \wedge \omega_h)g^{-1} \Longrightarrow \Omega_k = g\Omega_h g^{-1}. \end{gathered}$$

All these computations are local and depend on the bases $h, k$. Consider $\det(I_n - \Omega_h)$; since the entries are forms of even degree, the wedge products commute and the determinant is well defined. Moreover since $\det(I_n - \Omega_h) = \det(I_n - \Omega_k)$, the expression is well-defined on $X$ without dependence on the local frames. Let us denote

$$\det\left(I_n - \frac{1}{2\sqrt{-1}\pi}\Omega\right) = c_0^\nabla + c_1^\nabla + \cdots + c_{n-1}^\nabla + c_n^\nabla, \quad c_j^\nabla \in \mathcal{E}^{2j}(X), \quad c_0^\nabla = 1.$$

**Example 7.2.** Let us consider the tautological fiber bundle $\tau$ over $\mathbb{P}^1(\mathbb{C})$. We consider the following two charts, $U^+ = \mathbb{D}_2$ (the open disk of radius 2) and $U^- = \{\infty\} \cup (\mathbb{C} \setminus \bar{\mathbb{D}}_{\frac{1}{2}})$. We consider the sections $s_\pm \in \mathcal{C}^\infty(\tau_{|U^\pm})$, given by

$$s_+(x) = ((x,1)[x:1]), \qquad s_(x) = ((1, x^{-1})[x:1]).$$

Let $U := U^+ \cap U^-$. On $U$, we have $s_+ = xs_-$. We want to choose a connexion such that $\nabla s_- = 0$. It imposes a condition for $\nabla s^+$, namely

$$(\nabla s^+)_{|U} = dx \otimes s^-_{|U} = \frac{dx}{x} \otimes s^-_{|U}.$$

We can choose a form $\omega^+ \in \mathcal{E}^1(U^+)$ such that $\omega^+ = \frac{dx}{x}$ on $U$. In our case, $\Omega^+ = 0$ and $\Omega^+ = d\omega^+$. It is a well defined 2-form and note that it closed. In particular it determines an element $[c_1] = [\frac{\omega}{2\sqrt{-1}\pi}]$ in the de Rham cohomology group $H^2(\mathbb{P}^1;\mathbb{C})$. Let us compute its integral over the fundamental class:

$$\int_{\mathbb{P}^1} c_1 = \frac{-1}{2\sqrt{-1}\pi} = \frac{-1}{2\sqrt{-1}\pi}\int_{\bar{\mathbb{D}}_1} d\omega = \frac{-1}{2\sqrt{-1}\pi}\int_{\mathbb{S}^1}\frac{dx}{x} = -1.$$

This situation is not particular to this example.

**Theorem 7.3.** *Under the previous notations:*

(1) $c_j^\nabla$ *is a closed form;*
(2) *if* $\nabla_0, \nabla_1$ *are* 2*-connexions,* $c_j^{\nabla_0}$ *and* $c_j^{\nabla_1}$ *define the same de Rham cohomology class.*

Before the proof of the theorem we introduce some notation. Let $P : \mathrm{Mat}(n;\mathbb{C}) \to \mathbb{C}$ be a polynomial such that $P(AB) = P(BA)$ (or equivalently $P(ABA^{-1}) = P(B)$ for any invertible matrix $A$); it will be called a *symmetric polynomial.* Any of these polynomials can be expressed as follows. There exists a symmetric function $Q : (\mathrm{Mat}(n;\mathbb{C})^n \to \mathbb{C}$ which can be expressed in terms of traces of products of the entries such that $P(A) = Q(A,\dots,A)$.

Given such a polynomial $P$ we define $P' : \mathrm{Mat}(n;\mathbb{C}) \to \mathrm{Mat}(n;\mathbb{C})$ such that

$$P'(A) := \left(\frac{\partial P}{\partial A_{ji}}(A)\right)_{1\leq i,j\leq n}$$

**Lemma 7.4.** *The matrices* $P'(A)$ *and* $A$ *commute.*

*Proof.* The derivative of a function $P(A(t))$ at $t = 0$ is given by $\mathrm{tr}(P'(A(0)) \cdot A'(0))$, using the chain rule. For any matrices $A, B$, we have $P(A(I_n + tB)) - P((I_n + tB)A)$ vanishes. It is also the case for its derivative with respect to $t$, for $t = 0$, hence

$$\mathrm{tr}\, P'(A)AB = \mathrm{tr}\, P'(A)BA = \mathrm{tr}\, AP'(A)B.$$

Since $\mathrm{tr}(AB)$ is a symmetric non-degenerated paring, we deduce that $P'(A)A = AP'(A)$. □

*Proof of the Theorem* 7.3. Note first that $dP(\Omega) = \mathrm{tr}(P'(\Omega)d\Omega)$. On the other hand $d\omega = \omega\wedge\Omega - \Omega\wedge\omega$. Then

$$dP(\Omega) = \mathrm{tr}(\omega\wedge\Omega - \Omega\wedge\omega)$$

For the second part note that $\nabla^t = (1-t)\nabla^0 + t\nabla^1$ can be interpreted either as a connexion in $X\times\mathbb{R}$ or a family of connexions in $X$. If its interpreted in $X\times\mathbb{R}$, it determines a de Rham class $C_t \in H^*(X\times\mathbb{R})$; the maps $i_\varepsilon : X \to X\times\mathbb{R}$, $\varepsilon = 0, 1$, are homotopic, and then $[P(\Omega^0)] = i_0^* C_t = i_1^* C_t = [P(\Omega^1)]$. □

**Definition 7.5.** The Chern-Weil characteristic classes ($j^{\mathrm{rm\ th}}$ and total) are defined as follows. The *total class* is the class of $\det(I_n - \Omega)$, and the $j^{\mathrm{rm\ th}}$ is the homogeneous component of degree $2j$ (i.e. in $H^{2j}(X; \mathbb{C})$).

The following theorem is easy.

**Theorem 7.6.** *The Chern-Weil characteristic classes satisfy:*

(1) *They behave well with respect to pull-backs.*
(2) $c(L_1 \oplus \cdots \oplus L_n) = (1 + c_1(L_1)) \cdot \cdots \cdot c_1(L_n)$.
(3) *The Chern-Weil first class of the tautological bundle over* $\mathbb{P}^{\nVdash}(\mathbb{C})$ *is the opposite of the positive generator of* $H^1(\mathbb{P}^1(\mathbb{C}); \mathbb{Z})$.

*In particular they match with the map* $H^*(X; \mathbb{Z}) \to H^*(X; \mathbb{C})$ *with respect to the usual Chern classes.*