Topology of Polynomials and Low Dimensional Algebraic Geometry

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Contents

LECTURE 1

Germs of curves and Puiseux expansions

The main object of study in this series of lectures is the topology of polynomials $f : \mathbb{C}^2 \to \mathbb{C}$. The strategy to study these polynomials have several steps:

- Find the critical points of f and study the behaviour of f around them.
- Find the *critical values at infinity* and study f around them.
- Study the embedded topology of the fibers.
- Study the global behaviour of f.

1.1. Germs of curves

EXAMPLE 1.1.1. Let us consider the polynomials $f, g\colon \mathbb{C}^2 \to \mathbb{C}$ given by $f(x, y) := x^2 - y^2$ and $g(x, y) := x^2 - y^2 + x^3$. They have different global properties but near the origin they share the same analytical properties: note that $g(x, y) = (xu(x) - y)(x(u(x) + y))$ where $u(x)$ is a local determination of $\sqrt{1 + x}$ around 0.

There exist small neighbourhoods U, V of 0 in $\mathbb{C}^2$ such that $\psi : U \to V$, $\psi(x, y) := (xu(x), y)$, is a well-defined analytic isomorphism and $g_{|U} = f_{|V} \circ \psi$.

Let $f : U \to \mathbb{C}$, $g : V \to \mathbb{C}$ two analytical mappings such that U, V are neighbourhoods of 0 in $\mathbb{C}^2$. We say that $f \sim g$ if there exists $W \subset U \cap V$ neighbourhood of 0 in $\mathbb{C}^2$ such that $f_{|W} = g_{|W}$. The equivalence class of such an f is called the *germ of f at* 0. The study of such germs is the first ingredient in the comprehension of the global topology of polynomials.

The usual operations respect the equivalence relation and the germs form a $\mathbb{C}$-local algebra $\mathscr{O}_2 := \mathscr{O}_{(\mathbb{C}^2,0)}$ which is naturally isomorphic with the ring of convergent power series $\mathbb{C}\{x, y\}$. The maximal ideal $\mathscr{M}_2$ is the set of germs of functions vanishing at 0. In the same way we define the ring $\mathscr{O}_1 := \mathscr{O}_{(\mathbb{C},0)} \cong \mathbb{C}\{x\}$. The order of a non-zero element f of $\mathscr{O}_2$ (or $\mathscr{O}_1$) is the minimal $n \geq 0$ such that $f \in \mathscr{M}_j^n$ $(j = 1, 2)$.

EXAMPLE 1.1.2. The maximal ideal $\mathscr{M}_1$ is generated by x. The only proper ideals of $\mathscr{O}_1$ are $\mathscr{M}_1^m$, $m \in \mathbb{N}$, i.e., $\mathscr{O}_1$ is principal (and hence neotherian, factorial).

Let us fix a germ $f : (\mathbb{C}^2, 0) \to \mathbb{C}$. We will usually identify f with a representative for a suitable neighbourhood V of 0 in $\mathbb{C}^2$. After a change of coordinates, we may suppose that $f(0, y) \not\equiv 0$ and is of order n.

DEFINITION 1.1.3. A non-zero element of $\mathscr{O}_2$ is a *y-regular of order n* if $f(0, y) \not\equiv 0$ and of order n.

Using the **Principle of Analytic Continuation**, we can fix $\varepsilon > 0$ such that $f(0,y) \neq 0$ if $0 < |y| \leq \varepsilon$ and $\{0\} \times \bar{\Delta}_\varepsilon \subset V$, where Δ_ε (resp. $\bar{\Delta}_\varepsilon$) is the open (resp. closed) disk of radius ε in $\mathbb{C}$ centered at 0.

Using standard compactness arguments, $\exists \delta > 0$ such that $f(x,y) \neq 0$ if $|y| = \varepsilon$, $x \in \bar{\Delta}_\delta$ and $\bar{\Delta}_\delta \times \bar{\Delta}_\varepsilon \subset V$. Let us consider the mapping

$$V_f : \bar{\Delta}_\delta \to \mathbb{C}, \quad V_f(x_0) := \frac{1}{2i\pi} \int_{|y|=\varepsilon} \frac{\frac{\partial f(x_0,y)}{\partial y}}{f(x_0,y)} dy.$$

This mapping is well-defined and continuous by the above construction.

Let us recall the following classical theorem in complex analysis.

Theorem 1.1.4. *Let f be a homolomorphic function in a neighbourhood of $\bar{\Delta}_\varepsilon$, $\varepsilon > 0$ such that f does not vanish on its boundary. Let us suppose that $z_1, \dots, z_k$ are the zeroes of f in Δ_ε, with multiplicities $n_1, \dots, n_k$. Then,*

$$\frac{1}{2i\pi} \int_{|t|=\varepsilon} \frac{f'(t)dt}{f(t)} = \sum_{j=1}^{k} n_j.$$

Moreover, if g is also holomorphic in a neighbourhood of $\bar{\Delta}_\varepsilon$, then

$$\frac{1}{2i\pi} \int_{|t|=\varepsilon} \frac{g(t)f'(t)dt}{f(t)} = \sum_{j=1}^{k} n_j g(z_j).$$

Hence $V_f(\bar{\Delta}_\delta) \subset \mathbb{Z}$, i.e. it is a constant map and its value equals $V_f(0) = n$. Given $x \in \bar{\Delta}_\delta$, $n = V_f(x)$ is the number of zeroes of $y \mapsto f(x,y)$ in Δ_ε (counted with multiplicities); we call them $y_1(x), \dots, y_n(x)$ (we do not bother about the ordering). Let us consider now

$$V_{f,m} : \bar{\Delta}_\delta \to \mathbb{C}, \quad V_f(x_0) := \frac{1}{2i\pi} \int_{|y|=\varepsilon} y^m \frac{\frac{\partial f(x_0,y)}{\partial y}}{f(x_0,y)} dy.$$

These mappings are well-defined, continuous on $\bar{\Delta}_\delta$ and analytic on Δ_δ. Moreover

$$V_{f,m}(x) = \sum_{j=1}^{n} y_j(x)^m.$$

Remark 1.1.5. Given $j = 1, \dots, n$, let $\sigma_j : \bar{\Delta}_\delta \to \mathbb{C}$ the j^{th} elementary symmetric function in $y_1, \dots, y_n$. Recall that σ_j is the sum of all the possible products of j different terms in $\{y_1, \dots, y_n\}$. Following standard invariant theory, σ_j is a polynomial function (with rational coefficients) of $V_{f,1}, \dots, V_{f,j}$, and, in particular, it is continuous on $\bar{\Delta}_\delta$ and analytic on Δ_δ.

Let us consider $\tilde{f} : \bar{\Delta}_\delta \times \bar{\Delta}_\varepsilon \to \mathbb{C}$ given by

$$\tilde{f}(x,y) := y^n + \sum_{j=1}^{n} (-1)^j \sigma_j(x) y^{n-j} = \prod_{j=1}^{n} (y - y_j(x)).$$

which is holomorphic in $W_{\delta,\varepsilon} := \Delta_\delta \times \Delta_\varepsilon$. By construction the quotient $u := \dfrac{f}{\tilde{f}}$ is well-defined, continuous on $\bar{W}_{\delta,\varepsilon}$, holomorphic on $W_{\delta,\varepsilon}$ and vanishes nowhere. Note that $\sigma_j(0) =, \forall j = 1, \dots, n$.

DEFINITION 1.1.6. A non-zero element of $\mathscr{O}_1[y]$ is a *Weierstrass polynomial* if it is monic and the non-leading coefficients are in $\mathscr{M}_1$.

The above arguments prove the following classic result.

THEOREM 1.1.7 (Weierstrass Preparation Theorem). *Let $f \in \mathscr{O}_2$ regular of order n. There exist a unique Weierstrass polynomial $\tilde{f}$ (of degree n) and a unique unit $u \in \mathscr{O}_2 \setminus \mathscr{M}_2$ such that $f = u\tilde{f}$.*

COROLLARY 1.1.8 (Implicit Function Theorem). *Let $f \in \mathscr{O}_2$ such that $f(0) = 0$ and $\frac{\partial f}{\partial y}(0) \neq 0$. Then, there exists $g \in \mathscr{M}_1$ such and neighbourhoods U, V of 0 in $\mathbb{C}^2$ and $\mathbb{C}$ such that*

$$Z := \{(x, y) \in U \mid f(x, y) = 0\} = \{(x, g(x)) \mid x \in V\};$$

in particular, Z and V are analytically isomorphic.

PROOF. The conditions $f(0) = 0$ and $\frac{\partial f}{\partial y}(0) \neq 0$ imply that f is y-regular of order 1. Then $f(x, y) = u(x, y)(y - g(x))$. Choose a neighbourhood U where u does not vanish and $f(x, y) = 0 \Longleftrightarrow y = g(x)$. □

REMARK 1.1.9. The Weierstrass Preparation and Division Theorems (the second one is proved with similar techniques) allow to prove that $\mathscr{O}_2$ is noetherian and factorial. Moreover, if $p \in \mathscr{O}_2$ is a Weierstrass polynomial, its irreducible decompositions in $\mathscr{O}_2$ and $\mathscr{O}_1[y]$ coincide and all the factors are Weierstrass polynomials.

Passing to germs, $\tilde{f} \in \mathscr{O}_1[y]$ and we can consider its decomposition $\tilde{f} = f_1^{r_1} \cdot \ldots \cdot f_l^{r_l}$, where f_j are irreducible Weierstrass polynomials.

1.2. Some classical invariants in singularity theory

There are several analytic-topological invariants associated to f.

DEFINITION 1.2.1. The *Milnor number* of f is $\mu(f) := \dim_{\mathbb{C}} \mathscr{O}_2/\mathscr{J}_f$, where $\mathscr{J}_f$ is the *Jacobian ideal of* f, i.e., the one generated by $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

REMARK 1.2.2. The following fact is classical: $r_1 = \cdots = r_l = 1$ if and only if $\mu(f) < \infty$. We assume for the rest of the section the finiteness of $\mu(f)$.

PROPOSITION 1.2.3. *There exists $0 < \eta \ll \delta \ll \varepsilon$ such that if $0 < |t| \leq \eta$ then $F_t := f^{-1}(t) \cap (\bar{W}_{\delta,\varepsilon})$ verifies $\chi(F_t) = 1 - \mu(f)$.*

Let us denote

- $\mathbb{B}_{\eta,\delta,\varepsilon} := f^{-1}(\bar{\Delta}_\eta) \cap \bar{W}_{\delta,\varepsilon}$,
- $\mathbb{B}^*_{\eta,\delta,\varepsilon} := f^{-1}(\bar{\Delta}^*_\eta) \cap \bar{W}_{\delta,\varepsilon}$,
- $\mathbb{S}_{\eta,\delta,\varepsilon} := f^{-1}(\bar{\Delta}_\eta) \cap \partial\bar{W}_{\delta,\varepsilon} = f^{-1}(\bar{\Delta}_\eta) \cap (\partial\bar{\Delta}_\delta \times \bar{\Delta}_\varepsilon)$ and
- $\mathbb{S}^*_{\eta,\delta,\varepsilon} := f^{-1}(\bar{\Delta}^*_\eta) \cap \bar{W}_{\delta,\varepsilon} = f^{-1}(\bar{\Delta}^*_\eta) \cap (\partial\bar{\Delta}_\delta \times \bar{\Delta}_\varepsilon)$.

PROPOSITION 1.2.4.

(1) *The restriction* $f_| : \mathbb{B}^*_{\eta,\delta,\varepsilon} \to \bar{\Delta}^*_\eta$ *is a locally trivial fibration.*
(2) *The restriction* $f_| : \mathbb{S}_{\eta,\delta,\varepsilon} \to \bar{\Delta}_\eta$ *is a trivial fibration.*
(3) *Both fibrations are compatible on* $\mathbb{S}^*_{\eta,\delta,\varepsilon}$.
(4) $\mathbb{B}_{\eta,\delta,\varepsilon}$ *is homeomorphic to a* 4*-dimensional ball.*

Moreover, these fibrations do not depend on the choice of suitable $\eta, \delta, \varepsilon$.

The proof of Proposition 1.2.4(1) is based on the fact that it is a proper submersion. For (2), we use also that any fibration over a contractible space is trivial.

A first consequence of (2) is that, up to isotopy, there is a unique homeomorphism $\Phi : \Delta_\eta \times \partial F_\eta \to \mathbb{S}_{\eta,\delta,\varepsilon}$ such that $f_| \circ \Phi$ is the first projection. The other fibration is determined up to isotopy by the so-called *geometric monodromy* $\phi : F_\eta \to F_\eta$, whose construction will be explained in §1.3; the compatibility implies that ϕ is the identity over ∂F_η.

Note that $\partial \bar{W}_{\delta,\varepsilon}$ is a 3-dimensional sphere obtained as the union of two solid tori $\partial\bar{\Delta}_\delta \times \bar{\Delta}_\varepsilon$ and $\bar{\Delta}_\delta \times \partial\bar{\Delta}_\varepsilon$.

We fix some notation:

- $K_{f,\delta,\varepsilon} := \partial \bar{W}_{\delta,\varepsilon} \cap f^{-1}(0)$.
- $\mathbb{B}^4_\varepsilon := \{(x,y) \in \mathbb{C}^2 \mid |x|^2 + |y|^2 \leq \varepsilon^2\}$
- $\mathbb{S}^3_\varepsilon := \{(x,y) \in \mathbb{C}^2 \mid |x|^2 + |y|^2 = \varepsilon^2\}$ which is also a 3-dimensional sphere.
- $K_{f,\varepsilon} := \mathbb{S}^3_\varepsilon \cap f^{-1}(0)$.

Theorem 1.2.5 (Milnor). *The homeomorphism type of* $(\partial \bar{W}_{\delta,\varepsilon}, K_{f,\delta,\varepsilon})$ *and* $(\mathbb{S}^3, K_{f,\varepsilon})$ *coincide for* δ *and* ε *small enough. Moreover the homeomorphism types of their cones coincide with the ones of* $(\bar{W}_{\delta,\varepsilon}, f^{-1}(0))$ *and* $(\mathbb{B}^4_\varepsilon, f^{-1}(0))$.

Definition 1.2.6. The *link* of f is the homeomorphism type of $(\partial \bar{W}_{\delta,\varepsilon}, K_{f,\delta,\varepsilon})$.

Remark 1.2.7. All these invariants coincide for f and $\tilde{f}$.

1.3. Unramified coverings

We recall some definitions and results on coverings which will be used in these lectures. All topological spaces will be locally path-connected.

Definition 1.3.1. A continuous mapping $\pi : X \to Y$ is an *unramified covering* if $\forall y \in Y$, there exists an open neighbourhood V of y such that $\pi^{-1}(V_y)$ is a disjoint union $\coprod_{i \in I_y} U_i$ of open sets such that $\pi_| : U_i \to V_y$ is a homeomorphism $\forall i \in I_y$.

There are some easy criteria to deduce if a mapping is a covering.

Lemma 1.3.2. *Let* $\pi : X \to Y$ *be a surjective local homeomorphism such that* $\forall y \in Y$, $\#\pi^{-1}(y)$ *is finite and constant. Then,* π *is a covering.*

There is a close relationship between covering theory and fundamental groups via the lifting path and homotopy properties. Let us recall the definition of *classical monodromy* of a covering.

Let us suppose that we have a covering $\pi : X \to Y$ where Y connected. Fix $y \in Y$ and let $G := \pi_1(Y; y)$; let $F := \pi^{-1}(y)$ and let Σ_F be the group of permutations of F (with exponent notation). The main invariant of π is the monodromy $\rho : G \to \Sigma_F$: for each $x \in F$ and each loop γ based at y there is a unique lifting of the loop starting at x and $x^{\rho(\gamma)}$ is the ending point of this lifting. The following properties are important:

- If $x \in F$, then $\pi_1(X; x)$ is identified by π_* with $\{g \in G \mid x^{\rho(g)} = x\}$.
- X is connected if and only if ρ is transitive, i.e., if $\forall x_1, x_2 \in F$, there exists $g \in G$ such that $x_1^{\rho(g)} = x_2$.
- Moreover, each connected component of X is associated with the orbits of F by the action of G and the restriction of π to each connected component is a covering.

There are two main theorems associated with this construction.

THEOREM 1.3.3. *Let $\pi : X \to Y$ be a covering, X, Y. Let $h : Z \to Y$ be a continuous mapping, Z connected. Fix $x \in X$ and $z \in Z$ such that $\pi(x) = h(z) =: y \in Y$. Then, there exists $\tilde{h} : Z \to X$ continuous mapping such that $\pi \circ \tilde{h} = h$ and $\tilde{h}(z) = x$ if and only if $h_*(\pi_1(Z; z)) \subset \pi_*(\pi_1(X; x))$.*

It it exists, $\tilde{h}$ is unique.

THEOREM 1.3.4. *Let Y be a connected space, $y \in Y$. For any subgroup H of $\pi_1(Y; y)$ there exists a covering $\pi : X \to Y$, X connected, and $x \in \pi^{-1}(y)$ such that $\pi_*(\pi_1(X; x)) = H$. The covering π is* essentially *unique.*

EXAMPLE 1.3.5. Let us recall that $\pi_1(\mathbb{C}^*; 1) \cong \mathbb{Z}$, generated by the class of $\gamma : [0, 1] \to \mathbb{C}^*$ where $\gamma(t) := \exp(2i\pi t)$. The subgroups of $\mathbb{Z}$ are $\{0\}$ and $\{m\mathbb{Z}\}_{m\in\mathbb{N}}$. The associated coverings are $\mathbb{C} \to \mathbb{C}^*$, $t \mapsto \exp(2i\pi t)$ and $\mathbb{C}^* \to \mathbb{C}^*$, $t \mapsto t^m$.

If we replace $\mathbb{C}^*$ by $\bar{\Delta}^*_\eta$, $\eta > 0$, we have the same result for $\mathbb{H}_\eta := \{z \in \mathbb{C} \mid 2\pi\Im z \geq \log \eta\} \to \bar{\Delta}^*_\eta$ and $\bar{\Delta}^*_{\eta^{\frac{1}{m}}} \to \bar{\Delta}^*_\eta$.

In particular, if we have a continuous mapping $h : X \to \mathbb{C}^*$, X connected, let $h_* : \pi_1(X; x) \to \pi_1(\mathbb{C}^*; h(x)) \equiv \mathbb{Z}$. Then, h has an m^{th}-root if and only if $h_*(\pi_1(X; x)) \subset m\mathbb{Z}$.

REMARK 1.3.6. We can explain the construction of the monodromy of the fibration of Proposition 1.2.4. We consider the locally trivial fibration $f_| : \mathbb{B}^*_{\eta,\delta,\varepsilon} \to \bar{\Delta}^*_\eta$ and the covering $\mathbb{H}_\eta \to \bar{\Delta}^*_\eta$. Let us define

$$\widetilde{\mathbb{B}}_{\eta,\delta,\varepsilon} := \{((x, y), t) \in \mathbb{B}^*_{\eta,\delta,\varepsilon} \times \mathbb{H}_\eta \mid f(x, y) = \exp(2i\pi t)\}.$$

The restriction of the second projection is a locally trivial fibration with fiber F_η. Since $\mathbb{H}_\eta$ is contractible, this fibration is a trivial one and there is a homeomorphism $\tilde{\Phi} : \mathbb{H}_\eta \times F_\eta \to \widetilde{\mathbb{B}}_{\eta,\delta,\varepsilon}$ which identifies the fibrations.

The mapping $\widetilde{\mathbb{B}}_{\eta,\delta,\varepsilon} \to \widetilde{\mathbb{B}}_{\eta,\delta,\varepsilon}$ given by $((x, y), t) \mapsto ((x, y), t + 1)$ is an automorphism of the projection and induces via $\tilde{\Phi}$ the geometric monodromy $\phi : F_\eta \to F_\eta$ as follows.

Let $(x, y) \in F_\eta$ and let $t_\eta := \frac{\log \eta}{2i\pi} \in i\mathbb{R}$. Hence $((x, y), t_\eta) \in \widetilde{\mathbb{B}}_{\eta,\delta,\varepsilon}$ and it is also the case for $((x, y), t_\eta + 1)$. Then, $\tilde{\Phi}^{-1}((x, y), t_\eta + 1)) = (t_\eta + 1, \phi(x, y))$.

1.4. Picard-Lefschetz

Let $f(x, y) := x^2 + y^2$; we can choose $\varepsilon = 10$, $\delta = 2$ and $\eta = 1$.

- $\bar{W}_{\delta,\varepsilon} \cap f^{-1}(0) = \{(x, ix) \mid |x| \leq 2\} \cap \{(x, -ix) \mid |x| \leq 2\}$, i.e., a union of two disks intersecting at their centers.
- $F_1 = \{(x,y) \in \bar{W}_{\delta,\varepsilon} \mid x^2 + y^2 = 1\}$. This condition can be given by:

$$\begin{aligned}\Re x^2 + \Re y^2 =& 1 + \Im x^2 + \Im y^2\\ \Re x \Im x + \Re y \Im y =& 0.\end{aligned}$$

Let us recall that the tangent space $T\mathbb{S}^1$ is the space of points $(u, v, w, t) \in \mathbb{R}^4$ such that:

$$\begin{aligned}u^2 + v^2 =& 1\\ uw + vt =& 0.\end{aligned}$$

Let us consider the homeomorphism $\mathbb{C}^2 \to \mathbb{R}^2 \times \{(w,t) \in \mathbb{R}^2 \mid w^2 + t^2 < 1\}$ given by

$$(x,y) \mapsto \frac{1}{\sqrt{1 + \Im x^2 + \Im y^2}} (\Re x, \Re y, \Im x, \Im y).$$

We deduce that F_1 is homeomorphic to a cylinder, i.e., to a closed neighbourhood of $\mathbb{S}^1$ in $T\mathbb{S}^1$ which is a disk fibration over $\mathbb{S}^1$.
- $\mathbb{S}_{\eta,\delta,\varepsilon} = \{(x,y) \in \mathbb{C}^2 \mid |x^2 + y^2| \leq 1, |x| = 2\}$.

Let us denote $z = y^2$ and $\omega := x^2 + z$. Note that $\omega \in \bar{\Delta}_1$, and then z lives in the closed disk of radius 1 centered at $-x^2$.

LEMMA 1.4.1. *The function*

$$\hat{\rho} : T := \{(x,\omega) \in \mathbb{C}^2 \mid \omega \in \bar{\Delta}_1, x \notin [-\sqrt{\omega}, \sqrt{\omega}]\} \to \mathbb{C}^*, \quad x \mapsto \omega - x^2,$$

has two square roots.

PROOF. Let $G := \pi_1(T; (2,1))$ and $H := \pi_1(\mathbb{C}^*; -3)$. It is easily seen that $H \cong \mathbb{Z}$ at is generated by $\gamma : t \mapsto -3\exp(2i\pi t)$. In the same way, $G \cong \mathbb{Z}$ and is generated by the path $\beta : t \mapsto (2\exp(2i\pi t), 1)$.

It is not hard to check that $\hat{\rho}_*([\beta]) = [\gamma^2]$. Since $\hat{\rho}(G) \subset 2\mathbb{Z}$, we conclude the existence of the square roots. □

Let ρ the square root of $\hat{\rho}$ such that $\rho(1,0) = i$; note that $\rho(x,0) = ix$. Hence,

$$\mathbb{S}_{\eta,\delta,\varepsilon} = \{(x, \rho(x,\omega)) \in \mathbb{C}^2 \mid \omega \in \bar{\Delta}_1, |x| = 2\} \cup \{(x, -\rho(x,\omega)) \in \mathbb{C}^2 \mid \omega \in \bar{\Delta}_1, |x| = 2\}.$$

In particular,

$$\partial F_1 = \{(x, \rho(x,1)) \in \mathbb{C}^2 \mid |x| = 2\} \cup \{(x, -\rho(x,1)) \in \mathbb{C}^2 \mid |x| = 2\}$$

and we obtain a homeomorphism $\mathbb{S}_{\eta,\delta,\varepsilon} \cong \bar{\Delta}_\eta \times \partial F_1$.
- $\mathbb{H}_\eta = \{t \in \mathbb{C} \mid \Im t \geq 0\}$.
- $\mathbb{B}^*_{\eta,\delta,\varepsilon} := \{(x,y) \in \mathbb{C}^2 \mid x^2 + y^2 \in \bar{\Delta}^*_1, x \in \bar{\Delta}_2\}$.

- $\widetilde{\mathbb{B}}_{\eta,\delta,\varepsilon} := \{(x,y,t) \in \mathbb{C}^3 \mid t \geq 0, x \in \bar{\Delta}_2, x^2+y^2 = \exp(2i\pi t)\}$.

There is a trivial way of identifying $\mathbb{H}_\eta \times F_1$ with $\widetilde{\mathbb{B}}_{\eta,\delta,\varepsilon}$:

$$(t,(x,y)) \mapsto (x\exp(i\pi t), y\exp(i\pi t), t)$$

which produces a monodromy $F_1 \to F_1$ given by $(x,y) \mapsto (-x,-y)$, but unfortunately it does not respect the identifications in the boundary. Following the ideas of Arnol'd, Gussein-Zade and Varchenko, we can see the F_w as the Riemann surface of the function $\sqrt{w-x^2}$.

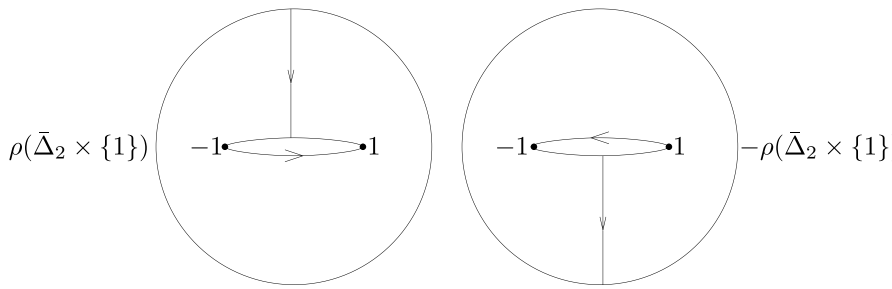

FIGURE 1.1. Riemann surface of $\sqrt{1-x^2}$

With this language, the *natural* trivialization is:

$$(t,(x,\pm\rho(x,1))) \mapsto (t,(x_t,\pm\rho(x_t,\exp(2i\pi t))), \quad x_t := \exp(i\pi t)x.$$

The trivialization in the boundary is:

$$(t,(x,\pm\rho(x,1))) \mapsto (t,(x,\pm\rho(x,\exp(2i\pi t))).$$

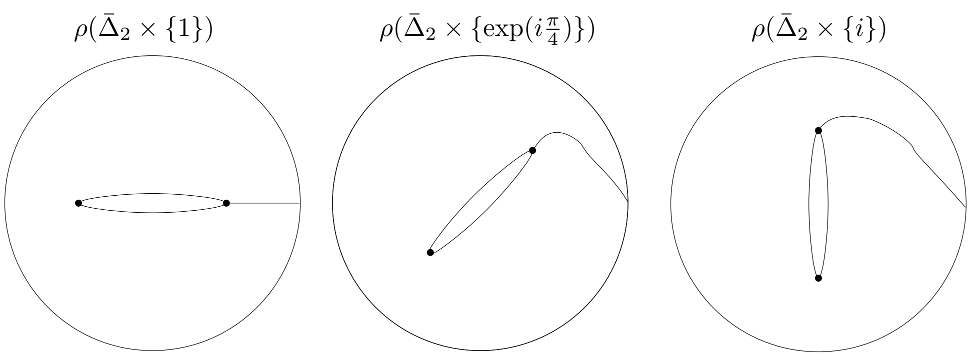

FIGURE 1.2. Monodromy

Let $\xi : [0,2] \times (0,1] \to \mathbb{R}$ a differentiable function such that $\xi(\bullet, v)$ is decreasing $\forall v \geq 0$, $\xi(u,v) = 1$ if $u \in [0,v]$ and $\xi(2v,v) = 1$. The good trivialization is:

$$(t,(x,\pm\rho(x,1))) \mapsto (t,(x_{t,\xi},\pm\rho(_{t,\xi},\exp(2i\pi t))), \quad x_{t,\xi} := \exp\left(i\xi\left(|x|,\exp(-2\pi|t|)\right)\pi t\right)x.$$

The monodromy of F_1 is given by

$$(x, \pm\rho(x,1)) \mapsto (x\xi(|x|,1), \pm\rho(x\xi(|x|,1),1))$$

1.5. Puiseux expansions

In this section we will fix the variables x, y. Fix $g \in \mathscr{O}_2$ an irreducible Weierstrass polynomial of degree m and consider ε and δ as above. Let $\operatorname{disc}_y(g) \in \mathscr{O}_1$ be the discriminant of $\tilde{f}$ with respect to y, which has a representative $\bar{\Delta}_\delta \to \mathbb{C}$.

Lemma 1.5.1. $\operatorname{disc}_y(g) \not\equiv 0$.

Proof. Since $\mathscr{O}_1$ is factorial, if $\operatorname{disc}_y(g) \equiv 0$, then g and $\frac{\partial f}{\partial y}$ have a common factor, which is impossible since g is irreducible and $\deg_y \frac{\partial f}{\partial y} < \deg_y g$. □

Remark 1.5.2. As a consequence, we can suppose that $\operatorname{disc}_y(g)(x_0) \neq 0$ if $0 < |x_0| \leq \delta$, i.e., $g(x_0, y) \in \mathbb{C}[y]$ has m distinct roots.

Let $Z := \{(x,y) \in \bar{W} \mid g(x,y) = 0\}$. We consider the projection $\pi_x : Z \to \bar{\Delta}_\delta$ onto the first variable.

Notation 1.5.3. If X is a subset of $\mathbb{C}$ or $\mathbb{C}^2$ containing the origin, $X^* := X \setminus \{0\}$.

Lemma 1.5.4. *The restriction* $\pi_{x|} : Z^* \to \bar{\Delta}^*_\delta$ *is an unramified covering of degree* m.

Proof. We use Lemma 1.3.2. The statement about cardinality comes from Remark 1.5.2. The assumption about local homeomorphism comes from the Implicit Function Theorem. □

Applying Example 1.3.5, we obtain that Z^* is a disjoint union of punctured disks. We are going to prove that Z^* is connected. Let Z_1^* be a connected component of Z^* and let m_1 be the degree of $\pi_{|Z_1^*}$. Given $x \in \bar{\Delta}^*_\delta$, we can reorder the roots of $g(x,y)$ such that $(x, y_1(x)), \dots, (x, y_{m_1}(x)) \in Z_1^*$.

Lemma 1.5.5. *The symmetric functions of* $y_1, \dots, y_{m_1}$ *define continuous functions on* $\bar{\Delta}_\delta$ *which are holomorphic on* Δ_δ.

Proof. First, we can proof the statement for $\bar{\Delta}^*_\delta$. Given $x \in \bar{\Delta}^*_\delta$ and fixing $y_j(x)$, $j = 1, \dots, m_1$, the Implicit Function Theorem allows us to find a local homeomorphism (analytic isomorphism if $x \in \bar{\Delta}_\delta$) of a connected neighbourhood of x in $\bar{\Delta}^*_\delta$ onto a neighbourhood of $y_j(x)$ in Z^* which by connectedness is contained in Z_1^*. This local homeomorphism may not respect the indices and this is why the statement is reduced to the symmetric functions.

Since all the roots tend to zero, these functions are locally bounded at the origin, and by Riemann's removal singularity Theorem, the symmetric functions extend to $\bar{\Delta}_\delta$. □

Proposition 1.5.6. $Z_1^* = Z^*$.

Proof. Let $\sigma_{1,1},\dots,\sigma_{1,m_1}$ be the elementary symmetric functions of $y_1,\dots,y_{m_1}$. We have proved that

$$h(x,y):=y^{m_1}+\sum_{j=1}^{m_1}(-1)^j\sigma_{1,j}(x)y^{m_1-j}=\prod_{j=1}^{m_1}(y-y_j(x)).$$

is a Weierstrass polynomial and it divides g; since g is irreducible hence $g=h$, $m=m_1$ and $Z_1^*=Z^*$. $\square$

Let us consider the following diagram:

$$\begin{array}{ccc} & & Z^* \\ & & \downarrow\pi \\ \bar\Delta^*_{\delta^{\frac{1}{m}}} & \longrightarrow & \bar\Delta^*_\delta \\ t & \longmapsto & t^m. \end{array}$$

This diagram induces a corresponding one for fundamental groups:

$$\begin{array}{ccc} & & \pi_1(Z^*;(\delta,y_1(\delta))) \\ & & \downarrow\pi_* \\ \mathbb{Z}\cong\pi_1(\bar\Delta^*_{\delta^{\frac{1}{m}}};\delta^{\frac{1}{m}}) & \longrightarrow & \pi_1(\bar\Delta^*_\delta;\delta)\cong\mathbb{Z} \\ 1 & \longmapsto & m. \end{array}$$

The subgroup corresponding to the vertical arrow is $m\mathbb{Z}$ and we have proved the following result.

Theorem 1.5.7 (Puiseux expansions). *There exists $h\in\mathscr{O}_1$ admiting a representative in a neighbourhood of $\bar\Delta_{\delta^{\frac{1}{m}}}$ such that $Z=\{(t^m,h(t))\mid t\in\bar\Delta_{\delta^{\frac{1}{m}}}\}$. The set of germs satisfying this property is $\{h(\zeta t)\mid\zeta^m=1\}$.*

Let us consider the expansion of h:

$$h(t)=\sum_{k=1}^{\infty}a_kt^k.$$

Lemma 1.5.8. $\gcd\{k\mid a_k\neq 0\}=1$.

Notation 1.5.9. Since the parametrization of Z provides the equality $x=t^m$, it is classical to write down $h(x^{\frac{1}{m}})=\sum_{k=1}^{\infty}a_kx^{\frac{k}{m}}$.

Let us consider the following construction:

- If $m=1$ the construction does not start.
- If $m>1$, using Lemma 1.5.8, there exists $n_1:=\min\{k\mid a_k\neq 0, m\nmid k\}$. The rational $\frac{n_1}{m}$ is the first Puiseux exponent of g with respect to x. Let $\frac{n_1}{m}=:\frac{q_1}{p_1}$, where $\gcd(p_1,q_1)=1$. If we see h as a series in $x^{\frac{1}{m}}$, it is the first term with non-integer exponent.
- If $p_1=m$, we are done.
- If not, let $n_2:=\min\{k\mid a_k\neq 0,\frac{m}{p_1}\nmid k\}$. Then, $\frac{n_2}{m}$ is the second Puiseux exponent of g with respect to x. Let $\frac{n_2}{m}=:\frac{q_2}{p_1p_2}$, where $\gcd(p_2,q_2)=1$. It is the first term such that its exponent is not in $\frac{1}{p_1}\mathbb{Z}$.
- If $p_1p_2=m$, we are done.

- If not, we continue the procedure constructing a sequence $\{\frac{n_i}{m}\}_{i=1}^r = \left\{\frac{q_i}{\prod_{j=1}^i p_j}\right\}_{i=1}^r$, where $\gcd(p_i, q_i) = 1$, $p_i > 1$, $\prod_{j=1}^r p_j = m$.

Definition 1.5.10. The sequence $\{\frac{n_i}{m}\}_{i=1}^r = \left\{\frac{q_i}{\prod_{j=1}^i p_j}\right\}_{i=1}^r$ is the set of *Puiseux exponents* of g.

Let g_1, g_2 be two irreducible Weierstrass polynomials of degrees m_1, m_2. Let $h_1, h_2 \in \mathscr{O}_1$ be their Puiseux expansions. For convenience we denote them by

$$h_j := \sum_{k \in \frac{1}{m_i}\mathbb{N}} a_{k,j} t^{km_i}.$$

Definition 1.5.11. The *coincidence exponent* of g_1, g_2 is

$$\min\{k \mid \forall \zeta_1, \zeta_2 \text{ s.t } \zeta_1^{m_1} = 1, \zeta_2^{m_2} = 1 \text{ one has } \zeta_1^k a_{k,1} \neq \zeta_2^k a_{k,2}\}.$$

Remark 1.5.12. Let $f = f_1 \cdot \ldots \cdot f_r$ be a reduced Weierstrass polynomial, f_j of degree m_j. For each j we consider a Puiseux expansion $h_j(x^{\frac{1}{m_j}}$. Let us denote

$$\{\tilde{h}_1(x^{\frac{1}{m}}), \ldots, \tilde{h}_1(x^{\frac{1}{m}})\} = \bigcup_{j=1}^r \{h_j(\zeta_j x^{\frac{1}{m_j}}) \mid \zeta_j^{m_j} = 1\},$$

where m is the least common multiple of $m_1, \ldots, m_r$ and $n = \deg_y f$. Then, we have that $f(x,y) = \prod_{j=1}^n (y - \tilde{h}_j(x^{\frac{1}{m}}))$ and the set of orders of $\tilde{h}_j(x^{\frac{1}{m}}) - \tilde{h}_k(x^{\frac{1}{m}})$, $j \neq k$, is the set of Puiseux exponents of the roots and the set of coincide exponents.

LECTURE 2

Braid monodromy of germs

2.1. Newton construction

The computation of the Weierstrass polynomial $\tilde{f}$ of f, the factorizacion of $\tilde{f}$ and the construction of Puiseux exponents for its irreducible components in §1.5 are not constructible.

There is method which comes from Newton to obtain them. Let us fix $f \in \mathscr{O}_2$, where we consider coordinates such that f is y-regular of order n,

$$f(x,y) := \sum_{\mathbf{n}:=(n_x,n_y)\in\mathbb{Z}^2_{\geq 0}} a_{\mathbf{n}} x^{n_x} y^{n_y}.$$

Let Δ_f be the convex closure of $\bigcup\{\mathbf{n}+\mathbb{R}^2_{\geq 0} \mid a_{\mathbf{n}} \neq 0\}$ in $\mathbb{R}^2_{\geq 0}$.

Definition 2.1.1. The *Newton polygon* Γ_f of f is the union of the compact faces of $\partial\Delta_f$.

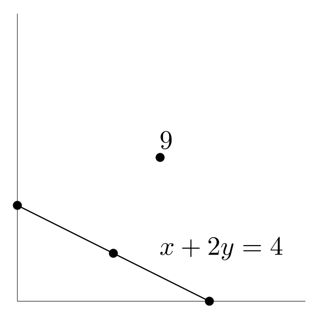

Figure 2.1. Newton polygon of $y^2+x^2y+x^4+x^3y^3$

Let γ be and edge of Γ_f. We have:

$$\gamma \subset \{(u,v) \mid qu+pv=N\}, \quad \gcd(p,q)=1, \quad p,q,N\in\mathbb{N}.$$

We consider a weighted-degree in $\mathscr{O}_2$ where $\deg x = q$ and $\deg y = p$; the order of f with respect to this degree equals N and the minimal form is $f_\gamma := \sum_{\mathbf{n}\in\gamma} a_{\mathbf{n}} x^{n_x} y^{n_y}$, which is a weighted homogeneous polynomial and can be decomposed in a unique way as

$$f_\gamma(x,y) = x^{N_x} y^{N_y} \prod_{j=1}^{r_\gamma} (y^q - w_{\gamma,j} x^p)^{m_{\gamma,j}},$$

where $N_x, N_y \geq 0$, $r_\gamma, m_{\gamma,j} \in \mathbb{N}$ and $w_{\gamma,j} \in \mathbb{C}^*$. The following properties hold:

- $N_x q + N_y p + pq \sum_{j=1}^{r_\gamma} m_{\gamma,j} = N$.
- $r_\gamma > 0$.
- $N_y + q \sum_{j=1}^{r_\gamma} m_{\gamma,j} \leq n$.

EXAMPLE 2.1.2. In the germ of Figure 2.1 we have only one edge γ and in this case $r_\gamma = 2$.

DEFINITION 2.1.3. We say that x, y are *good coordinates* for f if no edge γ of Γ_f satisfies $q = 1$ and $r_\gamma = 1$.

PROPOSITION 2.1.4. *There is a change of variables of the form $y \mapsto y + \alpha(x)$, $\alpha \in \mathscr{O}_1$, such that x, y are good coordinates for f.*

Fix $j = 1, \ldots, r_\gamma$. To the pair $(\gamma, w_{\gamma,j})$ we associate the following *change of variables*:

$$x = x_1^q, \quad y = x_1^p(y_1 + \tilde{w}_{\gamma,j}),$$

where $\tilde{w}_{\gamma,j}$ is a q^{th}-root of $w_{\gamma,j}$.

DEFINITION 2.1.5. The above change is the *Newton transformation* associated to γ and $w_{\gamma,j}$.

REMARK 2.1.6. It is not a change of variables since the mapping defined is not injective. It is generically $q:1$.

Note that $f(x, y) = x_1^N f_{\gamma, w_{\gamma,j}}(x_1, y_1)$, where

$$\begin{aligned} f_{\gamma,w_{\gamma,j}}(x_1,y_1) =& (y_1 + \tilde{w}_{\gamma,j})^{N_y} \left(\prod_{k \neq j} ((y_1 + \tilde{w}_{\gamma,j})^q - w_{\gamma,k})^{m_{\gamma,k}} \right) ((y_1 + \tilde{w}_{\gamma,j})^q - w_{\gamma,j})^{m_{\gamma,j}} + \cdots = \\ =& C y_1^{m_{\gamma,j}} + \ldots \end{aligned}$$

The new series $f_{\gamma,w_{\gamma,j}}(x_1, y_1) \in \mathbb{C}\{x_1, y_1\}$ satisfies that the order of $f_{\gamma,w_{\gamma,j}}(0, y_1)$ equals $m_{\gamma,j}$.

LEMMA 2.1.7. $m_{\gamma,j} < n$.

PROOF. It is clear that $m_{\gamma,j} \leq n$. The equality holds if $N_y = 0$, $r_\gamma = 1$ and $p = 1$, but this is impossible if we have good coordinates. $\square$

In this way we construct a tree of Newton transformations:

- We start with a root; the number of branches issued from this root is $\sum_\gamma r_\gamma$.
- The other vertex of these branches correspond to $f_{\gamma,w_{\gamma,j}}$.
- For each vertex, if $m_{\gamma,j} = 1$ we stop the process for this branch.
- If $m_{\gamma,j} > 1$, we repeat the above construction for $f_{\gamma,w_{\gamma,j}}$. First, we perform a change of variables as in Proposition 2.1.4. We attach again branches to this vertex in function of the new Newton polygon.

Let us consider a vertex which is at and end of the tree; we have series f_k in x_k, y_k such that:

- $x_k = x^{\frac{1}{l}}$, for a suitable l.
- $y = \sum_{j=1}^{\infty} a_j x^{\frac{j}{l}} + b x^{\frac{c}{l}} y_k$, where these terms come from the changes of variables in order to obtain good coordinates and the Newton transformations.

- Since f_k is of order 1 in y_k, using the Implicit Function Theorem we obtain that $f_k(x_k, y_k) = 0$ is equivalent to $y = h_k(x_k)$, $h \in \mathbb{C}\{x_k\}$.

The final result of this process is a series $\tilde{h}_k(x^{\frac{1}{l}})$ such that $f(x, \tilde{h}_k(x^{\frac{1}{l}})) \equiv 0$.

REMARK 2.1.8. At the Newton transformation there are q choices for $\tilde{w}_{\gamma,j}$ which are related with the indetermination of h. Taking into account all the possibles indeterminations, we obtain $\tilde{h}_k(\zeta x^{\frac{1}{l}})$, $\zeta^l = 1$.

REMARK 2.1.9. The *branch* of the tree associated with an edge γ and a root $w_{\gamma,j}$ such that $m_{\gamma,j} = 1$ stops after the Newton transformation.

PROPOSITION 2.1.10. *The number of irreducible components of f is at least $\sum_\gamma r_\gamma$.*

EXAMPLE 2.1.11. Let us consider $f(x,y) := (y^2 - x)^2 - 4x^2y - x^3$; we have $n = 4$. We have

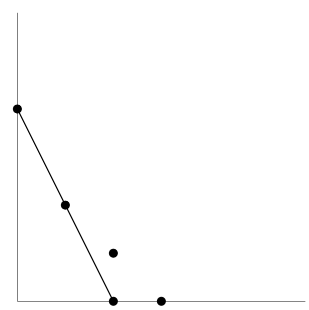

FIGURE 2.2. Newton polygon of $(y^2 - x)^2 - 4x^2y - x^3$

only one edge γ with one root 1. The Newton transformation is

$$x = x_1^2, \quad y = x_1(y_1 + 1).$$

We obtain

$$f(x,y) = x_1^4 \left(x_1 + 4y_1 + 4 + y_1^2\right)\left(y_1^2 - x_1\right)$$

Then, up to a unit, $f_{\gamma,1} = y_1^2 - x_1$. We have $y = x_1^{\frac{1}{2}}$, and then $y = x_1 + x_1 y_1 = x^{\frac{1}{2}} + x^{\frac{3}{4}}$.

2.2. Braid group

Artin's braid group $\mathbb{B}_n$ has a presentation

$$\left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \underset{1 \leq i < j < n}{[\sigma_i, \sigma_j]}, \quad \underset{1 \leq i < n-1}{\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}}, \right\rangle.$$

There is a canonical epimorphism $\Psi_n : \mathbb{B}_n \to \Sigma_n$, $\sigma_i \mapsto (i, i+1)$.

Let us denote by $\mathbf{X}$ the configuration space of n points in $\mathbb{C}$: it is defined as the quotient of $\mathbb{C}^n \setminus \{(x_1, \ldots, x_n) \in \mathbb{C}^n \mid \exists i < j \text{ s.t. } x_i = x_y\}$ by the coordinate-permutation action of Σ_n. It can be also interpreted as the set of subsets of $\mathbb{C}$ with cardinality n or else as

$$\{p(t) \in \mathbb{C}[t] \mid p \text{ monic }, \deg p = n, \operatorname{disc}(p) \neq 0\}.$$

Paths in $\mathbf{X}$ are identified with sets of n paths $\gamma_i : [0,1] \to \mathbb{C}$, $i = 1, \dots, n$, such that $\forall t \in [0,1]$, $\#\{\gamma_1(t), \dots, \gamma_n(t)\} = n$.

Let $\mathbf{n} := (-1, \dots, -n)$. It is possible to identify $\mathbb{B}_n$ with $\pi_1(\mathbf{X}; \mathbf{n})$, where σ_j is identified with the homotopy class of the set of n paths $[0,1] \to \mathbb{C}$ given by:

$$t \mapsto -k, k \neq j, j+1,$$
$$t \mapsto -\frac{2j+1}{2} + \frac{\exp(i\pi t)}{2},$$
$$t \mapsto -\frac{2j+1}{2} - \frac{\exp(i\pi t)}{2}.$$

In particular, for any $\mathbf{x} \in \mathbf{X}$, $\mathbb{B}_n$ is isomorphic to $\pi_1(\mathbf{X}; \mathbf{x})$. We can construct isomorphisms between $\mathbb{B}_n$ and $\pi_1(\mathbf{X}; \mathbf{x})$ in two essentially equivalent ways.

The first way is the standard one for identifying the fundamental group of a space at two different points. Choose a braid τ starting at $\mathbf{x}$ and ending at $\mathbf{n}$; the braid σ_j is sent to $\tau \sigma_j \tau^{-1}$.

The other way is as follows. Let us consider $n-1$ paths $\delta_1, \dots, \delta_{n-1}$ such that $\delta_j := [-j, -(j+1)]$. Note that $\sigma_j \in \mathbb{B}_n \equiv \pi_1(\mathbf{X}; \mathbf{n})$ consists on $n-2$ straight segments and two counter-clockwise half-circles having δ_j as the common diameter.

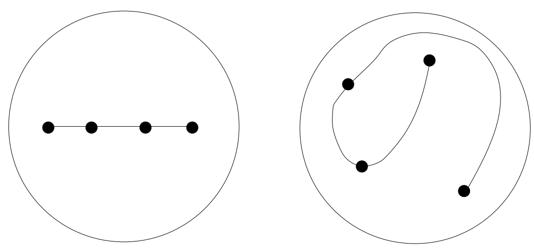

FIGURE 2.3. Basic segments

Let us order $\mathbf{x} := (x_1, \dots, x_n)$ and let us consider simple paths $\delta_j^{\mathbf{x}}$, starting at x_j and ending at x_{j+1}, $j = 1, \dots, n-1$, which intersect only at the extremities. We define an isomorphism which sends σ_j to a set of paths starting and ending at $\mathbf{x}$ such that:

- The paths are constant at x_k, $k \neq j, j+1$.
- Consider a small tubular neighbourhood of $\delta_j^{\mathbf{x}}$ and consider two paths that run counterclockwise the two semicircles in its boundary cut by $\delta_j^{\mathbf{x}}$.

DEFINITION 2.2.1. The ordered set of paths $\delta_j^{\mathbf{x}}$, $j = 1, \dots, n-1$, is a *basic segment* of $\mathbf{x}$.

REMARK 2.2.2. Note that a braid τ (seen as element of $\pi_1(\mathbf{X}; \mathbf{x})$) is represented by a set of paths starting and ending at $\mathbf{x}$; the path starting at x_j ends at x_{j^σ}, where $\Psi(\tau) = \sigma \in \Sigma_n$. It is usual to represent braids as the union of the graphs of these paths in $\Delta \times [0,1] (\subset \mathbb{R}^3)$ where Δ is a disk which contains the paths in its interior.

There is other useful representation of a braid if one identifies $\Delta \times \{0\}$ and $\Delta \times \{1\}$. The braid produces a link (called *the closed braid*) contained in a solid torus $\Delta \times \mathbb{S}^1 \subset \mathbb{S}^3$. Note that conjugate braids produce the same link.

DEFINITION 2.2.3. A braid is *positive* if can be written as word in $\sigma_1, \dots, \sigma_{n-1}$ having only positive exponents. A braid is *quasipositive* if it is conjugated to a positive braid.

DEFINITION 2.2.4. The *degree map* of $\mathbb{B}_n$ is the homomorphism $\deg : \mathbb{B}_n \to \mathbb{Z}$ such that $\deg(\sigma_j) := 1$.

2.3. Local braid monodromy

Let f be a Weierstrass polynomial of degree n; for convenience, we will suppose that f is reduced (there is no multiple irreducible component). We fix $0 < \delta \ll \varepsilon \ll 1$ as in §1.1 and let $Z := \{(x, y) \in \bar{\Delta}_\delta \times \bar{\Delta}_\varepsilon \mid f(x, y) = 0\}$.

DEFINITION 2.3.1. The *discriminant locus* of f relative to $\bar{\Delta}_\delta \times \bar{\Delta}_\varepsilon = \bar{W}_{\delta,\varepsilon}$ is $\operatorname{disc}_{f,\delta} := \{x_0 \in \bar{\Delta}_\delta \mid \operatorname{disc}_y(f)(x_0) = 0\}$.

The main property of the discriminant locus is that the projection of Z onto $\bar{\Delta}_\delta$ is an unramified covering outside $\operatorname{disc}_{f,\delta}$. In our case, $\operatorname{disc}_{f,\delta} = \{0\}$.

The covering $\pi : Z^* \to \bar{\Delta}^*_\delta$ is classified by the classical monodromy at δ; let $F := \pi^{-1}(\delta)$. Since $\#F = n$, ordering F we can identify Σ_F with Σ_n. The classical monodromy is

$$\rho : \pi_1(\bar{\Delta}^*_\delta; \delta) \equiv \mathbb{Z} \to \Sigma_n,$$

where $\rho(1)$ is the product of r disjoint cycles corresponding to the r irreducible components of f.

In fact, we have much more; any loop based at δ lifts in a set of n non intersecting-paths starting and ending at F, and (upon the choice of a basic segment of $\mathbf{x}$) we can define the braid monodromy of f as a morphism

$$\nabla : \pi_1(\bar{\Delta}^*_\delta; \delta) \equiv \mathbb{Z} \to \pi_1(\mathbf{X}; F) \equiv \mathbb{B}_n,$$

such that $\Psi_n \circ \nabla = \rho$.

EXAMPLE 2.3.2. If f is irreducible and h is a Puiseux expansion, then the image by ∇ of $t \mapsto \delta \exp(2i\pi t)$ is the set of paths $\{t \mapsto h(\zeta\delta \exp(2i\pi t)) \mid \zeta^n = 1\}$. In the general case, we consider the union of these paths for each irreducible component.

NOTATION 2.3.3. The image by of the standard generator of $\pi_1(\bar{\Delta}^*_\delta; \delta)$ is denoted $\tau_{f,\pi}$, because it depends both on f and the projection.

THEOREM 2.3.4. *The closed braid associated to τ_f is a model for the link of f. In particular, τ_f is only well-defined up to conjugation.*

We construct τ_f in several examples.

EXAMPLE 2.3.5. Let us consider $f(x, y) := y^2 - x$. We can choose $\delta = 1$ and hence $F := \{\pm 1\}$. The set of paths is formed by $t \mapsto \exp(i\pi t)$ and $t \mapsto -\exp(i\pi t)$. If the basic segment is formed by the segment $[1, -1]$, the braid is σ_1.

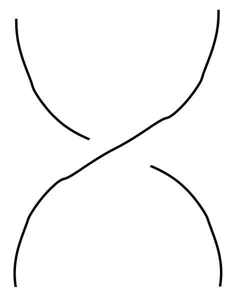

FIGURE 2.4. Braid for an ordinary tangent

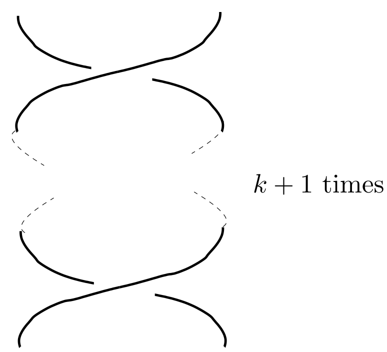

FIGURE 2.5. Braid for an $\mathbb{A}_k$-singularity

EXAMPLE 2.3.6. Let us consider $f(x,y) := y^2 - x^{k+1}$. We can choose again $\delta = 1$ and hence $F := \{\pm 1\}$. The set of paths is formed by $t \mapsto \exp((k+1)i\pi t)$ and $t \mapsto -\exp((k+1)i\pi t)$. If the braid in Example 2.3.5 is a single half-twist, in this case we have $k+1$ half-twists. If the basic segment is formed by the segment $[1,-1]$, the braid is σ_1^{k+1}.

EXAMPLE 2.3.7. Let us consider $f(x,y) := y^k - x$. We can choose again $\delta = 1$ and hence $F := \mu_k = \{\zeta \in \mathbb{C} \mid \zeta^k = 1\}$. The set of paths is formed by $t \mapsto \exp(2i\pi\frac{t+2j}{k})$, $j = 0, 1 \ldots, k-1$.

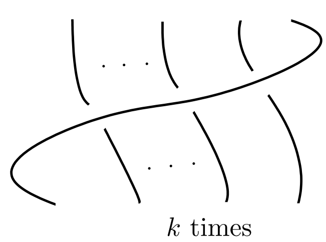

FIGURE 2.6. Braid for a flex

We choose as basic segment the arcs in the unit circle joining consecutively and counterclockwise the k-roots of unity and starting at 1. The braid is $\sigma_{k-1} \cdot \ldots \cdot \sigma_1$.

We can proceed in the same way for $f(x,y) := y^k - x^l$; in this case the braid is $(\sigma_{k-1} \cdot \ldots \cdot \sigma_1)^l$.

EXAMPLE 2.3.8. Let us consider $f(x,y) := x(y^k - x^l)$. We can proceed as in Example 2.3.7; in this case we have a braid in $k+1$ strings. There is another way of drawing this braid. We consider a cylinder with k vertical lines in the boundary and its core and we glue the bottom and the top faces by a rotation of angle $-\frac{2\pi l}{k}$, see Figure 2.7.

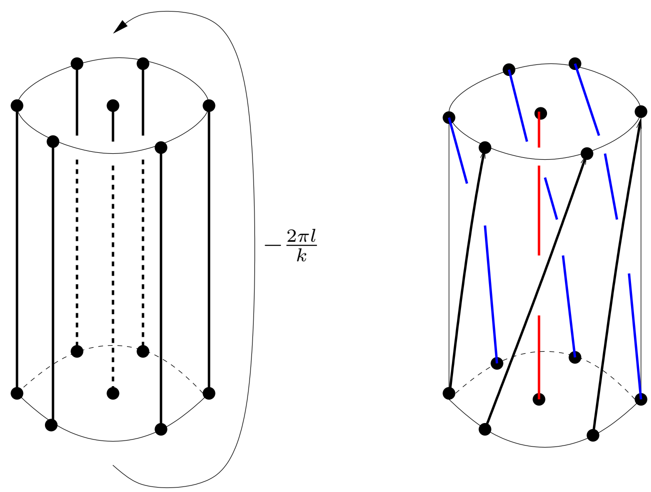

FIGURE 2.7. Braid with core

EXAMPLE 2.3.9. All the above examples have *nice* equations. In fact, their Puiseux expansions are minimal. What happens if we have a Puiseux expansion like $y = x^{\frac{1}{2}} + x^{\frac{3}{2}}$. The main idea is that the last term only produces some *noise* which can be homotopically avoided. Choosing δ small enough, we proceed as follows:

- Consider the braid associated to $y = x^{\frac{1}{2}}$.
- We can construct a *tube* around this braid such that the braid of $y = x^{\frac{1}{2}} + x^{\frac{3}{2}}$ is contained in this tube, since the term $x^{\frac{3}{2}}$ is very small with respect to $x^{\frac{1}{2}}$.
- Since the exponent of the second term has the same denominator, we can perform a homotopy of the *actual* braid to the first one.

EXAMPLE 2.3.10. Let us consider $f(x,y) := (y^2 - x)^2 - 4x^2y - x^3$ as in Example 2.1.11. Note that $n = 4$ and the Puiseux expansion is $h(t) = t^2 + t^3$. In the classical notation, we have $y = x^{\frac{1}{2}} + x^{\frac{3}{4}}$, i.e., we have two Puiseux exponents. We proceed in this example as in Example 2.3.9. We start with a curve having as parametrization $y = x^{\frac{1}{2}}$; the braid has two strings, producing the braid σ_1. If we choose δ small enough, we have a braid with 4 strings pairwise distributed in the two tubes. The core of these tubes is parametrized by $y = x^{\frac{1}{2}}$. We are drawing the braids in $\mathbb{C} \times [0,1]$ as usual, but they should be drawn in in $\mathbb{C} \times \mathbb{S}^1$, where

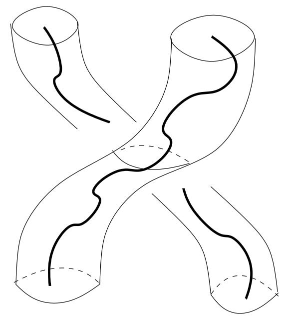

FIGURE 2.8. Perturbed braid

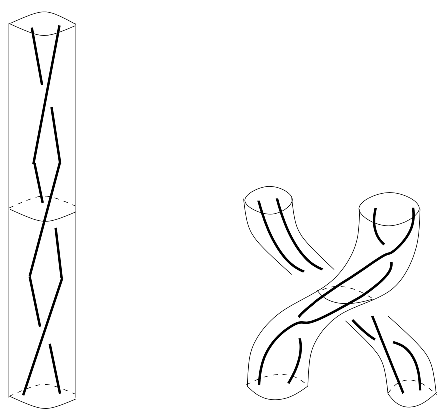

FIGURE 2.9. Two-pair braid

$\mathbb{S}^1 := \{t \in \mathbb{C} \mid |t| = 1\}$. The tube becomes a solid torus and the braid associated to $y = x^{\frac{1}{2}}$ is its core.

We can consider that the two parts of the tube glue in a tube $\Delta_\eta \times [0, 2]$, where the variables are $x_1 := x^{\frac{1}{2}}$ and $y_1 := y - x^{\frac{1}{2}}$. The new parametrization is $y_1 = x_1^{\frac{3}{2}}$ which is represented in the left-hand cylinder of Figure 2.9. Cutting this cylinder in two pieces give the right-hand braid. The basic segment is constructed as follows. First, we construct a basic segment in each small disk which is used for the construction of the tube. Second, we join these basic segments using the basic segment of the first Puiseux pair. The basic segments are those of Example 2.3.7.

We explain the inductive process to construct a braid associated to a germ f. First, consider good coordinates for f and let Γ_f be its Newton polygon. Consider the highest edge γ of Γ_f. Let $\frac{q}{p}$ be the anti-slope of γ, $\gcd(p, q) = 1$. If γ is the unique edge of Γ_f we consider the braid associated to $y^{pr_\gamma} - x^{qr_\gamma}$; if Γ_f has more edges, we consider the braid associated to $x(y^{pr_\gamma} - x^{qr_\gamma})$. We replace this braid by tubes; when closing the braids, we obtain r_γ tubes in the first case and $r_\gamma + 1$ in the second one.

Each tube different from the core is associated to a root $w_{j,\gamma}$. The germ $f_{\gamma, w_{\gamma,j}}$ is *simpler* than the one of f and we may suppose constructed its braid. We cut it in the p sub-tubes.

We forget now the first edge and we continue the process in the tube associated to the core.

THEOREM 2.3.11. *The braids associated to algebraic germs of curves are quasipositive.*

LECTURE 3

Global braid monodromy

3.1. Applications to the local case.

The construction of the local braid monodromy can be applied in a more general setting.

PROPOSITION 3.1.1. *Let g be a monic polynomial in y of degree n, whose coefficients are continuous functions on $\bar{\Delta}_\delta$ and holomorphic on Δ_δ. Let us suppose that:*

(1) $\forall x_0 \in \bar{\Delta}_\delta$*, the roots of* $g(x_0, y) = 0$ *are in* Δ_ε *(in particular,* $g^{-1}(0) \cap (\bar{\Delta}_\delta \cap \partial\bar{\Delta}_\varepsilon) = \emptyset$*).*
(2) $\operatorname{disc}_{f,\delta} \cap \partial\bar{\Delta}_\delta = \emptyset$.

Then, there exists a braid monodromy

$$\nabla : \pi_1(\bar{\Delta} \setminus \operatorname{disc}_{g,\delta}; \delta) \to \mathbb{B}_n.$$

We apply this proposition to the local braid monodromy of §2.3.

THEOREM 3.1.2. *Let $\tau := \tau_{f,\pi}$ be a braid associated to a reduce germ f, regular of order n. Then,* $\deg \tau = n + \mu(f) - 1$.

PROOF. For $|t|$ small enough, we can apply Proposition 3.1.1 to $f - t$. We may suppose that $f^{-1}(t)$ is smooth. For a generic choice of t, for each $x_0 \in \operatorname{disc}_{f-t,\delta}$ there exists a unique $y_0 \in \bar{\Delta}_\varepsilon$ (in fact in Δ_ε) such that $f(x_0, y_0) = t$ and the tangent line to $f^{-1}(t)$ at (x_0, y_0) is vertical. Let

$$\nabla_t : \pi_1(\bar{\Delta}_\delta \setminus \operatorname{disc}_{f-t,\delta}; \delta) \to \mathbb{B}_n$$

be the braid monodromy of $f - t$. Let $r_t := \# \operatorname{disc}_{f-t,\delta}$; the group $\pi_1(\bar{\Delta}_\delta \setminus \operatorname{disc}_{f-t,\delta}; \delta)$ is free of rank r_t and a base $\gamma_1, \dots, \gamma_{r_t}$ is obtained as in Figure 3.1.

Note that $\nabla_t(\gamma_i)$ is decomposed as $\beta_i \alpha_i \beta_i^{-1}$, where β_i is the braid associated to the *straight* part and α_i is the braid of the boundary of the small disk. The braid α_i is formed by $n-2$ unlinked straight braids and the two remaining strings behave as in Example 2.3.5, i.e., $\alpha_i = \sigma_j$ for some j. Let $\varphi := \prod_{j=1}^{r_t} \gamma_{r_t - j + 1}$, i.e., the counterclockwise boundary of $\bar{\Delta}_\delta$; note that $\deg \nabla_t(\gamma_i) = 1$ and then $\deg \nabla_t(\varphi) = r_t$.

It is easily seen that $\nabla_t(\varphi) = \nabla(\varphi) = \tau$. We conclude that $\deg \tau = r_t$.

If we compose ∇ with Ψ_n we obtain the classical monodromy of the unramified covering $f^{-1}(t) \setminus \pi^{-1}(\operatorname{disc}_{f-t,\delta}) \to \bar{\Delta}_\delta \setminus \operatorname{disc}_{f-t,\delta}$ which is composed of r_t transpositions. Since we have an n-sheeted covering,

$$\chi(f^{-1}(t) \setminus \pi^{-1}(\operatorname{disc}_{f-t,\delta})) = n\chi(\bar{\Delta}_\delta \setminus \operatorname{disc}_{f-t,\delta}) = n(1 - r_t) = n(1 - \deg \tau).$$

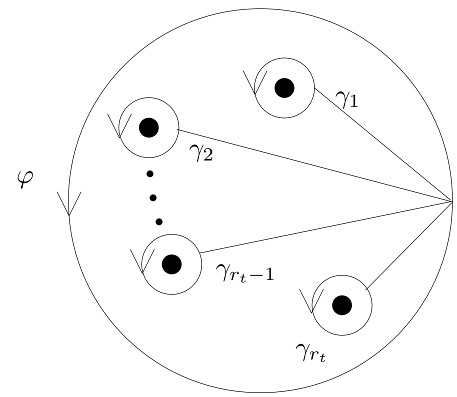

FIGURE 3.1. Base of $\pi_1(\bar{\Delta}_\delta \setminus \mathrm{disc}_{f-t,\delta};\delta)$

Over each point of $\mathrm{disc}_{f-t,\delta}$ we have $n-1$ points:

$$\mu = 1-\chi(f^{-1}(t)\setminus\pi^{-1}(\mathrm{disc}_{f-t,\delta}))-(n-1)\deg\tau = 1-n(1-\deg\tau)-(n-1)\deg\tau = 1-n-\deg\tau. \ \square$$

In general, it is also possible to make a change of variables such as n coincides with the multiplicity m of f. Let us suppose that we have f a germ y-regular of order n, which is bigger than the multiplicity of f. We construct ε and η as above, and we consider the braid $\tau_{f,\pi}$. Then, $f_t(x,y) := f(x+ty,y)$, for $|t|$ small enough,is y-regular of order m. We can apply Proposition 3.1.1 to f_t, δ and ε and we denote $\nabla_t : \pi_1(\bar{\Delta}_\delta \setminus \mathrm{disc}_{f_t,\delta}) \to \mathbb{B}_n$. For a generic choice of t the set disc_{f_t} contains 0 and $n-m$ other points.

Let us choose a base $\gamma_0, \gamma_1, \ldots, \gamma_{n-m}$ as in Figure 3.1 such that γ_0 runs around 0. Then

- $\nabla_t(\gamma_0)$ is conjugated to a braid obtained by adding constant maps to a braid of the type $\beta_{f_t,\pi}$.
- $\nabla_t(\gamma_j)$, $j>0$, are conjugated to σ_1.
- $\nabla_t(\gamma_{n-m}\cdot\ldots\cdot\gamma_1\cdot\gamma_0) = \tau_{f,\pi}$.

3.2. Braid monodromy of affine plane curves

Let us consider a reduced affine plane curve C with no vertical asymptotes and no vertical lines. This means that C has an equation $f(x,y)=0$, where f has no multiple factor and f is monic as an element of $\mathbb{C}[x,y]$; such a curve will be called *horizontal*. Let $n := \deg_y f$ (which may be smaller than $\deg f$).

Let $\mathrm{disc}_f := \{x \in \mathbb{C} \mid \mathrm{disc}_y f(x) \neq 0\}$. Since f has no multiple factor, then $\mathrm{disc}_y f$ is a non-zero element of $\mathbb{C}[x]$, and then it is a finite set $\{x_1,\ldots,x_r\}$. Let $M \gg 0$ such that $\mathrm{disc}_f \subset \Delta_M$. Since the roots of $f(x_0,y)$ vary continuously with x_0 then there exists $N \ggg 0$, such that if $x \in \bar{\Delta}_M$ and $f(x,y)=0$, then $y \in \Delta_\varepsilon$. In particular, f, M, N are in the hypothesis of Proposition 3.1.1.

DEFINITION 3.2.1. The homomorphism $\nabla : \pi_1(\Delta_M \setminus \mathrm{disc}_f; M) \to \mathbb{B}_n$ is the *braid monodromy morphism* of C.

The homomorphism ∇ is only well-defined up to conjugation in $\mathbb{B}_n$.

EXAMPLE 3.2.2. Let us consider the affine curves $y^k - x^l = 0$, where $\mathrm{disc}_f = \{0\}$. In this case, braid monodromy is defined as in the local case $\nabla : \mathbb{Z} \to \mathbb{B}_k$, where the image of the generator of $\mathbb{Z}$ equals $(\sigma_{k-1} \cdot \ldots \cdot \sigma_1)^l$.

EXAMPLE 3.2.3. Since braid monodromy is well-defined up to conjugation, the particular choice of a base point is irrelevant. Let us consider now $f(x, y) := y^d - x + 1$; it is clear that $\mathrm{disc}_f := \{1\}$. One can choose M big and $\pi_1(\bar{\Delta}_M \setminus \{1\}; 0)$ is generated by $\gamma : [0, 1] \to \bar{\Delta}_M \setminus \{1\}$, given by $\gamma(t) := 1 - \exp(2i\pi t)$. Following Example 3.2.2, we know that braid monodromy equals $\sigma_{d-1} \cdot \ldots \cdot \sigma_1$.

Let us consider now $f(x, y) := y^d - x^d + 1$; in this case, $\mathrm{disc}_f := \{\zeta \in \mathbb{C} \mid \zeta^d = 1\}$. Let us consider the mapping $x \mapsto x^d$. Taking preimages, γ produces d loops based at 0, as in Figure 3.2. We consider a basis $\gamma_1, \ldots, \gamma_d$ of $\pi_1(\bar{\Delta}_M \setminus \mathrm{disc}_f; 0)$ as in Figure 3.2, and it is easily seen that

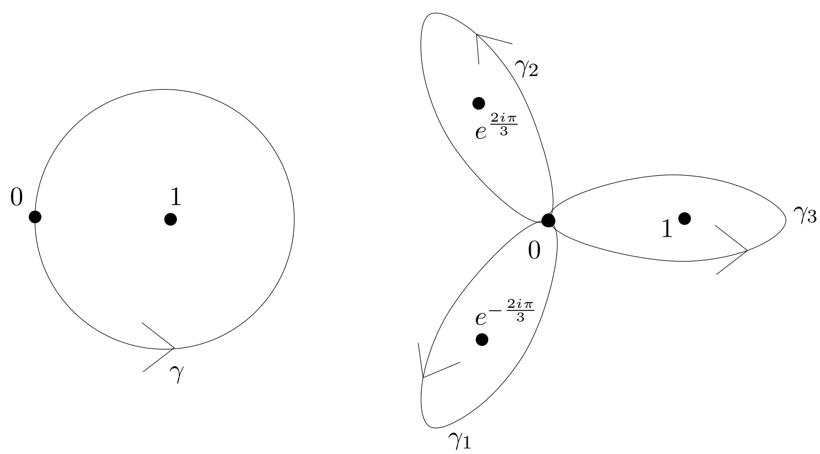

FIGURE 3.2. Basis for $d = 3$

$\nabla(\gamma_j) = \sigma_{d-1} \cdot \ldots \cdot \sigma_1$. Note that the product $\gamma_d \cdot \ldots \cdot \gamma_1$ is homotopic to the boundary of a (topological) disk containing disc_f.

DEFINITION 3.2.4. Let $M > 0$ and let $A \subset \Delta_M$, $\#A = d < \infty$. We say that an ordered basis $\gamma_1, \ldots, \gamma_d$ of the free group $\pi_1(\bar{\Delta}_M \setminus A)$ is *seudogeometric* if $\gamma_d \cdot \ldots \cdot \gamma_1$ is homotopic to the (positive) boundary of a (topological) disk containing A and it is constructed as in Figures 3.1 and 3.2. If the base point is in $\partial\bar{\Delta}_M$ and the above product is homotopic to $\partial\bar{\Delta}_M$ then the base is called *geometric*.

REMARK 3.2.5. Basic segments are closely related to geometric basis. Let us order A as $x_1, \ldots, x_d$ and let us consider a basic segment $\delta_1, \ldots, \delta_{d-1}$ associated respecting this ordering.

We choose small disks around $x_1, \dots, x_d$ and we choose a path δ_0 from the base point in the boundary to the first disk. The base $\gamma_1, \dots, \gamma_d$ is obtained as follows:

- Start from δ_0, run counterclockwise the boundary of the first disk and come back through δ_0: this is the construction of γ_1.
- Start again from δ_0, run counterclockwise the boundary of the first disk until encountering the path δ_1; run δ_1 until the boundary of the second disk. Then, run counterclockwise the boundary of the second disk and come back to the base point: this is the construction of γ_2.
- The other elements are constructed in the same way.

Note that all bases we have used up to now are seudogeometric.

Definition 3.2.6. Let C be a horizontal curve of degree n and equation $f = 0$, $r := \# \operatorname{disc}_f$. We say that an element $(\tau_1, \dots, \tau_r) \in (\mathbb{B}_n)^r$ *represents the braid monodromy* of C if there exists a geometric basis $\gamma_1, \dots, \gamma_r$ such that $\tau_i = \nabla(\gamma_i)$.

Example 3.2.7. Let us consider $f(x, y) = y^d - (x + ay)^d + 1$, $|a| \ll 1$. It is possible to find a geometric basis (with $d(d-1)$ elements) such that $(\sigma_1, \dots, \sigma_{d-1}, \dots, \sigma_1, \dots, \sigma_{d-1})$ represents the braid monodromy.

Definition 3.2.8. Let G be a group; the group $\mathbb{B}_r$ acts naturally on G^r by *Hurwitz moves.* If $\sigma_1, \dots, \sigma_{r-1}$ is the standard basis of $\mathbb{B}_r$, then the result of the action of σ_j on $(g_1, \dots, g_r)$ is

$$(g_1, \dots, g_{j-1}, g_{j+1}, g_{j+1} g_j g_{j+1}^{-1}, g_{j+2}, \dots, g_r).$$

This action commutes with the action of G on G^r by conjugation and it defines an action of $\mathbb{B}_r \times G$ on G^r.

Theorem 3.2.9 (Artin). *Two seudogeometric basis are related by an inner automorphism and a Hurwitz move.*

Corollary 3.2.10. *Let C be a horizontal curve of degree n and equation $f = 0$, $r := \# \operatorname{disc}_f$; fix a representative $(\tau_1, \dots, \tau_r) \in (\mathbb{B}_n)^r$ of the braid monodromy of C. Then, an element of $(\mathbb{B}_n)^r$ represents the braid monodromy of C if and only if is in the orbit of $(\tau_1, \dots, \tau_r)$ by the action of $\mathbb{B}_r \times \mathbb{B}_n$; this orbit will be called the* braid monodromy *of C.*

3.3. Zariski-van Kampen Theorem

The topology of a curve C is the homeomorphism type of the pair $(\mathbb{C}^2, C)$. Given M, N as in §3.2, let us denote $C_{M,N} := C \cap (\bar{\Delta}_M \times \bar{\Delta}_N)$ and

$$K_{M,N} := C \cap \partial(\bar{\Delta}_M \times \bar{\Delta}_N) = C \cap (\partial\bar{\Delta}_M \times \bar{\Delta}_N).$$

Proposition 3.3.1. *The pair $(\bar{\Delta}_M \times \bar{\Delta}_N, C_{M,N})$ is a strong deformation retract of $(\mathbb{C}^2, C)$. Moreover,*

$$(\bar{\Delta}_M \times \bar{\Delta}_N, C_{M,N}) \cup (\partial(\bar{\Delta}_M \times \bar{\Delta}_N), K_{M,N}) \times [0, 1) \cong (\mathbb{C}^2, C).$$

The topological type of $(\partial(\bar{\Delta}_M \times \bar{\Delta}_N), K_{M,N})$ does not depend on M, N big enough and it is called the link at infinity *of C.*

We can interpret $(\bar{\Delta}_M \times \bar{\Delta}_N, C_{M,N})$ as a compact model for $(\mathbb{C}^2, C)$. We are going to recover the classical Zariski-van Kampen theorem from braid monodromy. The goal is to compute the group:

$$G := \pi_1(\mathbb{C}^2 \setminus C) = \pi_1(\bar{\Delta}_M \times \bar{\Delta}_N \setminus C_{M,N}).$$

We denote $X := \bar{\Delta}_M \times \bar{\Delta}_N \setminus C_{M,N}$ and

$$\check{X} := X \setminus \{(x,y) \mid x \in \operatorname{disc}_f, y \in \bar{\Delta}_N\}, \quad \check{G} := \pi_1(\check{X}).$$

THEOREM 3.3.2. *The first projection defines a locally trivial fibration $\check{X} \to \Delta_M \setminus \operatorname{disc}_f$ whose fiber is a closed disk with n punctures.*

COROLLARY 3.3.3. *Let us denote $\mathbb{F}_m$ the free group with m generators. There is a short exact sequence*

$$1 \mapsto \mathbb{F}_n \to \check{G} \to \mathbb{F}_r \to 1. \tag{3.1}$$

PROOF. We consider the long exact sequence of homotopy of the fibration of Theorem 3.3.2. The result follows since $\pi_1(\Delta_M \setminus \operatorname{disc}_f)$ is free of rank r, $\pi_2(\Delta_M \setminus \operatorname{disc}_f)$ is trivial and the fundamental group of the fiber is free of rank n. ☐

In order to obtain a presentation of $\check{G}$ from (3.1) we proceed as follows:

(P1) Choose a geometric basis $\gamma_1, \ldots, \gamma_n$ of of $\pi_1(\{M\} \times (\bar{\Delta}_N \setminus F_M; (M,N))$, where $F_M := \{y \mid f(M,y) = 0\}$, the basis is associated to a basic segment.
(P2) Choose a geometric basis $\alpha_1, \ldots, \alpha_r$ of $\pi_1(\bar{\Delta}_M \setminus \operatorname{disc}_f; M)$.
(P3) Lift this geometric basis to $\bar{\Delta}_M \setminus \operatorname{disc}_f \times \{N\}$. The lifted elements are denoted by $\beta_1, \ldots, \beta_r$.
(P4) Compute $\beta_i^{-1}\gamma_j\beta_i$ as a word in $\gamma_1, \ldots, \gamma_n$.

Braid monodromy appears in (P4). We consider a braid monodromy morphism $\nabla : \pi_1(\bar{\Delta}_M \setminus \operatorname{disc}_f; M) \to \mathbb{B}_n$ which uses the basic segment of $\gamma_1, \ldots, \gamma_n$. The representative of the braid monodromy we will consider is $(\nabla(\alpha_1), \ldots, \nabla(\alpha_r))$.

There is a natural action of $\mathbb{F}_n \times \mathbb{B}_n \to \mathbb{F}_m$ given by:

$$\gamma_i^{\sigma_j} := \begin{cases} \gamma_i & \text{if } i \neq j, j+1, \\ \gamma_{j+1} & \text{if } i = j, \\ \gamma_{j+1}\gamma_j\gamma_{j+1}^{-1} & \text{if } i = j+1. \end{cases}$$

Note that $(\gamma_n \cdot \ldots \cdot \gamma_1)^{\sigma_j} = \gamma_n \cdot \ldots \cdot \gamma_1$.

This action can be explained geometrically as follows:

- Identify $\pi_1(\{M\} \times (\bar{\Delta}_N \setminus F_M; (M,N))$ with $\mathbb{F}_n$ via $\gamma_1, \ldots, \gamma_n$.
- A braid τ where we identify $\mathbb{B}_n$ with $\pi_1(\mathbf{x}; F_M)$ via the basic segment.

It is easily seen that

$$X_\tau := (\bar{\Delta}_N \times [0,1]) \setminus \tau \cong (\bar{\Delta} \setminus F_M) \times [0,1].$$

For $t \in \{0,1\}$, the mappings $j_t(x) := (x,t)$ induce isomorphisms $(j_t)_* : \mathbb{F}_n \to \pi_1(X_\tau; (N,t))$. If $\gamma \in \mathbb{F}_n$ then γ^τ is constructed as follows:

- Consider $(j_0)_*(\gamma) \in \pi_1(X_\tau; (N,1))$.
- If $v : [0,1] \to X_\tau$ is defined by $v(t) := (N,t)$, then $\gamma_v := v^{-1}(j_0)_*(\gamma)v \in \pi_1(X_\tau; (N,0))$.
- Finally $\gamma^\tau := (j_1)_*)^{-1}(\gamma_v)$.

LEMMA 3.3.4. $\beta_i^{-1}\gamma_j\beta_i = \gamma_j^{\nabla(\alpha_i)}$.

PROOF. The space X_τ is the pull-back of the fibration $\check{X} \to \bar{\Delta}_M \setminus \mathrm{disc}_f$ by the mapping α_i; by this pull-back, β_i is sent to v and the result follows. □

PROPOSITION 3.3.5 (Fibered version of Zariski-van Kampen). *The group $\check{G}$ is generated by $\gamma_1, \dots, \gamma_n, \beta_1, \dots, \beta_r$ with the relations $\beta_i^{-1}\gamma_j\beta_i = \gamma_j^{\nabla(\alpha_i)}$.*

As a consequence of the standard Seifert-van Kampen Theorem, we have the following result.

THEOREM 3.3.6 (Zariski-van Kampen). *The group G is generated by $\gamma_1, \dots, \gamma_n$ with the relations $\gamma_j = \gamma_j^{\nabla(\alpha_i)}$, $j = 1, \dots, n-1$, $i = 1, \dots, r$.*

LECTURE 4

Topological applications

4.1. Puiseux monodromy and topology

The usual way to construct braid monodromy is as follows. We decomposed $\alpha_i = \zeta_i^{-1}\tilde{\alpha}_i\zeta_i$, where $\tilde{\alpha}_i$ is the counterclockwise boundary of a small disk D_i and ζ_i joins $\tilde{\alpha}_i$ with the base point, as it is shown in Figure 3.1. A suitable choice of basic segments determines a decomposition $\nabla(\alpha_i) = \tau_i^{-1}\rho_i\tau_i$, where ρ_i is an unlinked union of the braids associated with the Puiseux expansions of the points of C with first variable x_i, see §2.3. The braids ρ_i have an algebraic meaning, but it is usually difficult to compute the braids τ_i.

Definition 4.1.1. A decomposition $\nabla(\alpha_i) = \tau_i^{-1}\rho_i\tau_i$ as above, $j = 1, \ldots, r$, is called a *Puiseux monodromy* of C.

Proposition 4.1.2. *Let us consider a Puiseux monodromy of C and let us assume that α_i is an unlinked union of r_i local braids $\alpha_{i,k}$, $k = 1, \ldots, r_i$. Let $m_{i,k}$ be the number of strings of $\alpha_{i,k}$ ($n = \sum_k m_{i,k}$). Then, the relations $\gamma_j = \gamma_j^{\nabla(\alpha_i)}$ can be replaced by $\gamma_j^{\rho_i\tau_i} = \gamma_j^{\tau_i}$. Moreover, for each bunch of $m_{i,k}$ relations one can drop one of them.*

Theorem 4.1.3 (Libgober). *If we replace the relations of Theorem* 3.3.6 *by the relations of Proposition* 4.1.2*, the resulting presentation has the homotopy type of X.*

From a Puiseux monodromy one can reconstruct the topology of $(\mathbb{C}^2, C)$.

- Above the base point M, we consider the pair $(\mathbb{C}, \mathbf{n})$.
- Over each path ζ_i we consider the pair $(\zeta_i \times \mathbb{C}, \tau_i)$.
- Over each loop $\tilde{\alpha}_i$, we consider $\tilde{\alpha}_i \times \mathbb{C}$ with the closed braid ρ_i.
- Over each disk D_i we consider $D_i \times \mathbb{C}$ with r_i cones based at different vertices and based on the r_i braids giving $\tilde{\alpha}_i$
- Above $\partial\bar{\Delta}_M$ we consider the braid $\nabla(\gamma_r \cdot \ldots \cdot \gamma_1$.
- We complete the rest of $\bar{\Delta}_M$ using the homotopy between the braids.
- We complete to $\mathbb{C}$ using a product structure.

This construction depends on the Puiseux monodromy but a stronger result is true.

Theorem 4.1.4 (Carmona). *Two curves having the same braid monodromy are topologically equivalent.*

4.2. Discriminant of polynomial mappings

Definition 4.2.1. A polynomial $f(x,y)$ is *good at infinity* if all the curves $C_t : f(x,y) = t$ have the same link at infinity.

Let $f(x,y) = f_d(x,y) + f_{d-1}(x,y) + \cdots + f_1(x,y) + f_0$ be the homogeneous decomposition of f. Consider the standard compactification $\mathbb{C}^2 \hookrightarrow \mathbb{P}^2$ given by $(x,y) \mapsto [x : y : 1]$ in the homogeneous coordinates $[x : y : z]$; the line at infinity $L_\infty := \mathbb{P}^2 \setminus \mathbb{C}^2$ has equation $z = 0$ and the compactifications of the curves C_t have the homogeneous equations:

$$f_d(x,y) + f_{d-1}(x,y)z + \cdots + f_1(x,y)z^{d-1} + f_0 z^d = t z^d.$$

We can decompose

$$f_d(x,y) = \prod_{j=1}^{r} (a_j y - b_j x)^{m_j}, \quad (a_j, b_j) \neq (0,0).$$

Lemma 4.2.2. *$\forall t \in \mathbb{C}$, $\bar{C}_t \cap L_\infty = \{[a_j : b_j : 0] \mid j = 1, \ldots, r\}$ and these points are called the* points at infinity *of f.*

Fix a point $P_j := [a_j : b_j : 0]$; for simplicity we suppose $P_j = [0 : 1 : 0]$. We consider the germs $\{(\bar{C}_t, P_j)\}_{t \in \mathbb{C}}$; in the coordinates x, z they are defined by

$$f_d(x,1) + f_{d-1}(x,1)z + \cdots + f_1(x,1)z^{d-1} + f_0 z^d = t z^d.$$

Proposition 4.2.3. *The polynomial f is good at infinity if and only if $\forall j = 1, \ldots, r$, the Milnor number $\mu_{P_j,t} := \mu(\bar{C}_t, P_j)$ is independent of $t \in \mathbb{C}$ (such a polynomial has isolated critical points).*

Example 4.2.4. If $m_j = 1$ for all j (in particular $r = d$), then f is good at infinity. In fact in that case the link at infinity is the Hopf link with d components.

Example 4.2.5 (Broughton). Let $f(x,y) := x(xy+1)$. We have two points at infinity $P_1 := [1 : 0 : 0]$ and $P_2 := [0 : 1 : 0]$. At P_1 we can consider coordinates y, z and the family is given by $y + z^2 - tz^3 = 0$ and $\mu_{P_1,t} = 0$. At P_2 we can consider coordinates x, z and the family is given by $x^2 + xz^2 - tz^3 = 0$ and $\mu_{P_2,t} = 2$ if $t \neq 0$, $\mu_{P_2,0} = 3$. In particular, f is not good at infinity.

Theorem 4.2.6 (Thom). *Let $f : \mathbb{C}^2 \to \mathbb{C}$ a polynomial. There exists a minimal finite set $B(f)$ (the bifurcation set of f) such that $f_| : \mathbb{C}^2 \setminus f^{-1}(B(f)) \to \mathbb{C} \setminus B(f)$ is a locally trivial fibration. If f is good at infinity, then $B(f)$ coincides with the set* $\mathrm{Crit}(f)$ *of critical values.*

Example 4.2.7. In Example 4.2.5, $B(f) = \{0\}$ and there is no critical value.

A nice invariant of the topology of f is the monodromy action associated to the fibration $f_| : \mathbb{C}^2 \setminus f^{-1}(B(f)) \to \mathbb{C} \setminus B(f)$. In the homological level, if we choose $t_0 \in \mathbb{C} \setminus B(f)$ we have a morphism

$$\rho : \pi_1(\mathbb{C} \setminus B(f); t_0) \to \mathrm{Aut}(H_1(C_{t_0}; \mathbb{Z})).$$

In the case of polynomials which are good at infinity, we can understand it with Picard-Lefscthez techniques. For technical reasons, we restrict ourselves to a narrower class of polynomials.

DEFINITION 4.2.8. A polynomial f is *tame* if the polynomials $f(x,y)+a\ell$ are good at infinity and have the same link at infinity, where ℓ is a generic linear form and $|a|$ is small enough.

EXAMPLE 4.2.9 (Cassou-Noguès). Of course, any tame polynomial is good at infinity. The polynomial $f(x,y) = (x^2+y^3)^2 + y$ is good at infinity but not tame.

From now on we fix a tame polynomial f and we keep the previous notations. Let us suppose that x is generic for Definition 4.2.8.

We fix a geometric basis $\gamma_1, \ldots, \gamma_r$ of $\pi_1(\mathbb{C} \setminus \mathrm{Crit}(f); t_0) = \pi_1(\mathbb{C} \setminus \bigcup_{t \in \mathrm{Crit}(f)} D_t; t_0)$ where $D(t)$ is a small disk centered at t, for each $t \in \mathrm{Crit}(f)$.

LEMMA 4.2.10. *If $|a|$ is small enough the mapping $f_a := f - ax : \mathbb{C}^2 \to \mathbb{C}$ is a Morse function, i.e., the critical points of f_a are ordinary double points (of local equation $u^2 + v^2 = 0$) and have pairwise distinct critical values. Moreover:*

(M1) *The generic fibers of f and f_a are isotopic.*

(M2) *Let $p \in f^{-1}(t_1)$ be a critical point of f; there are $\mu(f^{-1}(t_1), p)$ Morse points of f_a* close *to p (whose values are in $D(t_1)$).*

(M3) *The mappings f and f_a define isotopic locally trivial fibrations over $\mathbb{C} \setminus \bigcup_{t \in \mathrm{Crit}(f)} D(t)$.*

Picard-Lefscthez theory allows us to compute the main invariants of these fibrations: fiber, intersection form and monodromy. We consider a morsification f_a.

NOTATION 4.2.11. For $t \in \mathrm{Crit}(f)$, we call $S_{a,t}$ the set of values of the Morse points coming from points in $f^{-1}(t)$.

We choose a geometric basis of $\pi_1(\mathbb{C} \setminus \mathrm{Crit}(f_a); t_0)$ which is compatible with the one of $\pi_1(\mathbb{C} \setminus \mathrm{Crit}(f); t_0)$ as in Figure 4.1; namely the reversed product of loops associated with the values in $S_{a,t}$ equals the corresponding loop for t.

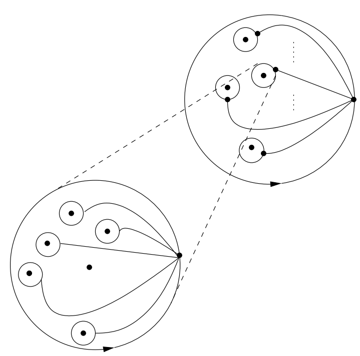

FIGURE 4.1. Geometric basis of the morsification

In this way, we obtain a *distinguished basis of vanishing cycles* for $H_1(f^{-1}(t_0), \mathbb{Z})$ (identified with $H_1(f_a^{-1}(t_0), \mathbb{Z})$ together with the intersection form $\langle \bullet, \bullet \rangle$ in terms of this basis. Following the results of §1.4, given a vanishing cycle a coming from an loop γ around the corresponding singular value of f_a, then the monodromy around γ is given by:

$$b \in H_1(f^{-1}(t_0), \mathbb{Z}) \mapsto b + \langle a, b \rangle b.$$

As in the local case, we obtain the monodromy action for the elements of the geometric basis $\pi_1(\mathbb{C} \setminus B(f); t_0)$ and the morphism ρ is computed.

In order to effectively compute the distinguished basis and the intersection form we will consider the polar mapping. Let $\psi := (f, x) : \mathbb{C}^2 \to \mathbb{C}^2$ (we will denote t the first variable of the second $\mathbb{C}^2$). This mapping is a submersion outside the polar locus Γ defined by $\frac{\partial f}{\partial y}(x, y) = 0$. Let $\Delta := \psi(\Gamma)$ be the discriminant locus of ψ. We have that $\psi_| : \mathbb{C}^2 \setminus \psi^{-1}(\Delta) \to \mathbb{C}^2 \setminus \Delta$ is an unramified covering of degree, say n. Moreover the equation of Δ is given by $\operatorname{disc}_y(f(x, y) - t) = 0$ (equation in x, t).

Remark 4.2.12. We do not need to compute $\operatorname{disc}_y(f_a(x, y) - t)$ since it is obtained from $\operatorname{disc}_y(f(x, y) - t)$ by the change $t \mapsto t + ax$. We denote Δ_a the curve $\operatorname{disc}_y(f_a(x, y) - t) = 0$.

Lemma 4.2.13. *The polynomial f is good at infinity if and only if Δ is horizontal and it is tame if and only if* $\deg(\operatorname{disc}_y(f(x, y) - t)) = \deg_x(\operatorname{disc}_y(f(x, y) - t))$.

Let us consider the *vertical* line L_{t_0} of equation $t = t_0$. We have a ramified covering $\phi_| : f^{-1}(t_0) \to L_{t_0}$ which is used as *multiparametrization* in Riemann's sense. The associated unramified covering is $\phi_| : f^{-1}(t_0) \setminus \phi^{-1}(\Delta) \to L_{t_0} \setminus \Delta$.

Lemma 4.2.14. *A value $t_1 \in \mathbb{C}$ is in* $\operatorname{Crit}(f)$ *if and only if there is a point $(x, t_1) \in \Delta$ such that the vertical line $t = t_1$ belongs to the tangent cone of Δ at (x, t_1).*

In particular, singular values of f_a correspond to ordinary vertical tangencies of Δ_a.

Let us fix a vertical tangency of Δ_a at (x, t_1). At the line $t = t_1 + \varepsilon$, $0 < \varepsilon \ll 1$, we can consider a *vanishing path* δ, whose extremities are two points in Δ which are *close* to the tangency.

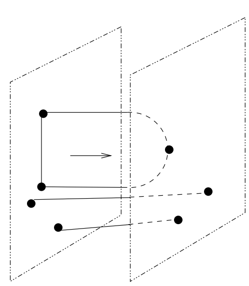

Figure 4.2. Vanishing paths

The preimage of this vanishing path in $f_a^{-1}(t_1 - \varepsilon)$ ($|\varepsilon|$ small) is composed by n edges; two of these edges share the extremities and its union define a vanishing cycle (up to orientation) in $f_a^{-1}(t_1 - \varepsilon)$.

This vertical tangency is related to a loop γ in the geometric basis of $\pi_1(\mathbb{C} \setminus B(f_a); t_0)$; let τ be the braid monodromy of Δ_a over γ. We can decompose $\tau = \beta\sigma_1\beta^{-1}$, where β is a braid which geometrically corresponds to a path from t_0 to $t_1 - \varepsilon$. This braid β allows us to transport the edge γ to a vanishing path $\tilde{\delta}$ in $f_a^{-1}(t_0) \cong f^{-1}(t_0)$.

Remark 4.2.15. It is possible to prove that the above construction does not depend on a particular decomposition $\tau = \beta\sigma_1\beta^{-1}$.

Theorem 4.2.16 (Escario). *The classical monodromy of* $\phi_| : f^{-1}(t_0) \setminus \phi^{-1}(\Delta) \to L_{t_0} \setminus \Delta$ *allows us to construct the corresponding vanishing cycle from* $\tilde{\delta}$.