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SEXTICS WITH SINGULAR POINTS IN SPECIAL POSITION

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ABSTRACT
In this paper we show a Zariski pair of sextics which is not a degeneration of the original example given by Zariski. This is the first example of this kind known. The two curves of the pair have a trivial Alexander polynomial. The difference in the topology of their complements can only be detected via finer invariants or techniques. In our case the generic braid monodromies, the fundamental groups, the characteristic varieties and the existence of dihedral coverings of $\mathbb{P}^2$ ramified along them can be used to distinguish the two sextics. Our intention is not only to use different methods and give a general description of them but also to bring together different perspectives of the same problem.

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The starting point of this paper is the existence of a certain pair of sextic curves $\mathcal{C}^{(1)},\mathcal{C}^{(2)} \subset \mathbb{P}^2$ in the complex projective plane. These curves have the same combinatorial properties and hence are candidates to form a Zariski pair. Roughly speaking, two curves form a Zariski pair if they have the same combinatorial data (degree of each irreducible component, local type of singularities,...) but different embeddings in $\mathbb{P}^2$ –cf. [1]. The combinatorial data of our pair are the following: both are reducible sextics, given by a smooth conic and a quartic; the quartic has two singular points of types $\mathbb{A}_1$ and $\mathbb{A}_3$; the conic and the quartic intersect at one single point of type $\mathbb{A}_{15}$. The non-combinatorial datum that distinguishes them is that the tangent line at $\mathbb{A}_{15}$ contains (for $\mathcal{C}^{(2)}$) or not (for $\mathcal{C}^{(1)}$) the point $\mathbb{A}_3$. The easiest and most common strategy to distinguish embeddings is to compute the

Alexander polynomial of the curves. Such a polynomial is known to be sensitive to the special position of some singular points of the curves – cf. [1], [2], [3]. In our case, even though the position of singularities distinguishes the two embeddings, it has no effect on the Alexander polynomial, which coincides in both cases. Examples of Alexander-equivalent Zariski pairs are already known – cf. [4], [6]. In this paper we will describe four different techniques to prove that two given curves constitute a Zariski pair and will apply them to the sextics $\mathcal{C}^{(1)}$ and $\mathcal{C}^{(2)}$:

- Computation of fundamental groups of the complement. Let $\mathcal{C}$ be an algebraic curve in the complex projective plane $\mathbb{P}^2$. The fundamental group of the complement $\mathbb{P}^2 \setminus \mathcal{C}$ is a topological invariant of the pair $(\mathbb{P}^2, \mathcal{C})$. Computation of such groups is in general a difficult task, and even if one manages to compute them, by finding a finite presentation, it might not be possible to decide whether or not the groups are isomorphic. In our case we use a non-generic braid monodromy to recover a presentation of the fundamental group.

- Computation of generic braid monodromy groups. This is also a topological invariant of the pair $(\mathbb{P}^2, \mathcal{C})$, but it is finer than the fundamental group. In fact, it determines the homotopy type of the complement $\mathbb{P}^2 \setminus \mathcal{C}$ – cf. [7]. It might be useful in cases where the fundamental groups are either isomorphic or at least difficult to compare. We think that it is worth looking for effective invariants, e.g. in the spirit of the work of Libgober – cf. [8].

- Computation of characteristic varieties. First defined by Libgober [9], they are a generalization of Alexander polynomials. If Alexander polynomials are related to (infinite) cyclic coverings of $\mathbb{P}^2$ ramified along a curve $\mathcal{C}$, the characteristic varieties are related to general abelian coverings of such kind. They might be computed from the fundamental groups; in this case they provide effective invariants. It is also possible to compute them without knowing the fundamental group using the theory of ideals of quasiadjunction, which generalizes computations of Alexander polynomials. This technique can be followed in detail in [9].

- Existence of dihedral coverings. The work of the fourth author has proven that it is also an effective way to obtain properties of the complements without an explicit computation of fundamental groups. The existence of a certain kind of dihedral coverings of $\mathbb{P}^2$ branched along $\mathcal{C}$ can be characterized by the existence of divisors satisfying certain algebraic properties. This allows to establish a connection between the position of singularities and the existence of non-abelian coverings of $\mathbb{P}^2$ ramified along $\mathcal{C}$.

As mentioned in the abstract, this example is the first known Zariski pair of sextics that is not a degeneration of the original example given by Zariski in [10]. Namely, there is no conic passing through six – maybe infinitely near – non-nodal singular points of either sextic.

The authors have found other pairs that might be worth studying. We won't discuss them here due to length considerations, but we include a brief description of them. The combinatorial data are the following: reducible sextics whose components are a quintic and a line; the quintic has two singular points, P_1 and P_2, of types $\mathbb{A}_1$ and $\mathbb{A}_9$ respectively; the quintic and the line intersect at a single point Q of type $\mathbb{A}_9$. There are three such curves up to projective change of coordinates. One of them, say $\mathcal{D}^{(2)}$, has the property that the tangent line at P_2 contains the point Q. This curve admits a birational transformation to the sextic $\mathcal{C}^{(2)}$ considered in this paper. The other two sextics, $\mathcal{D}^{(1,1)}$ and $\mathcal{D}^{(1,2)}$, are conjugated in $\mathbb{Q}(\sqrt{5})$ and don't admit a birational transformation to $\mathcal{C}^{(1)}$. Both pairs $(\mathcal{D}^{(1,1)}, \mathcal{D}^{(2)})$ and $(\mathcal{D}^{(1,2)}, \mathcal{D}^{(2)})$ are Zariski pairs and can be distinguished using the techniques presented in this paper, whereas the pair $(\mathcal{D}^{(1,1)}, \mathcal{D}^{(1,2)})$ remains indistinguishable for us.
A summarized description of the sections could be the following. In section 1 we define and construct the curves. In section 2 we briefly describe the braid monodromy of affine curves and the Zariski-van Kampen method as well as the connection between fundamental groups of affine and projective curves. Sections 3 and 4 are devoted to compute non-generic braid monodromies for $\mathcal{C}^{(1)}$ and $\mathcal{C}^{(2)}$, derive presentations for the fundamental groups of their complements and show that they are non-isomorphic. In section 5 we compute the generic braid monodromies of $\mathcal{C}^{(1)}$ and $\mathcal{C}^{(2)}$. In section 6 we introduce the characteristic varieties and briefly describe the procedure to calculate them in general following [9]. We apply such a method to our pair of sextics showing that both differ in the characteristic variety of depth one. The last section is devoted to describing the connection between a special type of dihedral coverings of $\mathbb{P}^2$ ramified along a curve and the existence of a divisor satisfying certain algebraic properties in terms of the Picard group. We prove that there exists such a dihedral covering of degree 16 ramified along one sextic whereas there is none for the other.

1. Definition of the curves

Let $\mathcal{C}$ be an algebraic curve in the complex projective plane $\mathbb{P}^2$ satisfying the following properties:

- P1. $\mathcal{C}$ has two irreducible components $\mathcal{C}_1$ and $\mathcal{C}_2$.
- P2. $\mathcal{C}_1$ is a smooth conic.
- P3. $\mathcal{C}_2$ is a curve of degree four having two singular points of types $\mathbb{A}_1$ (node) and $\mathbb{A}_3$ (tacnode). We will refer to them as $\mathbb{A}_1$ and $\mathbb{A}_3$.
- P4. $\mathcal{C}_1 \cap \mathcal{C}_2 = \{\mathbb{A}_{15}\}$, and $\mathbb{A}_{15}$ is a smooth point of $\mathcal{C}_2$. Then $(\mathcal{C}, \mathbb{A}_{15})$ is a singularity of type $\mathbb{A}_{15}$.

Example 1. Let us consider the curve $\mathcal{C}^{(1)} = \mathcal{C}_1^{(1)} \cup \mathcal{C}_2^{(1)}$ where:

$$\mathcal{C}_1^{(1)} : 8z^2 - 20yz + 36xz + 17y^2 - 18xy = 0,$$

$$\mathcal{C}_2^{(1)} : 8x^2z^2 - 16xy^2z + 52x^2yz - 36x^3z - 37x^2y^2 + 18x^3y - y^4 + 20xy^3 = 0.$$

This curve verifies the conditions P1-P4, where $\mathbb{A}_{15} = [1:0:0]$, $\mathbb{A}_1 = [1:1:0]$ and $\mathbb{A}_3 = [0:0:1]$. Note that the common tangent line at $\mathbb{A}_{15}$ is $y = 2z$ and it does not pass through $\mathbb{A}_3$.
The following figure shows the real part of $\mathcal{C}^{(1)}$ on the affine chart $\{x \neq 0\}$. We will denote it by $\mathcal{C}_{\text{af}}^{(1)}$. We also use the following conventions:

- The thin lines correspond to the conic $\mathcal{C}_{1,\text{af}}^{(1)}$. The thick lines correspond to the quartic $\mathcal{C}_{2,\text{af}}^{(1)}$.
- The dotted lines correspond to the real part of non-real complex conjugate points of the curves with real first coordinate.
- At the intersections of these dotted lines we put a thin string if the imaginary part of the points corresponding to the conic have a bigger absolute value. Otherwise, we put a thick string.

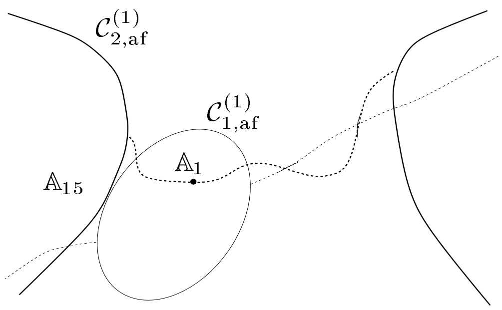

Fig. 1. Affine curve in (y, z).

Example 2. Let us consider the curve $\mathcal{C}^{(2)} = \mathcal{C}_1^{(2)} \cup \mathcal{C}_2^{(2)}$ where:

$$\mathcal{C}_1^{(2)} : 3x^2 + 2xy + 108z^2 = 0,$$

$$\mathcal{C}_2^{(2)} : 2xy^3 + 3x^2y^2 + 108y^2z^2 - x^4 = 0.$$

This curve also verifies P1-P4 above, where $\mathbb{A}_{15} = [0:1:0]$, $\mathbb{A}_3 = [0:0:1]$ and $\mathbb{A}_1 = [1:-1:0]$. Note that the common tangent line at $\mathbb{A}_{15}$ is $x = 0$ and it passes through $\mathbb{A}_3$.

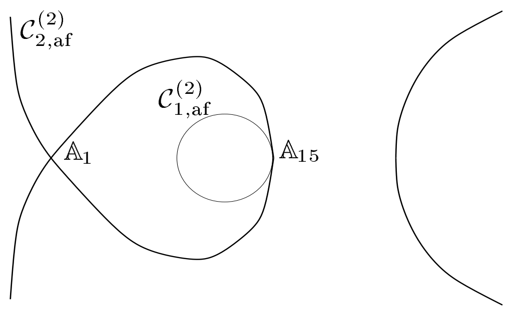

Fig. 2. Affine curve in (x, z).

Following the conventions from the previous example, we show the real part of $\mathcal{C}^{(2)}$ on the affine chart $\{y \neq 0\}$.

The way these examples can be constructed will be useful in the future. Let us suppose that $\mathcal{C}$ satisfies P1-P4. We want to find a suitable Cremona transformation to $\mathbb{P}^2$ that simplifies the problem. Let us consider the two dimensional family $\mathbb{P}$ of conics passing through $\mathbb{A}_3$ and $\mathbb{A}_{15}$ which are tangent to $\mathcal{C}$ at $\mathbb{A}_3$. There is a birational map $\mathbb{P}^2 \dashrightarrow \check{\mathbb{P}}$, where $\check{\mathbb{P}}$ is the dual of $\mathbb{P}$. Let us identify $\mathbb{P}^2$ with $\check{\mathbb{P}}$. There is an easy description of the mapping in terms of blowing-ups and blowing-downs as follows:

- Let us denote by T the tangent line to $\mathcal{C}$ at $\mathbb{A}_3$ and by L the line through $\mathbb{A}_{15}$ and $\mathbb{A}_3$. Blow up $\mathbb{A}_{15}$ and $\mathbb{A}_3$. Let us denote A_{15} and A_3 the corresponding exceptional components. Let us denote by A the intersection of T and A_3, and by P' the intersection of the proper transform of $\mathcal{C}$ and A_{15}.
- Let us blow up A and blow down L. The corresponding exceptional component will be also denoted by A.
- Let us blow down A_3 and T and denote accordingly the contracted points.

The above family of conics has been transformed into the family of conics passing through T and tangent to A_{15} at A_3.

It is an easy exercise to verify that $\mathcal{C}_2$ has been transformed onto a nodal cubic (nothing happened near $\mathbb{A}_1$) transversal to A and tangent to A_{15} at A_3. Moreover, A_3 is an inflection point of this cubic if and only if $\mathcal{C}$ is tangent to L at $\mathbb{A}_{15}$. On the other hand, the transform of $\mathcal{C}_1$ is an irreducible cubic also tangent to A_{15} at A_3 and having a double point at T. There are two possible cases for the intersection of the transforms of $\mathcal{C}_1$ and $\mathcal{C}_2$:

(a) If $\mathcal{C}$ is not tangent to L at $\mathbb{A}_{15}$, then the transforms are ordinary tangents at A_3 and the contact order at P' is equal to seven.

(b) If $\mathcal{C}$ is tangent to L at $\mathbb{A}_{15}$, then the contact order at A_3 is equal to nine.

Heuristically the second case is a degeneration of the first one and they may be treated together. We will also denote by $\mathcal{C}_1$ and $\mathcal{C}_2$ the transforms of the given curves by the above Cremona transformation.

Proposition 1. Up to projective transformation there are only two pairs of curves matching either (a) or (b) above. Namely:

1. There exists a pair of nodal cubics $\mathcal{C}_1^{(1)}$ and $\mathcal{C}_2^{(1)}$ satisfying:
   (a) $\mathcal{C}_1^{(1)} \cap \mathcal{C}_2^{(1)} = \{P_1, P'\}$ where P_1 and P' are singular points of $\mathcal{C}^{(1)} = \mathcal{C}_1^{(1)} \cup \mathcal{C}_2^{(1)}$ of type $\mathbb{A}_{13}$ and $\mathbb{A}_3$ respectively, and
   (b) the tangent line to $\mathcal{C}^{(1)}$ at P' contains the point P_1.

   These cubics generate a pencil containing a reducible curve having as components the common tangent line at P' and the conic which has intersection number six at P_1 with both cubics.
2. There exists a pair of nodal cubics $\mathcal{C}_1^{(2)}$ and $\mathcal{C}_2^{(2)}$ such that $\mathcal{C}_1^{(2)} \cap \mathcal{C}_2^{(2)} = \{P\}$ is a singular point of $\mathcal{C}^{(2)} = \mathcal{C}_1^{(2)} \cup \mathcal{C}_2^{(2)}$ of type $\mathbb{A}_{17}$.

   These cubics generate a pencil containing a triple line which is the common tangent line at P, an inflection point for both curves.

Proof. Let $\mathcal{C}_2$ be any nodal cubic. We will denote by $\varphi : \mathbb{C}^* \to \operatorname{Reg}(\mathcal{C}_2)$ any parametrization inducing an isomorphism of groups. The group structure of $\operatorname{Reg}(\mathcal{C}_2)$ is the geometrical one where the zero element is an inflection point. Such a parametrization is well defined up to a different choice of inflection point (obtained by multiplying by a cubic root of unity) and interchanging the branches of the nodal point (obtained by taking the inverse). In other words, there is a transitive and free action by the dihedral group $\mathcal{D}_6$ of order 6 on the space of all such parametrizations.
We recall that the geometrical group structure can be described as follows: let $P_1, \ldots, P_r \in \operatorname{Reg}(\mathcal{C}_2)$ and let $m_1, \ldots, m_r \in \mathbb{N}$ such that $m_1 + \ldots + m_r = 3n$. Then $m_1 P_1 + \ldots + m_r P_r = 0$ if and only if there exists a curve D of degree n such that $(D \cdot \mathcal{C}_2)_{P_j} = m_j$, $j = 1, \ldots, k$.
Let us denote $t_1 = \varphi^{-1}(P')$ and $t_2 = \varphi^{-1}(A_3)$. Then:

$$t_1 t_2^2 = 1, \quad t_1^7 t_2^2 = 1,$$

i.e., $t_1 = t_2^{-2}$ and $t_2^{12} = 1$. The factorization $t^{12} - 1 = (t^3 - 1)(t^3 + 1)(t^6 + 1)$ represents the space of orbits of the roots of $t^{12} - 1$ by the action of $\mathcal{D}_6$.
Let t_2 be a 12-th root of unity. Consider the curve $\mathcal{C}_2$ and consider the pencil of cubics having contact order equal to seven (resp. two) with $\mathcal{C}_2$ at P' (resp. A_3). To prove the existence of $\mathcal{C}_1$ we need to find a nodal cubic in this pencil (whose node is not A_3).
One has several possibilities for t_2:

- If $t_2^3 = -1$, then $t_1^3 = 1$. Then P' is an inflection point of $\mathcal{C}_2$. As it is readily seen, the product of the double tangent to P' and the line A_{15} joining P' and A_3 form a member of the pencil. Blowing up the base points results in the elliptic fibration associated to the pencil. An easy argument of euler characteristics shows that there is no such a fiber as $\mathcal{C}_1$.

- If $t_2^6 = -1$, one obtains (1) as follows. Denote $P' = \varphi(t_2)$. Note that

$$t_1^6 = (-t_2^4)^6 = 1.$$

Moreover, t_1 has order six. Hence, the tangent line L' to $\mathcal{C}_2^{(1)}$ at P' contains the point $\varphi(t_1) = P_1$. In fact, there is smooth conic Q intersecting $\mathcal{C}_1^{(1)}$ only at P_1 (i.e. with multiplicity six). The cubics $\mathcal{C}' = Q + L'$ and $\mathcal{C}_2^{(1)}$ define a pencil and, after blowing up the base points, an elliptic fibration from a rational surface X_{8211} on $\mathbb{P}^1$. This fibration has three evident exceptional fibers: the one given by $\mathcal{C}_2^{(1)}$ (I_1 in Kodaira's notation), the one given by $\mathcal{C}'$ (I_8) and a nodal cubic with the node on P' (I_2). Since the euler characteristic of X_{8211} is 12 and $\chi(I_n) = n$, there must be another special fiber I_1. This fiber corresponds to a nodal cubic $\mathcal{C}_1^{(1)}$, whose node is not a base point of the pencil. Hence, this is the required pair of cubics.

- If $t_2^3 = 1$, one obtains (2) as follows. In this case $P' = \varphi(t_2)$ is an inflexion point of $\mathcal{C}_2^{(2)}$. Let L' be the tangent line of $\mathcal{C}_2^{(2)}$ at P. Consider the pencil defined by $\mathcal{C}_2^{(2)}$ and $3L'$. As in the previous case, after blowing up the nine base points of the pencil (counted with multiplicity), we obtain a rational surface X_{211} defining an elliptic fibration over $\mathbb{P}^1$. In this case, we have two evident special fibers, namely the one determined by $\mathcal{C}_2^{(2)}$ (I_1) and the one determined by the triple line (II^*). Since $\chi(II^*) = 10$, $\chi(I_1) = 1$ and $\chi(X_{211}) = 12$, there has to be another special fiber whose euler characteristic equals one, that is, a nodal cubic $\mathcal{C}_1^{(2)}$ whose node is not on P. The pair $\mathcal{C}_1^{(2)}$, $\mathcal{C}_2^{(2)}$ is the required pair of cubics. $\square$

With a suitable choice of equations for $\mathcal{C}_i^{(j)}$ and for the Cremona transformations one can obtain the equations given in Examples 1 and 2.

Remark 1. The existence of the elliptic surfaces X_{8211} and X_{211} can also be derived directly from [14]. Both rational surfaces will be useful in section 7.

In the remaining part of the paper we will prove the following

Main Theorem 1. The sextic curves $\mathcal{C}^{(1)}$ and $\mathcal{C}^{(2)}$ form a Zariski pair.

2. Some facts about fundamental groups and braid monodromy of affine curves

In the first part of this section we will define the basic concepts required to work with braid monodromies of affine curves. In the second part we will prove the relationship between fundamental groups of projective and affine curves.

Let $a_1, \ldots, a_k$ be k distinct complex numbers and put $\mathcal{U} = \mathbb{C} \setminus \{a_1, \ldots, a_k\}$. We may suppose that, for $1 \leq i < k$ we have either $\Re a_i > \Re a_{i+1}$ or $\Re a_i = \Re a_{i+1}$ and $\Im a_i > \Im a_{i+1}$. Let us choose $a_0 \in \mathbb{R}$ such that $a_0 \gg \max\{|a_1|, \ldots, |a_k|\}$. One can construct an oriented piecewise linear path Γ which is the union of the segments $[a_{j-1}, a_j]$. Take $0 < \varepsilon \ll 1$ such that the closed disks D_i of radius ε centered at a_i, $i = 1, \ldots, k$, are pairwise disjoint and do not contain a_0. Let us orient $\alpha_i := \partial D_i$ counterclockwise. Let us denote by p_i^+ the first point of intersection of Γ and α_i. The remaining point of intersection of Γ and α_i will be denoted by p_i^-. The path Γ cuts α into two components α_i^+ and α_i^-, where α_i^+ starts at p_i^+. We also construct Γ_j, $j = 1, \ldots, k$ as follows:

- Γ_1 is the subpath in Γ from a_0 to p_1^+.
- If $j > 1$ then Γ_j is the subpath in Γ from p_{j-1}^- to p_j^+.

For $j = 1, \ldots, s$ let $\beta_j := \Gamma_1 \cdot \prod_{i=2}^{j} (\alpha_i^+ \cdot \Gamma_i)$, which is a path from a_0 to p_j^+. We define:

$$\gamma_j := \beta_j \cdot \alpha_j \cdot \beta_j^{-1}.$$

It is well known that the homotopy classes of $\gamma_1, \ldots, \gamma_k$ form a basis of the free group $\pi_1(\mathcal{U}, a_0)$.

Definition 1. An ordered family of lassos $\mu_1, \ldots, \mu_k$ is called a **standard basis** if these paths are homotopy equivalent to $\gamma_1, \ldots, \gamma_k$.

Definition 2. Let $a_1, \ldots, a_k$ be as above and let $b_1, \ldots, b_k$ be another set of k distinct points in $\mathbb{C}$. A **motion** from $a_1, \ldots, a_k$ to $b_1, \ldots, b_k$ is a set of paths $\delta_i : [0,1] \to \mathbb{C}$, $i = 1, \ldots, k$ such that:

- $\delta_i(0) = a_i$, $\delta_i(1) = b_{\rho(i)}$, $i = 1, \ldots, s$, where ρ is a permutation of $\{1, \ldots, s\}$.
- For all $t \in [0,1]$, the set $\{\delta_i(t)\}_{i=1}^{s}$ contains s distinct points.

If $a_i = b_i$, $i = 1, \ldots, k$, we say that the motion is based on $a_1, \ldots, a_k$.

Remark 2. Note that the order of the k points is not relevant.

Definition 3. Two motions $\delta_i^0 : [0,1] \to \mathbb{C}$ and $\delta_i^1 : [0,1] \to \mathbb{C}$, $i = 1, \ldots, s$ from on $a_1, \ldots, a_s$ to $b_1, \ldots, b_k$ are homotopic if there exists a set of homotopies $H_i : [0,1] \times [0,1] \to \mathbb{C}$, $i = 1, \ldots, k$, such that:

- $\delta_i^0 = H_i(-,0)$, $\delta_i^1 = H_i(-,1)$, $\{1, \ldots, k\}$, and
- if we define $\delta_i^s := H_i(-,s)$, then $\{\delta_i^s\}_{i=1}^{k}$ is also a motion from $a_1, \ldots, a_k$ to $b_1, \ldots, b_k$.

A braid from $a_1, \ldots, a_k$ to $b_1, \ldots, b_k$ (resp. a braid based on $a_1, \ldots, a_k$) is an equivalence class of motions by this relation. The set of braids based on $a_1, \ldots, a_k$ is denoted $B_{a_1,\ldots,a_k}$ and is a group which is of course isomorphic to the braid group B_k.

The canonical presentation of B_k is given by

$$|\sigma_1, \ldots, \sigma_{k-1} : [\sigma_i, \sigma_j] = 1 \text{ if } |i-j| \geq 2, \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, i = 1, \ldots, k-2| .$$

We can define a model for these generators in our group. Take $1 \leq i < k$. Consider the segment $[a_i, a_{i+1}]$ and take a small (topological) disk such that this segment is a diameter. Then:

- Move from a_i to a_{i+1} along one of the arcs of the boundary of this disk counterclockwise;
- move from a_{i+1} to a_i along the other arc;
- take constant paths for the other points.

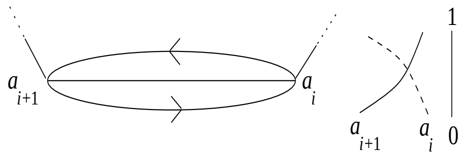

Fig. 3. Motion between a_i and a_{i+1}.

There are several equivalent definitions of the action of $B_{a_1,\ldots,a_k}$ on the free group $\pi_1(\mathcal{U}, a_0)$. In our case we have for $i = 1, \ldots, k-1$:

$$\begin{array}{rcll} \gamma_i^{\sigma_i} & = & \gamma_{i+1}, & \\ \gamma_{i+1}^{\sigma_i} & = & \gamma_{i+1}\gamma_i\gamma_{i+1}^{-1}, & \\ \gamma_j^{\sigma_i} & = & \gamma_j, & \text{if } j \neq i, i+1. \end{array}$$

Now let us recall what is meant by braid monodromy for affine curves. Let $\mathcal{C}_{\text{af}} := \{(x, y) \in \mathbb{C}^2 \mid f(x, y) = 0\}$, where $f(x, y) \in \mathbb{C}[x, y]$ is of the form:

$$f(x, y) = y^d + a_1(x)y^{d-1} + \ldots + a_{d-1}(x)y + a_d(x), \quad a_j \in \mathbb{C}[x],$$

and such that no vertical line is contained in $\mathcal{C}_{\text{af}}$.

Let $S := \{x_1, \ldots, x_r\}$, the set of zeros of $\Delta_y(x)$, which is the discriminant of f with respect to y. The lines $x = x_j$, $j = 1, \ldots, r$ are the vertical lines which are not transversal to $\mathcal{C}_{\text{af}}$, i.e., such that the number of points of intersection with $\mathcal{C}_{\text{af}}$ is less than d. Let $x_0 \in \mathbb{C} \setminus S$ such that $x_0 \in \mathbb{R}$ and $x_0 \gg \max\{|x_1|, \ldots, |x_r|\}$. Let $y_1, \ldots, y_d$ the d distinct roots of $f(x_0, y) = 0$. We will use the following notation:

- $\mathcal{F}$ for the vertical line $x = x_0$;

- $\check{\mathcal{F}} := \mathcal{F} \setminus \mathcal{C}_{\text{af}}$, i.e., $\check{\mathcal{F}} = \{(x_0, y) \in \mathbb{C}^2 \mid y \neq y_j, j = 1, \ldots, d\}$. In order to have a suitable base point we choose $y_0 \in \mathbb{R}$ such that $y_0 \gg \max\{|y_1|, \ldots, |y_d|\}$.
- B_d for the braid group based on $y_1, \ldots, y_d$.

Since outside S the curve defines a multi-valuated function, lassos based on x_0 define motions based on $y_1, \ldots, y_d$, which respect homotopy equivalences. This defines a homomorphism

$$\Phi : \pi_1(\mathbb{C} \setminus S) \to B_d,$$

which is called the **braid monodromy of the curve $\mathcal{C}_{\text{af}}$ with equation $f(x, y) = 0$** –cf. [11].
Let us recall the Zariski-Van Kampen theorem with this notation. Let us consider the group $F := \pi_1(\check{\mathcal{F}}, (x_0, y_0))$ which is a free group on d generators. Fix a standard basis $\mu_1, \ldots, \mu_d$ and consider the action of B_d. Then,

$$\pi_1(\mathbb{C}^2 \setminus \mathcal{C}_{\text{af}}; (x_0, y_0)) = |a \in \mathcal{F} : a = a^{\Phi(b)}, a \in \mathcal{F}, b \in \pi_1(\mathbb{C} \setminus S, x_0)|.$$

Taking a standard basis $\varphi_1, \ldots, \varphi_r$ of $\pi_1(\mathbb{C} \setminus S)$ one obtains a finite presentation of this group as follows:

$$\pi_1(\mathbb{C}^2 \setminus \mathcal{C}_{\text{af}}; (x_0, y_0)) = |\mu_1, \ldots, \mu_d : \mu_i = \mu_i^{\Phi(\varphi_j)}, i = 1, \ldots, d, \ j = 1, \ldots, r|.$$

Remark 3. Let us consider a path $\beta : [0, 1] \to \mathbb{C} \setminus S$ and let us denote R_t the set of roots of $f(\beta(t), y)$. This path defines a motion from R_0 to R_1. The conventions we have followed in the construction of standard basis allow us to construct well-defined braids from $y_1, \ldots, y_d$ to any subset of d distinct points of $\mathbb{C}$. Then joining $y_1, \ldots, y_d$ to R_0 and R_1 one can associate to every γ a braid in B_d. If a lasso γ is presented as a product of non-closed paths $\beta_1 \cdot \ldots \cdot \beta_l$, the braid associated to γ is the product of the braids associated to each β_i.
One can interpret motions of k points as sets of k non-intersecting curves in $\mathbb{C} \times [0, 1]$ such that each of these curves cut each horizontal plane $\mathbb{C} \times \{t\}$ in one point, $t \in [0, 1]$. In order to draw effectively this three dimensional picture, one chooses a projection $\mathbb{C} \to \mathbb{R}$ such that: (i) at most two points of the motion have the same intersection point and (ii) the set of such points is isolated; the *visible* part of $\mathbb{C}$ is the negative imaginary semiplane. We will use the following criteria in order to choose the projection:

- Generically the projection $\mathbb{C} \to \mathbb{R}$ is defined by $z \mapsto \Re z$. If two points have the same real part, we mark as visible the one with smaller imaginary part.
- Whenever this projection is not generic (for example if one has couples of conjugate non-real points), we deform slightly the projection as in figure , where vertical lines indicate complex points sent to same real point. That is, if we deform near the imaginary axis, positive (resp. negative) imaginary numbers are sent to positive (resp. negative) real numbers.

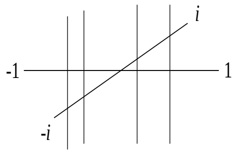

Fig. 4.

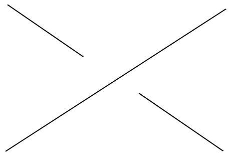

Fig. 5. Braid corresponding to $\sqrt{z}$ around the unit circle.

We end this section with some comments on the relationship of the fundamental group of affine and projective curves. Let $\mathcal{C} \subset \mathbb{P}^2$ be a reduced projective curve of degree d. Let G be the fundamental group $\pi_1(\mathbb{P}^2 \setminus \mathcal{C})$.
Let L_∞ be a line in $\mathbb{P}^2$ transversal to $\mathcal{C}$. Let us identify $\mathbb{C}^2$ with $\mathbb{P}^2 \setminus \mathrm{L}_\infty$ and set $\mathcal{C}_{\mathrm{af}} := \mathcal{C} \setminus \mathrm{L}_\infty$, which is an affine curve. Let us denote by H the fundamental group of the affine curve, that is, $H = \pi_1(\mathbb{C}^2 \setminus \mathcal{C}_{\mathrm{af}})$. Let $i_* : H \to G$ be the homomorphism induced by the inclusion. It is easily seen that this homomorphism is surjective. With the notation of this section, the kernel of this mapping is the cyclic infinite central subgroup K of H generated by $\mu_d \cdot \ldots \cdot \mu_1$, the meridian of the line at infinity. These facts are easy consequences of the Zariski-Van Kampen method.
If one drops the assumption of transversality on L_∞, the product $\mu_d \cdot \ldots \cdot \mu_1$ is no longer a meridian of L_∞. One can still describe a meridian of L_∞ as a word in $\mu_1, \ldots, \mu_d$. Note that such meridian is not necessarily central in H. An example of this situation will be carried out in §3. For a more general description see [12].
These groups are generated by the meridians of the curves. Let $\mu : G \to \mathbb{Z}/d\mathbb{Z}$ be the group homomorphism defined by sending each meridian to [1]. Let us consider the quotient $\pi : \mathbb{Z} \to \mathbb{Z}/d\mathbb{Z}$ and the pull-back diagram for μ and π:

$$\begin{array}{ccc} \tilde{H} & \xrightarrow{\tilde{\mu}} & \mathbb{Z} \\ \downarrow \tilde{\pi} & & \downarrow \pi \\ G & \xrightarrow{\mu} & \mathbb{Z}/d\mathbb{Z}. \end{array}$$

Proposition 2. There exists an isomorphism $\varphi : H \to \tilde{H}$ such that $\tilde{\pi} \circ \varphi = i_*$. In particular, the pair $(\mathbb{P}^2, \mathcal{C})$ determines H providing L_∞ is transversal to $\mathcal{C}$.
Proof. We know that

$$\tilde{H} : \{(\alpha, x) \in G \times \mathbb{Z} |\ \mu(\alpha) \equiv x \ \mathbf{mod}\ d\}.$$

In particular, it is the kernel of

$$\begin{array}{ccc} G \times \mathbb{Z} & \longrightarrow & \mathbb{Z}/d\mathbb{Z} \\ (\alpha, x) & \mapsto & \mu(\alpha) - [x]. \end{array}$$

The remarks above concerning Zariski-Van Kampen method and Reidemeister-Schreier method give the result. □

3. Computation of braid monodromy for $\mathcal{C}^{(1)}$ and $\pi_1(\mathbb{P}^2 \setminus \mathcal{C}^{(1)})$

Consider the equation of $\mathcal{C}^{(1)}_{\text{af}}$ given in Example 1, and ints real part as shown in figure . Its discriminant set with respect to y will be denoted by S_1. It is easily seen that $\mathcal{C}^{(1)}_{\text{af}}$ has only real non-transversal vertical lines and that $\#S_1 = 6$. We order these points $x_1 > \cdots > x_6$ and choose $x_0 > x_1$. In what follows we will use the path Γ and the disks D_i of the standard construction described in the previous sections.

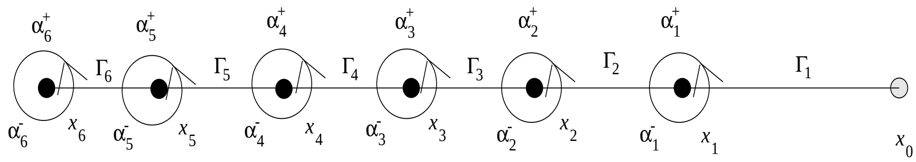

Fig. 6. Standard basis for $\mathbb{C}$ and S_1.

Using figure we list in a table the braids associated with each path. To obtain these braids we have followed the conventions presented in Example 1.
Let $\mathcal{F} := \{x = x_0\}$, where $x_0 \gg 0$, be a generic vertical line. Let also choose generators for the fundamental group of $\check{\mathcal{F}} := \mathcal{F} \setminus \mathcal{C}^{(1)}_{\text{af}}$. From right to left we call them a_1, a_2, a_3 and a_4. Note that a_1 and a_4 are meridians of $\mathcal{C}^{(1)}_{2,\text{af}}$ whereas a_2 and a_3 are meridians of $\mathcal{C}^{(1)}_{1,\text{af}}$. Also note that a_2 corresponds to a point with positive imaginary part.

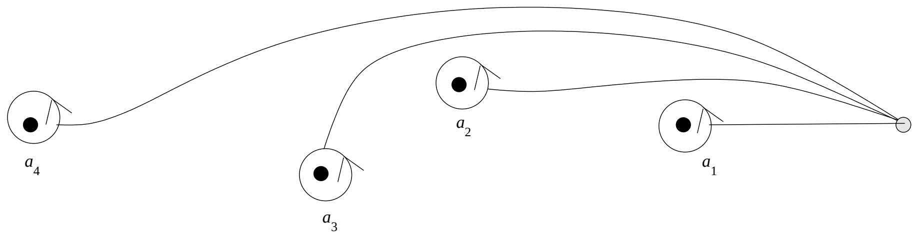

Fig. 7. Generic fiber with generators of the fundamental group homotopic to a standard basis.

The braids are:

Paths	Braids
Γ_1	$\sigma_3^{-1}\sigma_2$
α_1	σ_1
α_1^+	1
Γ_2	$\sigma_2^{-1}(\sigma_3^{-1}\sigma_1)^2\sigma_2$
α_2	σ_3
α_2^+	σ_3
Γ_3	$\sigma_2^{-1}\sigma_1$
α_3	σ_2^2
α_3^+	σ_2
Γ_4	$\sigma_1^{-1}\sigma_2$
α_4	σ_1
α_4^+	σ_1
Γ_5	1
α_5	σ_2^{16}
α_5^+	σ_2^8
Γ_6	1
α_6	σ_3

The braid monodromy is:

$$\begin{array}{llcc}
\Phi_1 : & \pi_1(\mathbb{C}^2 \setminus S_1) & \to & B_4 \\
& \gamma_1 & \mapsto & (\sigma_3^{-1}\sigma_2)\cdot\sigma_1 \\
& \gamma_2 & \mapsto & (\sigma_3^{-1}\sigma_2\sigma_2^{-1}\sigma_3^{-2}\sigma_1^2\sigma_2)\cdot\sigma_3 \\
& \gamma_3 & \mapsto & (\sigma_3^{-1}\sigma_2\sigma_2^{-1}\sigma_3^{-2}\sigma_1^2\sigma_2\sigma_3\sigma_2^{-1}\sigma_1)\cdot\sigma_2^2 \\
& \gamma_4 & \mapsto & (\sigma_3^{-1}\sigma_2\sigma_2^{-1}\sigma_3^{-2}\sigma_1^2\sigma_2\sigma_3\sigma_2^{-1}\sigma_1\sigma_2\sigma_1^{-1}\sigma_2)\cdot\sigma_1 \\
& \gamma_5 & \mapsto & (\sigma_3^{-1}\sigma_2\sigma_2^{-1}\sigma_3^{-2}\sigma_1^2\sigma_2\sigma_3\sigma_2^{-1}\sigma_1\sigma_2\sigma_1^{-1}\sigma_2\sigma_1)\cdot\sigma_2^{16} \\
& \gamma_6 & \mapsto & (\sigma_3^{-1}\sigma_2\sigma_2^{-1}\sigma_3^{-2}\sigma_1^2\sigma_2\sigma_3\sigma_2^{-1}\sigma_1\sigma_2\sigma_1^{-1}\sigma_2\sigma_1\sigma_2^8)\cdot\sigma_3
\end{array}$$

where $a \cdot b := aba^{-1}$.
This braid monodromy allows us to construct a presentation of the fundamental group of the complement of the affine curve. Our goal is to compute $\pi_1(\mathbb{P}^2 \setminus \mathcal{C}^{(1)})$. In order to do this it is enough to add any meridian of L_∞ as a relation.
In order to construct a meridian of the line at infinity, we blow up the projection point (the $\mathbb{A}_3$ singular point). The result is a ruled surface $\mathbb{F}_1$ with a (-1)-section E, where the strict transform M of the line at infinity is one of the fibers of the

ruling (the other ones being the compactified vertical lines). The real picture near M is shown in the figure below.

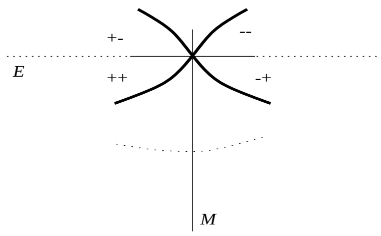

Fig. 8. Dotted lines are real parts of the two branches of the conic.

Take the boundary of a tubular neighborhood $\mathcal{T}$ of E in $\mathbb{F}_1$. It is an S^1-bundle $\pi : \mathcal{T} \to \mathbb{P}^1$ over $\mathbb{P}^1$ with Euler number equal to -1. This means the following: take disks $\Delta_1, \Delta_2 \subset \mathbb{P}^1$ which cover $\mathbb{P}^1$ and whose intersection is the common boundary and take a section $S : \Delta_1 \to \mathcal{T}$. Let $a \in \partial\Delta_1$ and $b := S(a)$. Let s be the path which runs counterclockwise $\partial S(\Delta_1)$ and is based at b. Let e be the oriented fiber over a based at b. Then, note that s and e are homotopic on the solid torus $\pi^{-1}(\Delta_2)$.
On the base $\mathbb{P}^1$, we choose a (complex) coordinate u centered at ∞ and on the fiber we choose a (complex) coordinate v centered at E. Note that we can do this preserving real coordinates. We show the four quadrants of the real parts in figure . We can suppose that near $M \cap E$ (which is the origin of this system) $\mathcal{T}$ is the closed polydisk of radius 2ε for some $\varepsilon > 0$. Fix the point $(\varepsilon, \varepsilon)$ as base point for the various fundamental groups. Note that Δ_1 can be chosen to be the disk of radius ε in the u-coordinate, and S the section $u \mapsto (u, \varepsilon)$. Preserving the notation introduced above, $s = e$ in the fundamental group of a solid torus. We note that this solid torus (contained in the original copy of $\mathbb{C}^2$) does not intersect the curve $\mathcal{C}^{(1)}_{\text{af}}$ if $\mathcal{T}$ is chosen small enough. Then, one has $s = e$ in $\mathbb{C}^2 \setminus \mathcal{C}^{(1)}_{\text{af}}$. Let us look at the relative position of the image of S, $\mathcal{C}^{(1)}_{\text{af}}$ and M. Figure shows the situation in the disk $v = \varepsilon$.

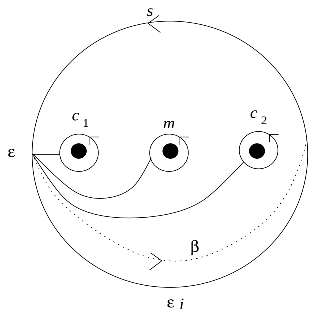

Fig. 9. Situation in S.

The path m is a meridian of M, and the paths c_1 and c_2 are meridians of $\mathcal{C}^{(1)}$. In

$\mathbb{C}^2 \setminus \mathcal{C}^{(1)}_{\text{af}}$ one has the equality $e = s = c_2 m c_1$. Note that c_1 is homotopic in $\mathbb{C}^2 \setminus \mathcal{C}^{(1)}_{\text{af}}$ to a path in the line $\mathcal{F}_\varepsilon$ of equation $x = \varepsilon^{-1}$. We can suppose that $\mathcal{F}_\varepsilon = \mathcal{F}$, i.e., $\varepsilon x_0 = 1$. It is easily seen that $c_1 = a_1$, where $a_1, \dots, a_4$ is the standard basis in $\mathcal{F}$ – see figure .
Let us choose a standard basis on the line $\mathcal{F}_{-\varepsilon}$ of equation $x = -\varepsilon^{-1}$. Let β be the dotted path in figure . Let us take a standard basis in $\mathcal{F}_{-\varepsilon}$ and denote by $a'_1, \dots, a'_4$ the paths obtained by applying the mapping $* \mapsto \beta \cdot * \cdot \beta^{-1}$ to this basis. It is easily seen that c_2 is homotopic to a'_1 in $\mathbb{C}^2 \setminus \mathcal{C}^{(1)}_{\text{af}}$.
There is a braid relating these two basis in $\mathbb{C}^2 \setminus \mathcal{C}^{(1)}_{\text{af}}$:

$$a'_i = (a_i)^{\sigma_1\sigma_2\sigma_3\sigma_2\sigma_1}, \quad i = 1, \dots, 4.$$

We deduce that $a'_1 = a_4$. The properties of standard basis imply that already in the fiber $\mathcal{F} = \mathcal{F}_\varepsilon$ we have $a_4 a_3 a_2 a_1 e = 1$. Then:

$$m = (a_1 a_4 a_3 a_2 a_1 a_4)^{-1}.$$

Therefore a presentation for $\pi_1(\mathbb{P}^2 \setminus \mathcal{C}^{(1)})$ may be given as follows:

- Generators: a_1, a_2, a_3, a_4
- One class of relators of the type $a_i = a_i^{\Phi(\gamma_j)}$, $i = 1, \dots, 4$, $j = 1, \dots, 6$.
- The relator $a_1 a_4 a_3 a_2 a_1 a_4 = 1$.

In order to simplify this presentation, we have used the free software GAP (version 4, release 1, fix 6). In the appendix, we provide the GAP program we applied to get the result:
Theorem 1. The fundamental group G_1 of the complement of $\mathcal{C}^{(1)}$ in $\mathbb{P}^2$ has a presentation

$$|a, b : a^2(ab)^2 = 1, [a, b^2] = 1|,$$

where a is a meridian of the quartic and b is a meridian of the conic. The abelianization exact sequence is a central extension of $\mathbb{Z}/2\mathbb{Z}$ by $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Choosing ω the non-trivial element of $\mathbb{Z}/2\mathbb{Z}$ and $p, q \in G_1$ such that their images in $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ generate this group, the image of q being of order 2, then $q^2 = pqp^{-1}q^{-1} = \omega$.

4. Computation of braid monodromy for $\mathcal{C}^{(2)}$ and $\pi_1(\mathbb{P}^2 \setminus \mathcal{C}^{(2)})$

The procedure and notations will be according to the previous section. Let S_2 be the discriminant set in this case. Note – figure – that S_2 consists of four points. Let us choose the standard fiber $\mathcal{F} := \{x = x_0\}$ for $x_0 \gg 0$ and denote the standard generators again by a_1, a_2, a_3 and a_4. Both a_1 and a_4 are meridians of $\mathcal{C}^{(2)}_{2,\text{af}}$, whereas a_2 and a_3 are meridians of $\mathcal{C}^{(2)}_{1,\text{af}}$. Note that a_2 corresponds to a point with positive imaginary part.

We use again a standard notation for the paths in the base:

Paths	Braids
Γ_1	$\sigma_1^{-1}\sigma_3^{-1}$
α_1	σ_2
α_1^+	1
Γ_2	1
α_2	$(\sigma_1\sigma_3)^7(\sigma_2\sigma_1\sigma_3\sigma_2)$
α_2^+	$(\sigma_1\sigma_3)^4(\sigma_2\sigma_1\sigma_3\sigma_2)$
Γ_3	1
α_3	σ_2
α_3^+	1
Γ_4	$\sigma_1^{-1}\sigma_3^{-1}$
α_4	σ_2^2

The braid monodromy is:

$$\begin{array}{rccc} \Phi_1: & \pi_1(\mathbb{C}^2\setminus S_2) & \to & B_4 \\ & \gamma_1 & \mapsto & (\sigma_1^{-1}\sigma_3^{-1})\cdot\sigma_2 \\ & \gamma_2 & \mapsto & (\sigma_1^{-1}\sigma_3^{-1})\cdot((\sigma_1\sigma_3)^7(\sigma_2\sigma_1\sigma_3\sigma_2)) \\ & \gamma_3 & \mapsto & ((\sigma_1\sigma_3)^3(\sigma_2\sigma_1\sigma_3\sigma_2))\cdot\sigma_2 \\ & \gamma_4 & \mapsto & ((\sigma_1\sigma_3)^2(\sigma_2\sigma_1\sigma_3\sigma_2))\cdot\sigma_2^2 \end{array}$$

The construction of a meridian m of the line at infinity is exactly as in the previous section and we obtain

$$m = (a_1a_4a_3a_2a_1a_4)^{-1}.$$

Since we have chosen as base fiber $\mathcal{F}$ the line of equation $x = x_0$, we must translate this element to the standard basis $a_1, \dots, a_4$ in $\mathcal{F}$. Since

$$a_i' = a_i^{(\tau^{-1})},$$

we deduce that

$$m = (a_1a_4a_3a_2a_1a_3a_2a_3^{-1})^{-1}.$$

We construct a presentation of $\pi_1(\mathbb{P}^2\setminus\mathcal{C}^{(2)})$ and we simplify it with GAP:

Theorem 2. The fundamental group G_2 of the complement of $\mathcal{C}^{(2)}$ in $\mathbb{P}^2$ has a presentation

$$|a, b : b^2 = (ab)^4|,$$

where a is a meridian of the quartic and b is a meridian of the conic. The abelianization exact sequence is an extension of $\mathbb{F}_3$ (the free group in three generators r, s, t)

by $\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$. One can choose $p, q \in G_2$ such that their images in $\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ generate this group, the image of q being of order 2, and:

- $q^2 = r$, $pqp^{-1}q^{-1} = t$.
- $prp^{-1} = ts$, $psp^{-1} = t^{-1}$, $ptp^{-1} = tr^{-1}$.
- $qrq^{-1} = r$, $qsq^{-1} = tr^{-1}$, $qtq^{-1} = rs$.

In particular, the groups G_1 and G_2 are not isomorphic. This provides a proof of the Main Theorem 1.

5. From special to generic braid monodromy

In the previous sections we have computed some braid monodromies where the projection point is in the projective curve. In fact the projection point is a tacnode of the curve and the line at infinity is tangent to this tacnode. We will apply this computation to find a generic braid monodromy in both cases. The idea is to slide the projection point somewhere close to the former one but not on the curve. In the first part, we study the situation near the infinity; in the second part we concentrate the curves near the transversal lines to the primitive projection.

Step 1 We keep the affine curve, but we move the projection point along the line at infinity.

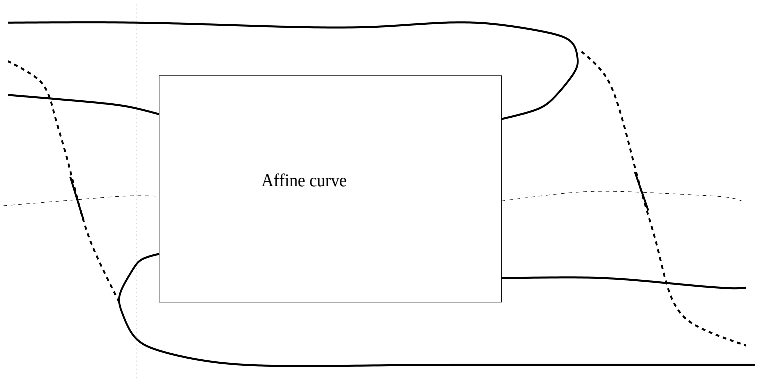

Fig. 10.

As in the introduction, the dotted lines represent the real part of the complex conjugated roots. When two couples of roots have the same real part the non-dotted line represents the one having biggest absolute value for the imaginary part.

Step 2 We take the dotted vertical line as the new line at infinity and preserve the projection.

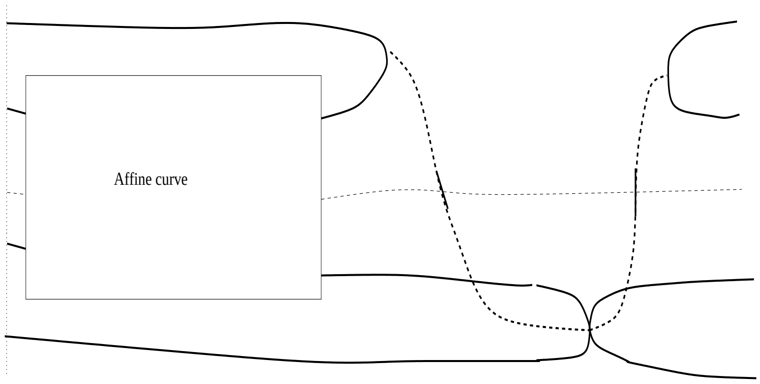

Fig. 11.

Step 3 Finally we move the projection in order to have a generic projection at the tacnode.

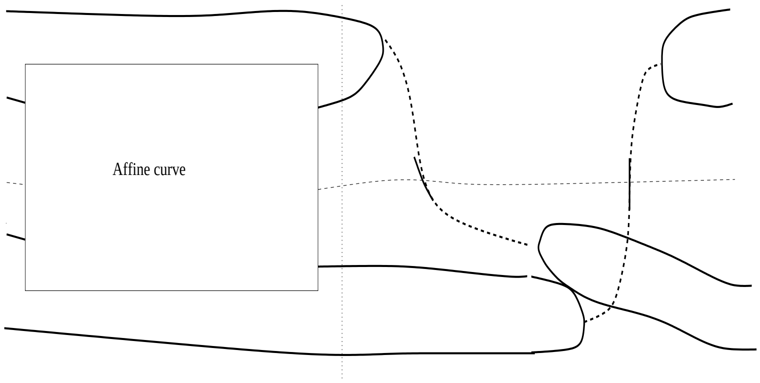

Fig. 12.

We fix the dotted line as the generic fiber and compute the braid monodromy for the part which is to the right of the generic fiber. The paths are decomposed as in the previous sections:

Paths	Braids
$\tilde{\Gamma}_1$	1
$\tilde{\alpha}_1$	σ_1
$\tilde{\alpha}_1^+$	1
$\tilde{\Gamma}_2$	$\sigma_2^{-1}(\sigma_3^{-1}\sigma_1)\sigma_2$
$\tilde{\alpha}_2$	σ_3
$\tilde{\alpha}_2^+$	σ_3
$\tilde{\Gamma}_3$	1
$\tilde{\alpha}_3$	σ_4^4
$\tilde{\alpha}_3^+$	σ_4^2
$\tilde{\Gamma}_4$	1
$\tilde{\alpha}_4$	σ_5
$\tilde{\alpha}_4^+$	1
$\tilde{\Gamma}_5$	$\sigma_4^{-1}\sigma_5\sigma_3^{-1}\sigma_4\sigma_2^{-1}\sigma_3\sigma_1^{-1}\sigma_2$
$\tilde{\alpha}_5$	σ_1

This part is common for both curves. Note that the projection of $\mathcal{C}_{\mathrm{af}}^{(1)}$, coincides with the non-generic case. One can recover the generic monodromy just by shifting $\sigma_i \mapsto \sigma_{i+1}$, for $i = 1, 2, 3$, since a new thread has appeared in the upper part.
For the curve $\mathcal{C}_{\mathrm{af}}^{(2)}$, we may have the same braid monodromy near the infinity and outside the point $P_{15}^{(2)}$. For this point, the real deformation looks as follows.

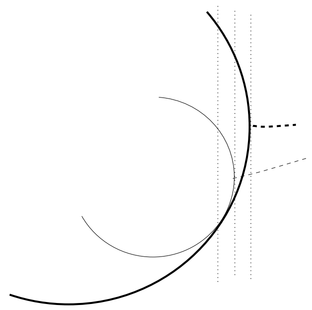

Fig. 13.

We recall that in the special monodromy, near the point $P_{15}^{(2)}$ all solutions were imaginary complex conjugated. Since the projection onto the real axis was not generic, we deformed this projection in order such that the order of the points was $((P')^+, P_4^+, P_4^-, (P')^-)$, where sub-index indicates de degree of the curve and the super-index indicates the sign of the imaginary part. Let us consider the line in the special monodromy where the braid Γ_2 starts. Take a line in this generic monodromy close to that line in order to start a braid Γ_{2a}. This braid will be very close to the braid Γ_2; then, one can only have interchanging of positive or negative points and finally a crossing to be in the situation of figure . Since there are only conjugated interchanges of positive and negative points, one has the following:

Paths	Braids
Γ_{2a}	$\sigma_1^{-2k}\sigma_3^{2k}\sigma_1\sigma_2$
α_{2a}	σ_1
α_{2a}^+	σ_1
Γ_{2b}	$\sigma_2^{-1}\sigma_3$
α_{2b}	σ_2
α_{2b}^+	σ_2
Γ_{2c}	1
α_{2c}	σ_3^{16}
α_{2c}^+	σ_3^8

Finally, since we performed a deformation, the following equality must be satisfied in the braid group:

$$\alpha_2^+ = \Gamma_{2a}\alpha_{2a}^+\Gamma_{2b}\alpha_{2b}^+\Gamma_{2c}\alpha_{2c}^+.$$

A straightforward computation gives:

$$\Gamma_{2a}\alpha_{2a}^+\Gamma_{2b}\alpha_{2b}^+\Gamma_{2c}\alpha_{2c}^+ = \sigma_1^{8-2k}\sigma_3^{2k}\sigma_2\sigma_1\sigma_3\sigma_2.$$

Then $k = 2$ is the only possible solution. Replacing $\Gamma_2, \alpha_2, \alpha_2^+$ by these braids and applying the shifting $\sigma_i \mapsto \sigma_{i+1}$ one obtains the generic braid monodromy for the affine part.

Note that, using the generic braid monodromy and Zariski-van Kampen method, one would be able to obtain a presentation of G_i in a more direct way, avoiding the calculations at the end of section 3.

6. Characteristic varieties

Notations and definitions in this section will mainly follow [9] and [13]. We first want to recall what the characteristic varieties of a plane curve are. Let $\mathcal{C} = \mathcal{C}_1 \cup \ldots \cup \mathcal{C}_r$ be an algebraic curve in $\mathbb{P}^2$ and consider L_∞ a transversal line to $\mathcal{C}$. It is well known that $X = \mathbb{P}^2 \setminus (\mathcal{C} \cup \mathrm{L}_\infty)$ has the homotopy type of a finite 2-dimensional CW-complex and $H_1(X) = \mathbb{Z}^r$. We will denote by $\tilde{X}$ the universal abelian cover of X. This has again the homotopy type of a 2-dimensional CW-complex, but no longer finite. In order to study its homology, we can use the action of $H_1(X)$ over $\tilde{X}$, which makes any $H_k(\tilde{X})$ into a $\mathbb{Z}[H_1(X)]$-module. Tensoring by $\mathbb{C}$ we are led to study the structure of the modules $H_k(\tilde{X}, \mathbb{C})$ over the noetherian ring $\Lambda = \mathbb{C}[\mathbb{Z}^r]$. We define the k-th characteristic variety of $\mathcal{C}$ as follows

$$\mathrm{Char}_k(\mathcal{C}) := \mathrm{Supp}_\Lambda(\wedge^k H_1(\tilde{X}, \mathbb{C})) = \mathrm{Supp}_\Lambda(\Lambda/F_k) \subset \mathrm{Spec}\Lambda = (\mathbb{C}^*)^r,$$

where F_k is the k-th Fitting ideal of the module $H_1(\tilde{X}, \mathbb{C})$. In other words, $\mathrm{Char}_k(\mathcal{C})$ is a subvariety of the r-dimensional torus given by the zeroes of the k-th Fitting ideal of $H_1(\tilde{X}, \mathbb{C})$. Observe that $H_1(\tilde{X}) = \pi_1(X)'/\pi_1(X)''$, and it is easy to check that the aforementioned action of $H_1(X) = \pi_1(X)/\pi_1(X)'$ on $H_1(\tilde{X})$ corresponds to conjugation. As a result of this remark we have that the varieties $\mathrm{Char}_k(\mathcal{C})$ are invariants of the fundamental group $\pi_1(X)$. In the case where $r = 1$, the first characteristic varieties are nothing but the zeroes of the Alexander polynomial. In the non-irreducible case, the Fitting ideals F_k are no longer principal and as a result $\mathrm{Char}_k(\mathcal{C})$ will in general not be a hypersurface.

Characteristic varieties can also be thought of in connection with homology with coefficients in rank one local systems of the fundamental group $\pi_1(X)$. This connection makes it possible to calculate the characteristic varieties of a curve $\mathcal{C}$ using the theory of adjoint and quasiadjoint ideals.

Our purpose is to calculate $\mathrm{Char}_k(\mathcal{C}^{(1)})$ and $\mathrm{Char}_k(\mathcal{C}^{(2)})$. They will turn out to be different sets of points. Since the characteristic varieties are invariants of the fundamental groups and by 2 the Main Theorem 1 will be proved.

One possible way to calculate the characteristic varieties is to compute the Fitting ideals F_k by means of Fox calculus. For this purpose one needs a finite presentation of the fundamental group. We will denote the Alexander modules of $\mathcal{C}^{(1)}$ and $\mathcal{C}^{(2)}$ by $M^{(1)}$ and $M^{(2)}$ respectively. We will also consider the 2-dimensional torus $\mathrm{Spec}\Lambda = (\mathbb{C}^*)^2$, where the first coordinate will represent a meridian around the quartic and the second coordinate will represent a meridian around the conic. Applying Fox calculus to the presentations given in Theorems 1 and 2 one can compute that

$$F_1(M^{(1)}) = (t_1 - 1, t_2 - 1)$$

$$F_1(M^{(2)}) = (t_1 - 1, t_2 - 1) \cdot ((t_2 t_1 + 1)(t_2^2 t_1^2 + 1), (t_2 + t_2^2 t_1 + t_2^3 t_1^2 - 1)).$$

Hence

$$\mathrm{Char}_1(\mathcal{C}^{(1)}) = \{(1,1)\}$$

$$\mathrm{Char}_1(\mathcal{C}^{(2)}) = \{(1,1), (-1,1), (\sqrt{-1}, -1), (-\sqrt{-1}, -1)\}.$$

Another way to calculate characteristic varieties is by computing the cohomology of the universal abelian covers $\tilde{X}^{(1)}$ and $\tilde{X}^{(2)}$ with coefficients in rank one local systems as mentioned before. These computations involve both local and global analyses and do not need a presentation of the fundamental group:

1. Local calculations

   (a) Calculate the ideals of quasiadjunction for each singular point, the local polytopes of quasiadjunction and their faces.

   (b) Put all the local data together in the global polytope of quasiadjunction.

   (c) Determine the faces of the global polytope of quasiadjunction that are contributing faces. Each contributing face has associated with it an integer number $l(\delta)$ called the level of δ.

2. Global calculations

   (a) Each contributing face δ has an ideal sheaf $\mathcal{A}_\delta$ over $\mathbb{P}^2$ associated with it. The contributing face will determine a subvariety of the k-th characteristic variety if and only if the irregularity of $\mathcal{A}_\delta(d-3-l(\delta))$ is greater or equal to k, where d is the degree of the curve $\mathcal{C}$ and $l(\delta)$ is the level of the contributing face δ.

   (b) Let δ be a contributing face from the previous step for which the irregularity is positive, that is $\dim H^1(\mathcal{A}_\delta(d-3-l(\delta))) = k > 0$. Let $L_s(x_1, ..., x_r) = \beta_s$ be a system of equations defining $\delta \cup \overline{\delta}$ where

   $$\overline{\delta} = \{\overline{1} - \overline{x} \mid \overline{x} \in \delta\}.$$

   Suppose that each $L_s(x_1, ..., x_r)$ is a linear form with integer coefficients and that the g.c.d. of the maximal order minors in the matrix of coefficients is one. Then

   $$\exp(2\pi\sqrt{-1}L_s) = \exp(2\pi\sqrt{-1}\beta_s)$$

   is the equation of a component of $\operatorname{Char}_k(\mathcal{C})$, where the coordinates in the r-dimensional torus $\operatorname{Spec}\Lambda$ are given by $t_i = \exp(2\pi\sqrt{-1}x_i)$. Moreover, all non-trivial components of $\operatorname{Char}_k(\mathcal{C})$ can be obtained in this way.

In the following, we will work out each step for our curves $\mathcal{C}^{(1)}$ and $\mathcal{C}^{(2)}$.

1. The local computations can be carried out for both curves together, since they both have the same local configuration of singularities.

   (a) The ideals of quasiadjunction of a singular point $\mathbb{A}_{15}$ given by two different components are

   $$m_{1,7} \subset m_{1,6} \subset ... \subset m_{1,2} \subset m_{1,1} = m \subset \mathcal{O},$$

   where $m_{1,i} = (y, x^i)$ $(i = 1, ..., 7)$, y denotes the tangent line at the singular point and x is independent to y. The local face of quasiadjunction of $m_{1,i}$ is given by the equation

   $$8x_1 + 8x_2 = 7 - i$$

and hence the local polytope of quasiadjunction is

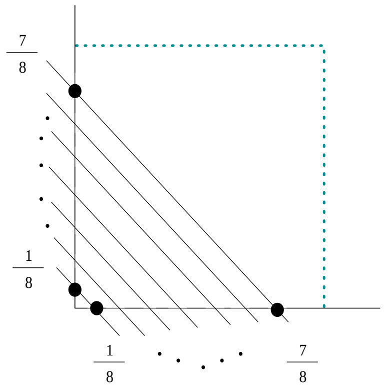

Fig. 14.

The only proper ideal of quasiadjunction for a tacnode whose local branches belong to the same global irreducible component is the maximal ideal. There is only one face of quasiadjunction $x_1 = 1/4$ and it is associated with the maximal ideal.

Nodes have no ideals of quasiadjunction attached to them.

(b) The global polytope of quasiadjunction looks as follows,

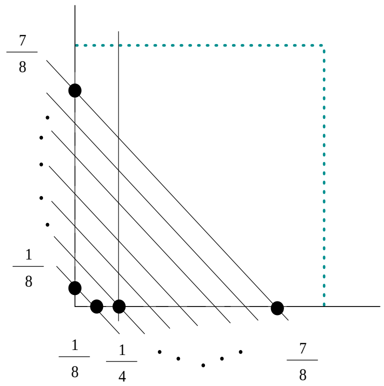

Fig. 15.

(c) The contributing faces are the points in the intersection of faces in the above picture that belong to hyperplanes of type

$$4x_1 + 2x_2 = l \in \mathbb{N}.$$

Hence the only contributing face is $\delta = \{(\frac{1}{4}, \frac{1}{2})\}$ and its level $l(\delta) = 2$.

2. Global calculations do depend on the curve. We will distinguish each case by superindices.

   (a) The ideal sheaf associated with δ can be defined as follows

   $$(\mathcal{A}_\delta^{(i)})_P = \begin{cases} m_{1,2} & \text{if } P \text{ is the singularity } \mathbb{A}_{15} \text{ in } \mathcal{C}^{(i)} \\ m & \text{if } P \text{ is the singularity } \mathbb{A}_3 \text{ in } \mathcal{C}^{(i)} \\ \mathcal{O}_{\mathbb{P}^2,P} & \text{otherwise} \end{cases}$$

   Let us calculate the irregularity of both ideal sheaves.

   $$\dim H^1(\mathcal{A}_\delta^{(1)}(6-3-2)) = \dim H^0(\mathcal{A}_\delta^{(1)}(1)) - \chi(\mathcal{A}_\delta^{(1)}(1)) = 0$$

   since

   $$\chi(\mathcal{A}_\delta^{(1)}(1)) = \chi(\mathcal{O}_{\mathbb{P}^2}(1)) - \dim(m_{1,2} \oplus m) = 3 - 3 = 0.$$

   Analogously

   $$\dim H^1(\mathcal{A}_\delta^{(2)}(6-3-2)) = \dim H^0(\mathcal{A}_\delta^{(2)}(1)) - \chi(\mathcal{A}_\delta^{(2)}(1)) = 1.$$

   (b) While $\mathrm{Char}_k(\mathcal{C}^{(1)})$ has only trivial components, $\mathrm{Char}_1(\mathcal{C}^{(2)})$ has a non-trivial component coming from the face δ. The defining equations of δ are

   $$L_1(x_1, x_2) = 2x_2 = 1 = \beta_1$$

   and

   $$L_2(x_1, x_2) = 2x_1 + x_2 = 1 = \beta_2.$$

   Hence,

   $$\mathrm{Char}_1(\mathcal{C}^{(2)}) = \{t_2^2 = 1, t_1^2 t_2 = 1\} = \\ = \{(1,1), (-1,1), (\sqrt{-1}, -1), (-\sqrt{-1}, -1)\}$$

These computations provide another proof for 1.

7. Dihedral covers

Let Y be a smooth projective variety and let X be a normal variety with finite morphism $\pi : X \to Y$. We denote the rational function fields of X and Y by $\mathbb{C}(X)$ and $\mathbb{C}(Y)$ respectively. The field $\mathbb{C}(X)$ is a finite extension of $\mathbb{C}(Y)$ with $[\mathbb{C}(X) : \mathbb{C}(Y)] = \deg \pi$. Let us denote by $\Delta(X/Y)$ the branch locus of the cover. We call X a $\mathcal{D}_{2n}$ cover if

1. $\mathbb{C}(X)$ is a Galois extension of $\mathbb{C}(Y)$ and
2. its Galois group is the dihedral group $\mathcal{D}_{2n} = \langle \sigma, \tau | \ \sigma^2 = \tau^n = (\sigma\tau)^2 = 1 \rangle$.

Note that, given a $\mathcal{D}_{2n}$ cover X, a double cover of Y is canonically determined by considering the τ-invariant field of $\mathbb{C}(X)$. We denote this double cover by $D(X/Y)$. In [15], the last author studied such covers, and proved the following result for $\mathcal{D}_{2n}$ (n even) covers.

Proposition 3. Let Y be a simply connected smooth projective variety and let $f : Z \to Y$ be a smooth double cover, and σ denotes the covering transformation. Suppose that there exist effective divisors, D_1 and D_2 on Z satisfying the following conditions:

1. D_1 and $\sigma^* D_1$ have no common components. Moreover, if we let $D_1 = \sum_i a_i D_{i,1}$ denote the irreducible decomposition of D_1, then the greatest common divisor of a_i's and n is one.
2. If D_2 is not empty, D_2 is a reduced divisor and there exists a divisor, B_2, on Y such that $D_2 = f^* B_2$.
3. There exists a line bundle $\mathcal{L}$ on Z such that $D_1 + \frac{n}{2} D_2 - \sigma^* D_1 \sim n\,\mathcal{L}$ (n even).

Then there exists a $\mathcal{D}_{2n}$ cover, X, of Y such that

(a) $D(X/Y) = Z$ and

(b) $\Delta(X/Z) \subset \mathrm{Supp}(D_1 + \sigma^* D_1 + D_2)$.

Proof. By Remark 3.1, in [15] and a similar argument to the proof of Proposition 1.1 in [16], Proposition 3 follows. □

The converse of Proposition 3 also holds in the following sense.

Proposition 4. Let $\pi : X \to Y$ be a $\mathcal{D}_{2n}$ cover such that $D(X/Y)$ is smooth. Then there exist (possibly empty) effective divisors, D_1 and D_2, and a line bundle $\mathcal{L}$ on $D(X/Y)$ satisfying the following conditions:

(i) If $D_1 \neq \emptyset$, D_1 and $\sigma^* D_1$ have no common components.

(ii) If D_2 is not empty, D_2 is a reduced divisor and there exists a divisor, B_2, on Y such that $D_2 = f^* B_2$.

(iii) $D_1 + \frac{n}{2} D_2 - \sigma^* D_1 \sim n\,\mathcal{L}$.

(iv) $\Delta(X/D(X/Y)) = \mathrm{Supp}(D_1 + \sigma^* D_1 + D_2)$.

See [15] for a proof.

7.1. $\mathcal{D}_{2n}$ *(n even) covers of* $\mathbb{P}^2$

In this subsection, we confine ourselves to studying a $\mathcal{D}_{2n}$ (n: even) cover, S, of $\mathbb{P}^2$ satisfying the following properties:

($*$) $\Delta(S/\mathbb{P}^2)$ consists of a reduced curve, B, of even degree and a line, L_∞, such that

(i) $\Delta(D(S/\mathbb{P}^2)) = B$,

(ii) B has at most simple singularities, and

(iii) B meets L_∞ transversally.

Let $f' : Z' \to \mathbb{P}^2$ be the double cover branched at B, where f' is finite and Z' is normal. Let $\mu\, Z \to Z'$ be

$$\begin{array}{ccc} Z' & \overset{\mu}{\leftarrow} & Z \\ f' \downarrow & & \downarrow \\ \mathbb{P}^2 & \overset{q}{\leftarrow} & \Sigma \end{array}$$

the canonical resolution, where $q : \Sigma \to \mathbb{P}^2$ is a sequence of blowing-ups so that the induced morphism $Z \to \Sigma$ gives a smooth finite double cover. We denote $f' \circ \mu$ by f. Let $\mathrm{NS}(Z)$ be the Néron-Severi group of Z. Let T be the subgroup of $\mathrm{NS}(Z)$ generated by $f^*\mathrm{L}_\infty$ and all the irreducible components of the exceptional divisor of $\mu : Z \to Z'$.

Under the assumption ($*$), Z is simply connected. Hence $\mathrm{NS}(Z) = \mathrm{Pic}(Z)$ and $\mathrm{NS}(Z)$ is a lattice with respect to the intersection pairing. Also T is regarded as a sublattice of $\mathrm{NS}(Z)$ and has a orthogonal decomposition

$$T = \mathbb{Z} f^*\mathrm{L}_\infty \bigoplus \bigoplus_{x \in \mathrm{Sing}(Z')} \left(\bigoplus_i \mathbb{Z}\Theta_{i,x} \right),$$

where $\Theta_{i,x}$ denotes an irreducible component of the exceptional divisor arising from x. We recall that $\mathrm{Sing}(Z')$ is in bijection with $\mathrm{Sing}(B)$.

Under this notation, we have

Proposition 5. If there exits a $\mathcal{D}_{2n}$ cover, S, of $\mathbb{P}^2$ such that

(i) $\Delta(S/\mathbb{P}^2) = B + \mathrm{L}_\infty$ and

(ii) $D(S/\mathbb{P}^2) = Z'$.

Then $\mathrm{NS}(Z)/T$ has n-torsion.

Proof. Let $\tilde{S}$ be the $\mathbb{C}(S)$-normalization of Σ and let $\tilde{\pi} : \tilde{S} \to \Sigma$ be the induced morphism. Then $\tilde{\pi} : \tilde{S} \to \Sigma$ satisfies:

(i) $\tilde{\pi} : \tilde{S} \to \Sigma$ is a $\mathcal{D}_{2n}$ cover of Σ,

(ii) $D(\tilde{S}/\Sigma) = Z$, and

(iii) the branch locus of $\tilde{S} \to Z$ is contained in $f^*\mathrm{L}_\infty$ and the exceptional set of μ.

Let D_1 and D_2 be the effective divisors on Z as in Proposition 4. Then by the proof of Proposition 0.7, [15], there exists a rational function φ such that

(a) $(\varphi) = D_1 + \frac{n}{2}D_2 - \sigma^* D_1 + nD_0$, where D_0 is some divisor on Z, and

(b) the polynomial $X^n - \varphi$ is irreducible in $\mathbb{C}(Z)[X]$, i.e. $\mathbb{C}(S) = \mathbb{C}(Z)(\sqrt[n]{\varphi})$.

Since $\mathrm{Supp}(D_1 + \sigma^* D_1 + D_2) = \Delta(S/Z)$ and (iii), the equality in (a) shows that D_0 gives rise to an element, α, of $\mathrm{NS}(S)/T$ whose order divides n. Our assertion follows from the following claim:

Claim 1. The order of α is n.

Proof of Claim. Suppose that the order of α is d and put $d_1 = n/d$. Hence $dD_0 \in T$ and therefore $dD_0 \sim af^*\mathrm{L}_\infty + \sum_x \sum_i b_{i,x}\Theta_{i,x}$. As we may assume that

$$\begin{array}{rcl} D_1 & = & \sum_{i,x} a_{i,x}\Theta_{i,x}, \quad 0 \le a_{i,x} < n, \\ D_2 & = & mf^*\mathrm{L}_\infty + \sum_{i,x} m_{i,x}\Theta_{i,x}, \quad m, m_{i,x} \text{ are either 0 or 1.}, \end{array}$$

we have

$$D_1 + \frac{n}{2}D_2 - \sigma^* D_1 + nD_0 = \sum_{i,x} a_{i,x}\Theta_{i,x} + \frac{n}{2}(mf^*\mathrm{L}_\infty + \sum_{i,x} m_{i,x}\Theta_{i,x}) -$$

$$- \sum_{i,x} a_{i,x}\sigma^*\Theta_{i,x} + d_1 af^*\mathrm{L}_\infty + d_1 \sum_{i,x} b_{i,x}\Theta_{i,x} \sim 0.$$

Since D_1, $\sigma^* D_1$ and D_2 have no common components, the above relation is non-trivial, unless all the coefficients of $f^*\mathrm{L}_\infty$ and $\Theta_{i,x}$'s are 0. This implies that d_1 divides $n/2$ and all $a_{i,x}$'s, and we can write $D_1 = d_1 D_1'$ for some effective divisor D_1' on Z. Hence we have

$$d_1(D_1' + \frac{n}{2d_1}D_2 - \sigma^* D_1' + \frac{n}{d_1}D_0) \sim 0.$$

As Z is simply connected, this is reduced to

$$D_1' + \frac{n}{2d_1}D_2 - \sigma^* D_1' - \frac{n}{d_1}D_0 \sim 0.$$

Let ψ be a rational function on Z such that

$$(\psi) = D_1' + \frac{n}{2d_1}D_2 - \sigma^* D_1' + \frac{n}{d_1}D_0.$$

Thus φ/ψ^{d_1} is a constant. Hence, multiplying by a suitable constant, we have $\varphi = \psi^{d_1}$. But this contradicts the condition (b) about φ. □

This finishes the proof of Proposition 5. □

7.2. *Non-existence of a certain $\mathcal{D}_{16}$ cover of $\mathbb{P}^2$ branched at $\mathcal{C}^{(1)} + \mathrm{L}_\infty$*

We keep the notation from §1. Let $\mathcal{C}^{(1)}$ be the sextic from Example 1. In this section, we disprove the existence of a $\mathcal{D}_{16}$ cover, S, of $\mathbb{P}^2$ such that (*i*) $\Delta(S/\mathbb{P}^2) = \mathcal{C}^{(1)} + \mathrm{L}_\infty$ and

(*ii*) $D(S/\mathbb{P}^2) = Z'$.

Suppose that such a cover exists. Then $\mathrm{NS}(Z)/T$ has eight torsion and $\mathrm{NS}(Z)/T$ is a torsion group. Denoting the absolute value of the determinant of the intersection matrix of $\mathrm{NS}(Z)$ (resp. T) by $\mathrm{disc}(\mathrm{NS}(Z))$ (resp. $\mathrm{disc}(T)$), we have

$$\mathrm{disc}(\mathrm{NS}(Z)) \leq \frac{\mathrm{disc}(T)}{64} = 4.$$

On the other hand, we have the following lemma which leads us to a contradiction; and no $\mathcal{D}_{16}$ cover as above exists.

Lemma 1. $\mathrm{disc}(\mathrm{NS}(Z)) = 16$.

Proof. Let X_{8211} be the rational elliptic surface in 1, and let $\hat{X}_{8211} \to X_{8211}$ be the morphism resulting from blowing-up the two nodes of the two I_1-fibers. By what we have seen in subsection , Σ coincides with $\hat{X}_{8211}$, and Z is a double cover branched at the proper transforms of I_1-fibers. Hence there exists an elliptic fibration on $\varphi : Z \to \mathbb{P}^1$ with a section having singular fibers $4I_2$, $2I_8$. This implies that $\mathrm{disc}(\mathrm{NS}(Z)) = 16$ by [17]. $\square$

7.3. *Existence of a certain $\mathcal{D}_{16}$ cover branched at $\mathcal{C}^{(2)} + \mathrm{L}_\infty$*

Let $\mathcal{C}^{(2)}$ be the sextic curve described in Example 2, and let L_1 be the tangent line at $\mathbb{A}_{15}$. The strict transform of L_1 splits up into two components, L_1^+ and L_1^-. We label the irreducible components of the exceptional divisor of μ as follows:

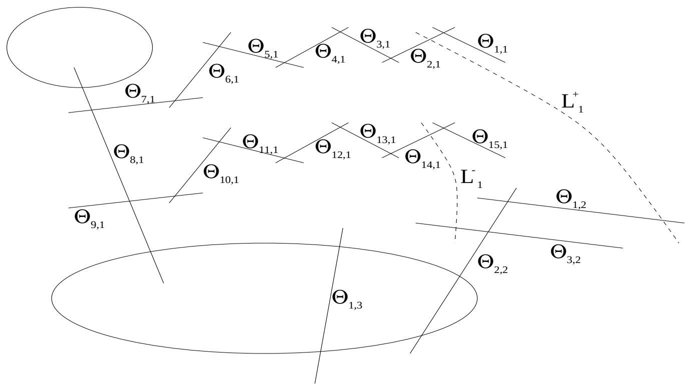

Fig. 16.

With this notation, we have

Lemma 2.

$$L_1^+ \sim_{\mathbb{Q}} \frac{1}{2} f^* \mathrm{L}_\infty - \frac{1}{8}\left(7\Theta_{1,1} + \sum_{k=2}^{15}(16-k)\Theta_{k,1}\right) - \frac{1}{4}\sum_{k=1}^{3}(4-k)\Theta_{k,2}.$$

Proof. Put

$$D_\alpha = L_1^+ - \frac{1}{2}f^*\mathrm{L}_\infty + \frac{1}{8}\left(7\Theta_{1,1} + \sum_{k=2}^{15}(16-k)\Theta_{k,1}\right) + \frac{1}{4}\sum_{k=1}^{3}(4-k)\Theta_{k,2}.$$

By [18], Lemma 1.2 and Lemma 1.4, D_α satisfy that

(i) $D_\alpha = 0 \bmod T \otimes \mathbb{Q}$ and

(ii) $D_\alpha \perp T$ with respect to the intersection pairing.

We can easily check $D_\alpha^2 = 0$. Hence $D_\alpha \approx_{\mathbb{Q}} 0$ by Lemma 1.3 [18]. Since Z is simply connected, we are done. □

Lemma 3. Put

$$\begin{array}{rcc} D_1 & = & \sum_{k=1}^{7}(8-k)\Theta_{k,1} + 6\Theta_{1,2}, \\ D_2 & = & f^*\mathrm{L}_\infty + \Theta_{2,2} \\ D_0 & = & L_1^+ - f^*\mathrm{L}_\infty + \Theta_{15,1} + \Theta_{8,1} + \sum_{k=1}^{7}(\Theta_{k,1} + \Theta_{16-k,1}) + \Theta_{3,2} \end{array}$$

Then D_1, D_2 and D_0 satisfies

$$D_1 + 4D_2 - \sigma^* D_1 + 8D_0 \sim 0.$$

Proof. By straightforward computation and Lemma 2, we have Lemma 3. □

The three divisors in Lemma 3 satisfy the following conditions:

(i) D_1 and D_2 are effective divisors satisfying the conditions in Proposition 3.

(ii) $\mathrm{Supp}(D_1 + \sigma^* D_1 + D_2)$ is contained in $\mathrm{Supp}(f^*\mathrm{L}_\infty)$ and the exceptional set of μ. Hence by Proposition 3, we infer that there exists a $\mathcal{D}_{16}$ cover, S, of $\mathbb{P}^2$ branched at $\mathcal{C}^{(2)} + \mathrm{L}_\infty$ and having $D(S/\mathbb{P}^2) = Z'$.

From the results obtained in this section, one has the Main Theorem 1.

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Appendix A: GAP programs

Both programs have common beginnings and ends:

```
g:=FreeGroup(4,"a");
lista:=GeneratorsOfGroup(g);
a1:=lista[1];
a2:=lista[2];
a3:=lista[3];
a4:=lista[4];
lista1:=[a2,a2*a1/a2,a3,a4];
k1:=GroupHomomorphismByImages(g,g,lista,lista1);;
lista1m:=[a2^a1,a1,a3,a4];;
k1m:=GroupHomomorphismByImages(g,g,lista,lista1m);;
lista2:=[a1,a3,a3*a2/a3,a4];;
k2:=GroupHomomorphismByImages(g,g,lista,lista2);;
```

```
lista2m:=[a1,a3^a2,a2,a4];;
k2m:=GroupHomomorphismByImages(g,g,lista,lista2m);;
lista3:=[a1,a2,a4,a4*a3/a4];;
k3:=GroupHomomorphismByImages(g,g,lista,lista3);;
lista3m:=[a1,a2,a4^a3,a3];;
k3m:=GroupHomomorphismByImages(g,g,lista,lista3m);;
id:=GroupHomomorphismByImages(g,g,lista,lista);;
igual:=function(elemento,trenza)
  return Image(trenza,elemento)/elemento;
end;
```

$(\cdots)$

```
h:=g/rel;
P:=PresentationFpGroup(h);
TzGoGo(P);
TzPrintRelators(P);
```

The central part for G_1 is:

$(\cdots)$

```
camino:=k2m*k1m^2*k3^2*k2*k2m*k3;
caminom:=k3m*k2*k2m*k3m^2*k1^2*k2;
camino1:=camino*k3m*k2;;
camino1m:=k2m*k3*caminom;;
sing1:=k1;;
trenza1:=camino1*sing1*camino1m;;
rel:=List(lista,u->igual(u,trenza1));;
camino2:=camino1*k2m*k3m^2*k1^2*k2;;
camino2m:=k2m*k1m^2*k3^2*k2*camino1m;;
sing2:=k3;;
trenza2:=camino2*sing2*camino2m;;
rel:=Union(rel,List(lista,u->igual(u,trenza2)));;
camino3:=camino2*k3*k2m*k1;;
camino3m:=k1m*k2*k3m*camino2m;;
sing3:=k2^2;;
trenza3:=camino3*sing3*camino3m;;
rel:=Union(rel,List(lista,u->igual(u,trenza3)));;
camino4:=camino3*k2*k1m*k2;;
camino4m:=k2m*k1*k2m*camino3m;;
sing4:=k1;;
trenza4:=camino4*sing4*camino4m;;
rel:=Union(rel,List(lista,u->igual(u,trenza4)));;
```

```
camino5:=camino4*k1;;
camino5m:=k1m*camino4m;;
sing5:=k2^16;;
trenza5:=camino5*sing5*camino5m;;
rel:=Union(rel,List(lista,u->igual(u,trenza5)));;
camino6:=camino5*k2^8;;
camino6m:=k2m^8*camino5m;;
sing6:=k3;;
trenza6:=camino6*sing6*camino6m;;
rel:=Union(rel,List(lista,u->igual(u,trenza6)));;
rel:=Union(rel,[Image(caminom,a1*a4*a3*a2*a1*a4)]);;
```

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and the central part for G_2 is:

(· · ·)

```
camino:=id;
caminom:=id;
camino1:=camino*k3m*k1m;;
camino1m:=k1*k3*caminom;;
sing1:=k2;;
trenza1:=camino1*sing1*camino1m;;
rel:=List(lista,u->igual(u,trenza1));;
camino2:=camino1;;
camino2m:=camino1m;;
sing2:=(k1*k3)^7*(k2*k1*k3*k2);;
trenza2:=camino2*sing2*camino2m;;
rel:=Union(rel,List(lista,u->igual(u,trenza2)));;
camino3:=camino2*(k1*k3)^4*(k2*k1*k3*k2);;
camino3m:=(k2m*k3m*k1m*k2m)*(k1m*k3m)^4*camino2m;;
sing3:=k2;;
trenza3:=camino3*sing3*camino3m;;
rel:=Union(rel,List(lista,u->igual(u,trenza3)));;
camino4:=camino3*k1m*k3m;;
camino4m:=k3*k1*camino3m;;
sing4:=k2^2;;
trenza4:=camino4*sing4*camino4m;;
rel:=Union(rel,List(lista,u->igual(u,trenza4)));;
rel:=Union(rel,[Image(caminom,a1*a4*a3*a2*a1*a4)]);;
```

(· · ·)