Topology of arrangements of curves in surfaces

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§1.- Motivation and settings

Goal. X complex (projective) surface, $D \subseteq X$ curve. Relate topological and algebraic properties of (X, D)

Main cases. $X = \mathbb{P}^2$ (or $\mathbb{P}^2_w$ weighted projective plane, or Hirzebruch ruled surfaces or ...)

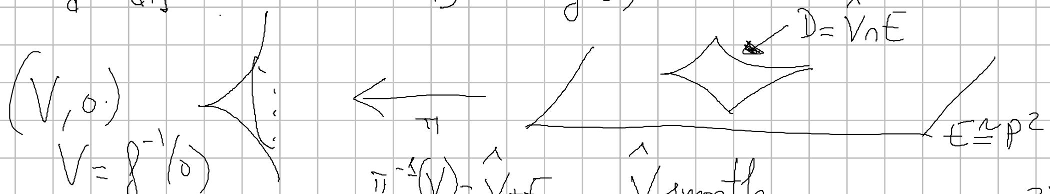
Curves in $\mathbb{P}^2$. ① Irreducible curves (e.g., rational cuspidal)
② Line arrangements ($\longrightarrow$ conic-line arrangements)
③ Rational arrangements (non-rational arrangements)

Why? Ⓐ X complex projective surface, $D \in \mathrm{Pic}(X)$ s.t. the map $X \xrightarrow{\;\delta_D\;} \mathbb{P}(H^0(X; \mathcal{O}(D))) \equiv \mathbb{P}^N$ is an embedding. Pick a (generic) $S \subseteq X$ $(N-3)$-projective surface, $\pi_S : \mathbb{P}^N \setminus S \longrightarrow \mathbb{P}^2$ projection and consider $\pi_X \circ \delta_D : X \xrightarrow{d:1} \mathbb{P}^2$; it is a ramified covering, $D \subseteq \mathbb{P}^2$ ramification curve

D is an irred. nodal-cuspidal curve, the monodromy $\pi_1(\mathbb{P}^2 \setminus D; x) \longrightarrow \mathrm{Sym}_d$ contains the main topological and numerical invariants of X.

Actually starting from any curve $D \subseteq \mathbb{P}^2$ and any monodromy $\pi_1(\mathbb{P}^2 \setminus D) \longrightarrow \mathrm{Sym}_d$ one can produce a compact surface as a ramified covering.

(B) $D \subseteq \mathbb{P}^2$ reduced curve, $D = \{F_d = 0\}$. Consider $f = F_d + F_{d+1} + \dots \in \mathbb{C}\{X, Y, Z\}$, $\operatorname{Sing}(D)$



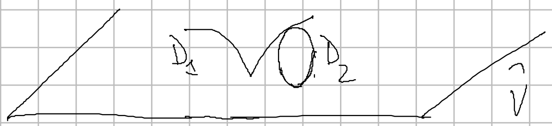
$V = f^{-1}(0)$ $\quad \pi^{-1}(V) = \hat{V} + E$ $\quad \hat{V}$ smooth

The $\binom{\text{abstract}}{\text{embedded}}$ topology of V is determined by $\binom{D \subseteq \mathbb{P}^2}{(\mathbb{P}^2, D)}$

Theorem 1 (Luengo) The link of V is the neighborhood of D in $\hat{V}$

$D = D_1 \cup \dots \cup D_r \quad \deg D_i = d_i$

$$D_i^2 = -d_i(d - d_i + 1) \qquad d = \sum_{i=1} d_i$$



Theorem 2 (Stevens) Δ characteristic polynomial of the monodromy of V

$$\Delta(t) = \frac{(t^d - 1)^{(d-1)^3 + 1 - \sum_p \mu_p}}{t-1} \prod_p \Delta^p(t^{d+1})$$

char. poly of the monodromy of (D, p)

Theorem 3 (_) The polynomial determining the 2-blocs of the monodromy of V is

$$\frac{\Delta_{\text{comb}}}{\Delta_C(t)}$$

Δ_{comb} depends only on $d_1, \dots, d_r$ and the topological type of the singularities of C

$\Delta_C(t)$ is the Alexander polynomial of C

$$X_D := \{T^d = F_d(X, Y, Z)\} \longrightarrow \mathbb{P}^2$$
$$[X:Y:Z:T] \longmapsto [X:Y:Z]$$

$$\sigma([X:Y:Z:T]) = [X:Y:Z:e^{\frac{2i\pi}{d}} T]$$

Δ_C = characteristic polynomial of σ_* on $H^1(X_D, \mathbb{C})$

(any birational model of (X, σ) works)

Consequence. The embedded topology may depend on something more than the topology of D and its numerical invariants.

Remark Similar statements for Lê-Yomdin singularities ($f = F_d + F_{d+k} + \dots$, $k \geq 1$) or weighted Lê-Yomdin singularities

§2. Zariski pairs (tuples)

Topological version of combinatorics

$D = D_1 \cup \ldots \cup D_r \subseteq \mathbb{P}^2$; let $V(D) \subseteq \mathbb{P}^2$ a compact regular neighborhood of D. The combinatorics of D is the oriented homeomorphism type of $(V(D), D)$.
$\partial V(D)$ is an oriented closed 3-manifold M_D!

Construction of M_D. Let $\pi : X \longrightarrow \mathbb{P}^2$ be the minimal sequence of blowing-ups such that $\pi^*(D)$ is a normal crossing divisor. We use π to construct $V(D)$.

$\pi^{-1}(D) = \bigcup_{i=1}^{t} A_i$, A_i smooth curve in X. Consider a tubular neighbourhood $T(A_i)$ of A_i in X with the following restrictions:

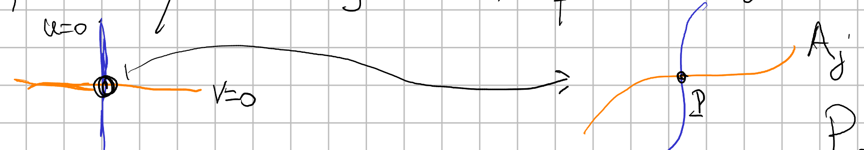
Step 1. Consider a tubular neighbourhood $T(A_i) \longrightarrow A_i$. Note that $\partial T(A_i) \longrightarrow A_i$ is an oriented S^1-bundle with Euler number A_i^2

Step 2. These tubular neighborhood will be constructed in a special way near the double points of $\pi^{-1}(D)$.

Let $P \in A_i \cap A_j$, $i \neq j$. We fix a holomorphic chart

$$\varphi : \{(u,v) \in \mathbb{C}^2 / |u|, |v| < 2\} \longrightarrow V_P \subseteq X$$

V_P open neighbourhood of P



Pairwise disjoint for double points

The neighborhoods satisfy: $\varphi(|u| \leq 1) = T(A_i) \cap V_P$
$\varphi(|v| \leq 1) = T(A_j) \cap V_P$

$W(D) := \bigcup T(A_i)$ is a regular neighborhood of $\pi^{-1}(D)$
and $V(D) := \pi(W(D))$ is a regular neighbourhood obtained by contracting the exceptional components of π.

<u>Remark</u> $\partial W(D) \cong \partial V(D)$

<u>Combinatorial alg-geom version of combinatorics</u>

Consider the dual graph of $\pi^*(D)$ weighted by $\begin{cases} A_i^2 = e_i \\ \text{genus}(A_i) = g_i \end{cases}$

From the construction and Waldhausen-Neumann:

<u>Fact</u> Both versions contain the same information.

<u>Definition</u>. Zariski pair: $C_1, C_2 \subseteq \mathbb{P}^2$, same combinatorics but $(\mathbb{P}^2, C_1) \not\cong_{\text{hom}} (\mathbb{P}^2, C_2)$

<u>Zariski's example</u>. Sextic curves with six cusps.

① For generic f_2, f_3, $\deg f_i = i$, $C_1 = \{f_2^3 + f_3^2 = 0\}$ Six cusps on a conic.

② $\exists\, C_2 \subseteq \mathbb{P}^2$ where the six cusps are not on a conic (deformation arguments from Zariski, explicit equations from Oka and _).

Zariski proved $\begin{cases} \pi_1(\mathbb{P}^2 \setminus C_1) \cong \mathbb{Z}_2 * \mathbb{Z}_3 & \Delta_{C_1} = t^2 - t + 1 \\ \pi_1(\mathbb{P}^2 \setminus C_2) \cong \mathbb{Z}_2 \times \mathbb{Z}_3 & \Delta_{C_2} = 1 \end{cases}$ (*)

(*) It can be computed without π_1

<u>Consequence</u> The SIS associated to C_1, C_2 have the same abstract topology, same char poly of monodromy, different embedded topology

§3.- Topological invariants of curves

Goals: Provide topological properties, distinguish Zariski pairs.

(1) $\pi_1(\mathbb{P}^2 \setminus C)$ • Computable via braid monodromy (see later)

• Via coverings it can be used to compute $\pi_1(X \setminus D)$ in general

• Rich invariant, but it may be difficult to derive properties from a presentation

What is algebraic about $\pi_1(\mathbb{P}^2 \setminus C)$?

(2) Braid monodromy (Zariski - van Kampen)

(1) Pick up $L \pitchfork C$, $P \in L \setminus C$; change coordinates s.t. $L = \{z=0\}$, $P = [0:1:0]$. Project $\mathbb{C}^2 \cong \mathbb{P}^2 \setminus L$

(2) In affine coordinates $(x,y) \hookrightarrow [x:y:1]$, the projection is $(x,y) \longmapsto x$. Let $\Delta = \{x_1 \in X / f(x_1,y)$ has less than d roots$\}$ (finite set, zeros of discriminant, $\operatorname{disc}_y f(x,y) \in \mathbb{C}[X] \setminus \{0\}$)

$F(x,y,z) = f_0(x,z) + f_1(x,z)y + \dots + f_{d-1}(x,z)y^{d-1} + y^d$

$\mathcal{P} = \{f(y) \in \mathbb{C}[Y] \mid \deg f = d, \operatorname{disc}_y f \neq 0\}$ $\pi_1(\mathcal{P})$ = braid group on d strands

The map $\mathbb{C} \setminus \Delta \longrightarrow \mathcal{P}$, $x_1 \longmapsto f(x_1, y)$ induces $\nabla_* : \pi_1(\mathbb{C} \setminus \Delta) \longrightarrow \pi_1(\mathcal{P}) = \mathbb{B}_d$

∇_* is braid monodromy. Let $\mathbb{F}_d \times \mathbb{B}_d \xrightarrow{\rho} \mathbb{F}_d$ the natural action of $\mathbb{B}_d$ on $\mathbb{F}_d$. Then

$$\pi_1(\mathbb{C}^2 \setminus C) \cong \left| \mu_1, \dots, \mu_d : \mu = \mu^{\rho(\alpha)}, \mu \in \mathbb{F}_d, \alpha \in \pi_1(\mathbb{C} \setminus \Delta) \right|$$

$$\pi_1(\mathbb{P}^2 \setminus C) \cong \left| \dots : \dots, \mu_1 \cdots \mu_d = 1 \right|$$

• Braid monodromy determines the topology (Kulikov-Teicher, Carmona)

• " " " homotopy type (Libgober)

• Under some extra conditions, braid monodromy determines topology (- Carmona - Cogolludo)

③ Alexander polynomial It can be computed from the fundamental group or following Zariski, Libgober, Esnault, Loeser-Vaquié, — as follows:

ⓐ $\Delta_C(t) = \prod_{k=1}^{d-1} \left(t - e^{2i\pi k/d}\right)^{\alpha_k + \alpha_{d-k}}$

$\alpha_k = \dim \operatorname{coker} \left(H^0(\mathbb{P}^2; \mathcal{O}(k-3)) \longrightarrow \bigoplus_{P \in \operatorname{Sing}(C)} \mathcal{O}_{\mathbb{P}^2,P} \Big/ \mathcal{J}_{\mathbb{P}^2,C}\left(\tfrac{k}{d}\right) \right)$

where $\mathcal{J}_{\mathbb{P}^2,C,\frac{k}{d}}(\alpha)$ is the α-multiplier ideal of $(\mathbb{P}^2, C)$

ⓑ Note that nodes do not contribute in the above formul.

ⓒ There are divisibility results:

- Libgober: $\Delta_C(t)$ divides the product of the Alexander polynomials of the singular points
- Zariski: if C irreducible, no cyclotomic factors of prime power order.

ⓓ From the definition, it can be extended to non-reduced curves, changing the ramified cover

④ Twisted Alexander polynomial (Cogolludo-Florens) It detects nodes

⑤ Characteristic varieties (abelian version) $E = \mathbb{P}^2 \setminus C$, $\mathbb{T} = H^1(E; \mathbb{C}^*)$ $(= \operatorname{Hom}(\pi_1(E); \mathbb{C}^*))$. A character $\xi \in \mathbb{T}$ induces $H^*(E; \mathbb{C}_\xi)$ twisted homology

$V_k^1(E) := \{ \xi \in \mathbb{T} \mid \dim H^1(E; \mathbb{C}_\xi) \geq k \}$, algebraic subvariety of $\mathbb{T}$, subject to many restrictions (Arapura, Dimca-Papadima-Suciu, — Cogolludo-Matei, Budur-Wang)

- V_k^1 can be computed with generalizations of multiplier ideals (Libgober)
- Some irreducible components are combinatorial.

⑥ As in ⑤ but looking for representations $\pi_1(E) \longrightarrow GL(n; \mathbb{C})$

7) Linking invariants (Guerville-Ballé, Cadigan-Schlieper, GB-Meilhan, _-Florens-GB, GB-Shirane, ...)

Consider $E = \mathbb{P}^2 \setminus V(D)$, the exterior of the curve; $\partial E = \partial V(D) = M_D$ is a graph manifold. Its fundamental group was computed by Mumford, Cohen-Suciu, ... and its homology contains 3-types of generators. Recalled, M_D is a union of (Seifert) S^1-fiber bundles:

- Lift of generators of H_1 of the base of the bundles, when genus ≥ 0
- Lift of the cycles of the graph

| Non-canonical lifts

- Fibers of the bundles $\longrightarrow$ meridians of the curves (and exceptional cps)

Though these lifts are non-canonical, there are solutions:

- _-F-GB. Pick-up a lift and a character non-sensitive to the indeterminacy of the lift. If the character is torsion it can be detected on coverings

- _, F-Adrien Rodau, for line arrangements.

8) Particular invariants:

- $C \subseteq \mathbb{P}^2$ sextic with simple singular points (A, D, E); the double cover has a smooth representative $\rightarrow$ K3 surface $S \Rightarrow$ study the lattice $\mathrm{Pic}(S)$ (Degtyarev, Shimada)

- Arrangement of curves containing a smooth component C, relationship of the other curves with $\mathrm{Jac}(C)$ (_-Bannai-Shirane-Tokunaga)

§4.- Realization spaces.

$D \subseteq \mathbb{P}^2$ algebraic curve of degree d, $\mathcal{C}$ combinatorial type

- $\mathbb{P}_d \equiv$ projective space of curves of degree d (of dimension $\frac{(d+1)(d+2)}{2}$)

$$\Sigma_{\mathcal{C}} := \{ D' \subseteq \mathbb{P}^2 \,/\, \deg D' = d,\ D' \text{ has } \mathcal{C} \text{ as combinatorial type} \}$$

$$\mathcal{M}_{\mathcal{C}} := \Sigma_{\mathcal{C}} / PGL(3, \mathbb{C})$$

- D_1, D_2 in the same connected component of $\mathcal{C} \Rightarrow D_1$ and D_2 (strongly) isotopic in $\mathbb{P}^2$

- Ordered version $(D_1, \dots, D_r)$ tuple of curves in $\mathbb{P}^2 \rightsquigarrow \mathcal{C}$ "ordered" combinatorics

Define $\Sigma_{\mathcal{C}} \subseteq \mathbb{P}_{d_1} \times \dots \times \mathbb{P}_{d_r}$ and $\mathcal{M}_{\mathcal{C}}$ in the same way.

How to look for Zariski pairs or tuples?

1. Find combinatorics with non connected realization spaces and apply invariants.
2. " ordered " " " " " " " "ordered", then add curves "breaking" the order.
3. Arithmetic cases coming from embeddings of number fields in $\mathbb{C}$

Problem Look for disconnected realization spaces with homeomorphic embedding of the curves

Example of triangular curves