Braid Monodromy Of Algebraic Plane Curves

José Ignacio COGOLLUDO-AGUSTÍN

Departamento de Matemáticas
Universidad de Zaragoza

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  such that:
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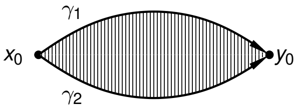

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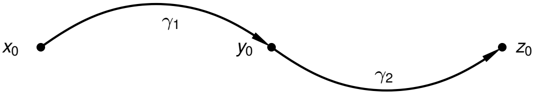

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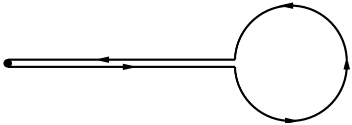

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- X connected $\Rightarrow \pi_1(X)$

Example

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Example (Ordered Configuration Spaces)

Let $X_n := \{(z_1, ..., z_n) \in \mathbb{C}^n \mid z_i \neq z_j, i \neq j\}$. Then $\pi_1(X_n) = \mathbb{P}_n$.

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Example (Non-ordered Configuration Spaces)

Let $\mathcal{P}_n := \{f(z) \in \mathbb{C}[z] \mid \deg(f) = n\}$, $Y_n := \mathbb{P}(\mathcal{P}_n \setminus \Delta_n)$, where $\Delta_n := \{f \in \mathcal{P}_n \mid f \text{ has multiple roots}\}$. Note that $Y_n \cong X_n/\Sigma_n$. Then $\pi_1(Y_n) = \mathbb{B}_n$. Analogously, if we consider $\bar{\mathcal{P}}_n := \{f(s,t) \in \mathbb{C}[s,t] \mid f \text{ homogeneous } \deg(f) = n\}$, $\bar{Y}_n := \mathbb{P}(\mathcal{P}_n \setminus \Delta_n)$, where $\bar{\Delta}_n := \{f \in \bar{\mathcal{P}}_n \mid f \text{ has multiple roots}\}$. Note that $\pi_1(\bar{Y}_n) = \mathbb{B}_n(\mathbb{S}^2)$.

Van Kampen Theorem

Theorem

Let U_1 and U_2 open subsets of X such that:

- $U_1 \cup U_2 = X$ *and*
- $U_{12} := U_1 \cap U_2$ *is path-connected.*

Then

$$\pi_1(X) = \pi_1(U_1) *_{\pi_1(U_{12})} \pi_1(U_2).$$

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Let $z_1, ..., z_n \in \mathbb{C}$, $Z_n := \{z_1, ..., z_n\}$. Then $\pi_1(\mathbb{C} \setminus Z_n) = \mathbb{F}_n$.

Locally trivial Fibrations

Definition

A surjective smooth map $\pi : X \to M$ of smooth manifolds is a *locally trivial fibration* if there is an open cover $\mathcal{U}$ of M and diffeomorphisms $\varphi_U : \pi^{-1}(U) \to U \times \pi^{-1}(p_U)$, with $p_U \in U$, such that φ_U is fiber-preserving, that is $pr_1 \varphi_U = \pi$. We denote $\pi^{-1}(p)$ by F_p.

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Consider $\pi : X \to M$ a locally trivial fibration and $s : M \to X$ a *section*. There is an action of $\pi_1(M, p)$ on $\pi_1(F_p, x_0)$ $(s(p) = x_0)$ called *monodromy action of* M on F_p.

Monodromy Actions

$$\pi^{-1}(\gamma) = \begin{array}{ccc} \tilde{X} & \hookrightarrow & X \\ \downarrow \tilde{\pi} & & \downarrow \pi \\ I & \xrightarrow{\gamma} & M \end{array}$$

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The fibration $\tilde{\pi}$ is trivial, and hence there exists

$$\varphi : I \times F_p \to \tilde{X}$$

such that $\varphi(0, x) = Id_{F_p}$.
If π is such that F_p is connected, then given a loop $\alpha \in \pi_1(F_p, x_0)$ and a loop $\gamma \in \pi_1(M, p)$, then one deforms $\varphi(t, \alpha)$ into a loop $\alpha_t \in \Gamma(F_{\gamma(t)}, s(\gamma(t)))$. Then $\alpha^\gamma := \alpha_1$ is the monodromy action of γ over α.

Remark

Another interesting scenario occurs when F_p is finite and π is a topological cover. In that case $\varphi(1, x)$ induces a permutation of F_p. This permutation is also called the *monodromy action of γ over F_p*.

Examples

Example

Let $\pi : X = M \times F \to M$ be a trivial fibration. Any continuous map $\omega : M \to F$, defines $s(x) = (x, \omega(x))$ a section of $\pi : X \to M$. In this case, φ is the identity. Let $\gamma \in \pi_1(M, p)$ and $\alpha \in \pi_1(F, x_0)$, then α_t is given by $(\omega_t \circ \gamma)^{-1}\alpha(\omega_t \circ \gamma)$, where $\omega_t \circ \gamma(\lambda) = \omega(\gamma(\lambda t))$. Therefore $\pi_1(M, p)$ acts on $\pi_1(F, \omega(p))$ by

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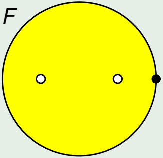

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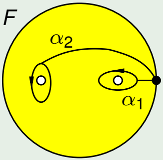

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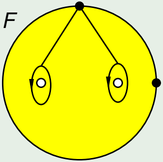

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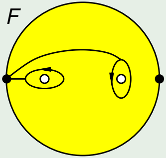

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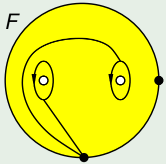

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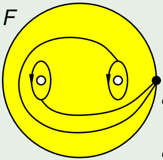

$$\alpha_1^\gamma = (\alpha_2\alpha_1)^{-1}\alpha_1(\alpha_2\alpha_1)$$

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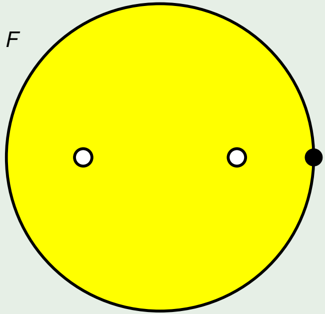

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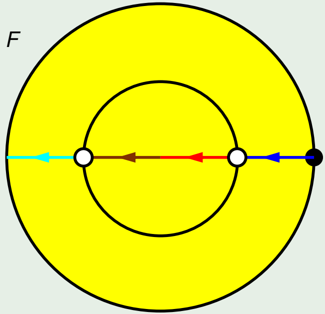

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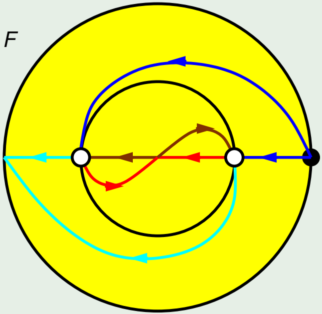

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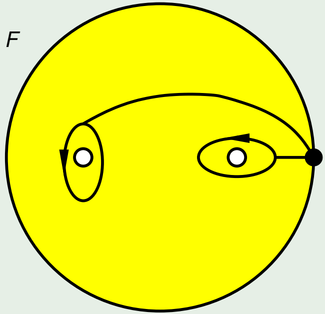

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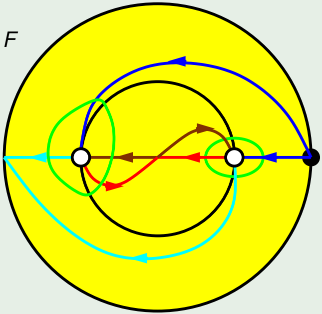

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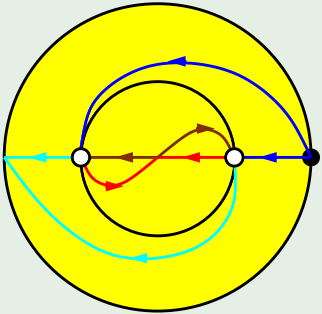

$$\alpha_1^\gamma = \alpha_2$$

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Mapping Class Group

Theorem

There is an isomorphism between the geometric group of braids on n-strings and the mapping class group of automorphisms on the punctured disc $\mathbb{D}_n := \mathbb{D} \setminus Z_n$ *modulo homotopy relative to the boundary, that is,* $\pi_0(Diff^+(X_n))$.

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$$g_j^{\sigma_i} = \begin{cases} g_{i+1} & j = i \\ g_{i+1} g_i g_{i+1}^{-1} & j = i+1 \\ g_i & \text{otherwise.} \end{cases}$$

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- Since $(g_n \cdot \ldots \cdot g_1) = \partial\mathbb{D}$, one obtains $(g_n \cdot \ldots \cdot g_1)^\sigma = (g_n \cdot \ldots \cdot g_1)$.

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Let M be an m-dimensional (connected) complex manifold. A *branched covering* of M is an m-dimensional irreducible normal complex space X together with a surjective holomorphic map $\pi : X \to M$ such that:

- every fiber of π is discrete in X,
- $R_\pi := \{q \in X \mid \pi^* : \mathcal{O}_{\pi(q),M} \to \mathcal{O}_{q,X} \text{is not an isomorphism}\}$ called the *ramification locus*, and $B_\pi = \pi(R_\pi)$ called the *branched locus*, are hypersurfaces of X and M, resp.
- $\pi| : X \setminus \pi^{-1}(B_\pi) \to M \setminus B_\pi$ is an unramified (topological) covering, and
- $\forall\, p \in M$ there is a connected open neighborhood $W^p \subset M$ such that for every connected component U of $\pi^{-1}(W)$:

  i) $\pi^{-1}(p) \cap U = \{q\}$
  ii) $\pi|_U : U \to W$ is surjective and proper.

Construction of branched coverings: smooth case

If B is a non-singular hypersurface, $B = D_1 \cup \ldots \cup D_n$, $e_1, \ldots, e_n \in \mathbb{N}$, $D = \sum n_i D_i$ on M.
$p_0 \in M \setminus B$ base point.

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Condition

If $\gamma_j^d \in J$ *then* $d \equiv 0 \ (mod\ e_j)\ \forall\ 1 \leq j \leq s$.

Theorem

There is a natural one-to-one correspondence between

$$\{\pi : X \to M \text{ Galois, finite, ramified along } D\} / \sim$$
$$\updownarrow$$
$$\{J \subset K \overset{f.i}{\lhd} \pi_1(M \setminus B) \text{ satisfying (1.4)}\}.$$

Moreover, there is a maximal Galois covering $\pi(M, D)$ *of* M *ramified along* D *iff* $K_\pi = \cap K \overset{f.i}{\lhd} \pi_1(M \setminus B)$ *satisfies (1.4).*

Construction of branched coverings: smooth case

Theorem (Riemann Existence Theorem)

Any monodromy action $\pi_1(\mathbb{P}^1 \setminus Z_n) \to \Sigma_s$ *can be realized by a branched covering of the projective line* $\mathbb{P}^1$.

Construction of branched coverings: general case

If B is a hypersurface, $B = D_1 \cup ... \cup D_n$, $e_1, ..., e_n \in \mathbb{N}$, $D = \sum n_i D_i$ on M. $p_0 \in M \setminus B$ base point.

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$i : W^{\tilde{p}_0} \setminus B \hookrightarrow M \setminus B$.

Condition

Let $K \triangleleft \pi_1(M \setminus B, p_0)$ *such that* $J \triangleleft K$. *For any point* $p \in \operatorname{Sing} B$,
$K_p = i_*^{-1}(K) \overset{f.i}{\triangleleft} \pi_1(W \setminus B, \tilde{p})$.

Theorem

There is a one-to-one correspondence:

$$\{\pi : X \to M \text{ Galois, finite, ramified along } D\} / \sim$$
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Example

Consider $M = \mathbb{P}^2$, $D_1 = \{zy^2 = x^3\}$, $D_2 = \{z = 0\}$. Let us study the possible Galois covers of $\mathbb{P}^2$ ramified along $D = e_1 D_1 + e_2 D_2$.

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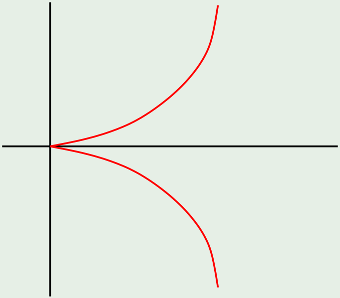

Figure: $y^2 = x^3$

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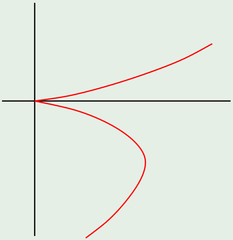

Figure: $y^2 = (x - y)^3$

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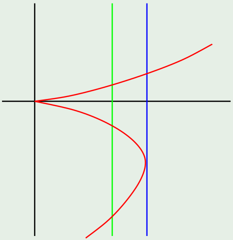

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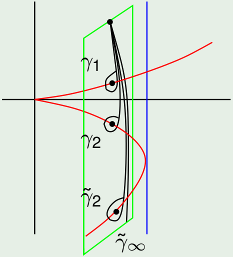

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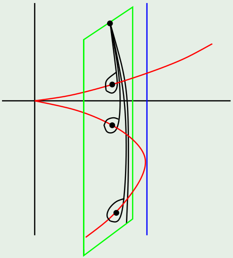

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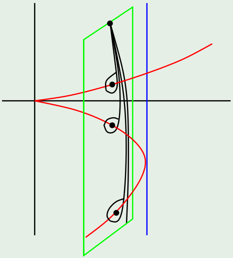

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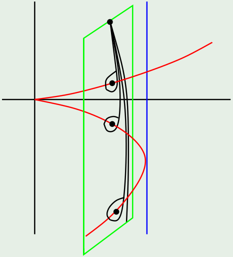

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$$\tilde{\gamma}_2 = \gamma_2,$$
$$\gamma_2\gamma_1\gamma_2 = \gamma_1\gamma_2\gamma_1.$$

Theorem

In the following cases there is a maximal Galois covering of $\mathbb{P}^2$ ramified along D:

(e_1, e_2)	$G = \pi_1(\mathbb{P}^2 \setminus D)/J$	$\|G\|$
$(2, 2)$	Σ_3	*6*
$(3, 4)$	$SL(2, \mathbb{Z}/3\mathbb{Z})$	*24*
$(4, 8)$	$\Sigma_4 \ltimes \mathbb{Z}/4\mathbb{Z}$	*96*
$(5, 20)$	$SL(2, \mathbb{Z}/5\mathbb{Z}) \times \mathbb{Z}/5\mathbb{Z}$	*600*

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$(5,20)$	$SL(2, \mathbb{Z}/5\mathbb{Z}) \times \mathbb{Z}/5\mathbb{Z}$	600

However, there is no maximal Galois cover of $\mathbb{P}^2$ ramified along $D = 6D_1 + 2D_2$.

Theorem

Let $B = D_1 \cup ... \cup D_n$. Then any representation of $\pi_1(M \setminus B)$ on a linear group $GL(r, \mathbb{C})$ such that the image of a meridian γ_i has order e_i, gives rise to a Galois cover of M branched along $D = e_1 D_1 + ... + e_n D_n$.

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Theorem (Hamm,Goreski-MacPherson)

Let $M \subset \mathbb{P}^n$ be a closed subvariety which is locally a complete intersection of dimension m. Let $\mathcal{A}$ be a Whitney stratification of M and consider $B \subset \mathbb{P}^n$ another subvariety such that $B \cap M$ is a union of strata of $\mathcal{A}$. Consider H a hyperplane transversal to $\mathcal{A}$ in $M \setminus B$, then the inclusion

$$(M \setminus B) \cap H \hookrightarrow M \setminus B$$

is an $(m-1)$-homotopy equivalence.

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- Chisini Problem:
  Let S be a nonsingular compact complex surface, let $\pi : S \longrightarrow \mathbb{P}^2$ be a finite morphism having simple branching, and let B be the branch curve; then "to what extent does the pair $(\mathbb{P}^2, B)$ determine π"?

Zariski-Van Kampen Method

Purpose:
Obtain a presentation for the fundamental group of the complement of a plane projective curve in $\mathbb{P}^2$.
We will put together several ingredients, among which, the *Van Kampen Theorem* is key.

Let $\pi : X \longrightarrow M$ be a locally trivial fibration with section $s : M \longrightarrow X$. Consider $p \in M$ and $x_0 \in F_p$.

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Theorem

$\pi_1(X, x_0) = \pi_1(F_p, x_0) \rtimes \pi_1(M, p)$, where the action of $\pi_1(M, p)$ on $\pi_1(F_p, x_0)$ is given by the monodromy of π.

Proposition

Meridians around the same irreducible components of B are conjugate in $\pi_1(M \setminus B)$. Moreover, the conjugacy class of a meridian coincides with the set of homotopy classes of meridians around the same irreducible component.

Proposition

The inclusion $M \setminus B \hookrightarrow M$ induces a surjective morphism, whose kernel is given by the smallest normal subgroup of $\pi_1(M \setminus B)$ containing meridians of all the irreducible components of B.

Zariski-Van Kampen Theorem

Let $\mathcal{C} \subset \mathbb{P}^2$ be a projective plane curve. Consider $P = [0 : 1 : 0] \in \mathbb{P}^2 \setminus \mathcal{C}$.

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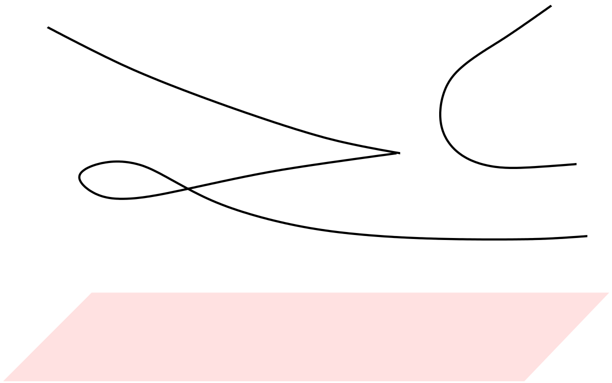

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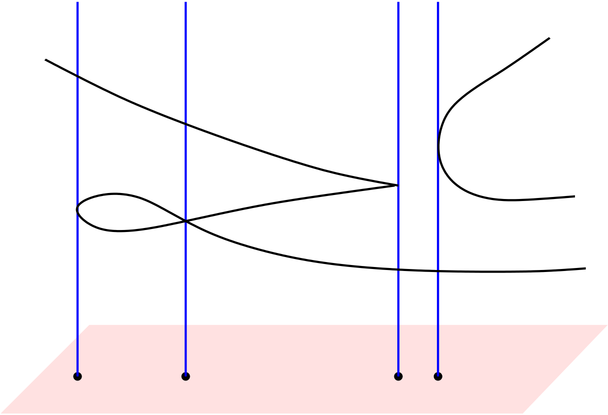
Let $\mathcal{C} \subset \mathbb{P}^2$ be a projective plane curve. Consider $P = [0:1:0] \in \mathbb{P}^2 \setminus \mathcal{C}$. Project $\pi : \mathbb{P}^2 \setminus \{P\} \to \mathbb{P}^1$ from P



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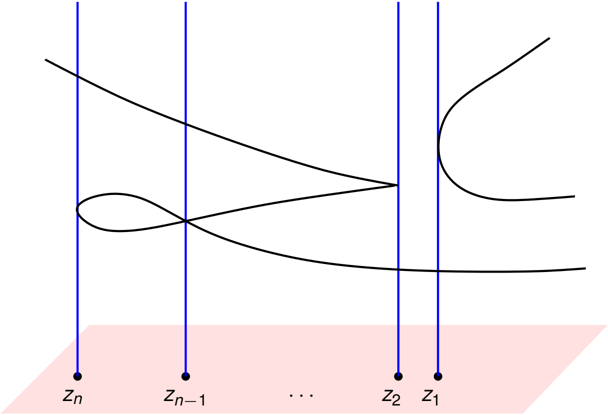

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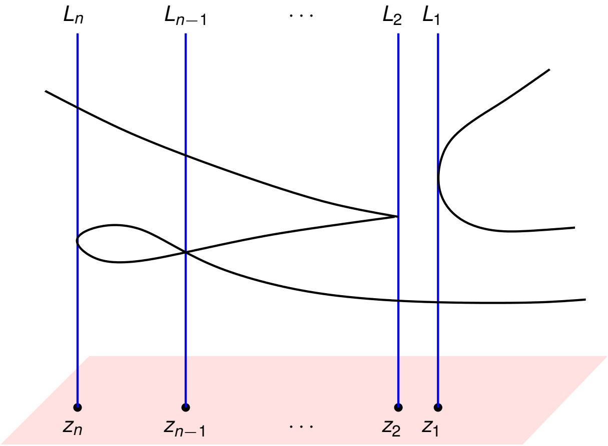

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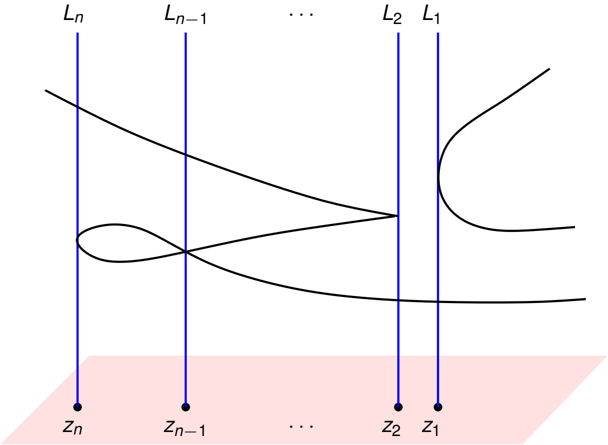
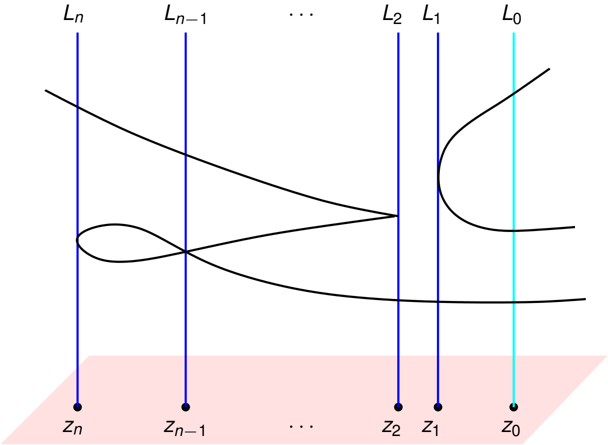


Remark (1)

Let $X = \mathbb{P}^2 \setminus (\mathcal{C} \cup L)$, then $\pi|_X : X \to \mathbb{P}^1 \setminus Z_n$ is a locally trivial fibration.

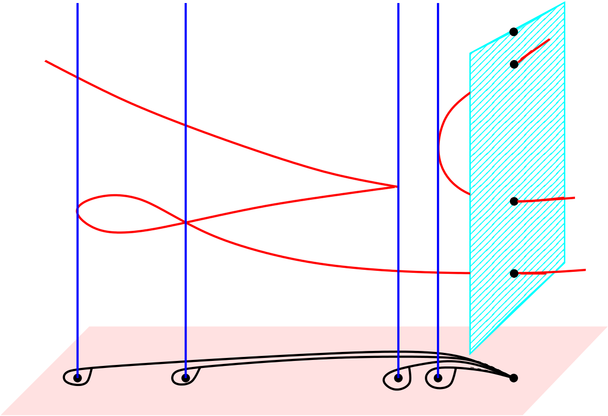
Zariski-Van Kampen Theorem



Remark (1)

Let $X = \mathbb{P}^2 \setminus (\mathcal{C} \cup L)$, then $\pi|_X : X \to \mathbb{P}^1 \setminus Z_n$ is a locally trivial fibration. Moreover, its fiber is $\mathbb{P}^1 \setminus Z_d$, where $d := \deg \mathcal{C}$.

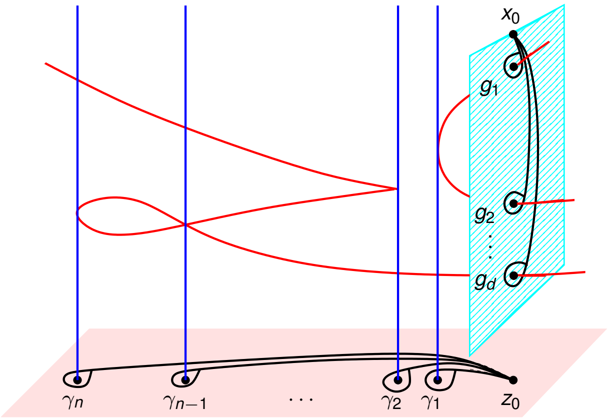
Zariski-Van Kampen Theorem



Remark (2)

By (2.1), $\pi_1(X, x_0) = \pi_1(F_{z_0}, x_0) \rtimes \pi_1(\mathbb{P}^1 \setminus Z_n, z_0)$. Action is given by the monodromy of $\pi_1(\mathbb{P}^1 \setminus Z_n, z_0)$ on $\pi_1(F_{z_0}, x_0)$.

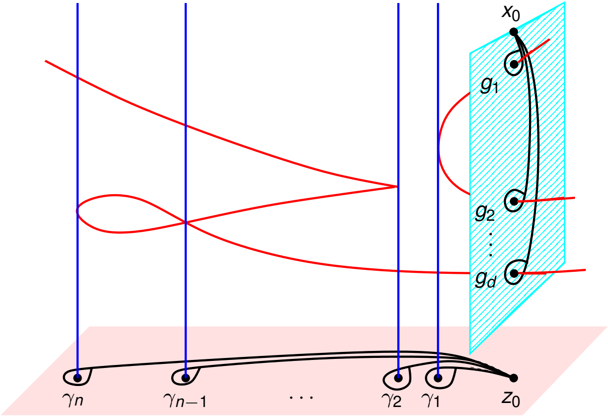
Zariski-Van Kampen Theorem



Remark (3)

Note that $\pi_1(F_{z_0}, x_0) = \langle g_1, ..., g_d : g_d g_{d-1} \cdots g_1 = 1\rangle$ and $\pi_1(\mathbb{P}^1 \setminus Z_n, z_0) = \langle \gamma_1, ..., \gamma_n : \gamma_n \cdots \gamma_1 = 1\rangle$.

Zariski-Van Kampen Theorem

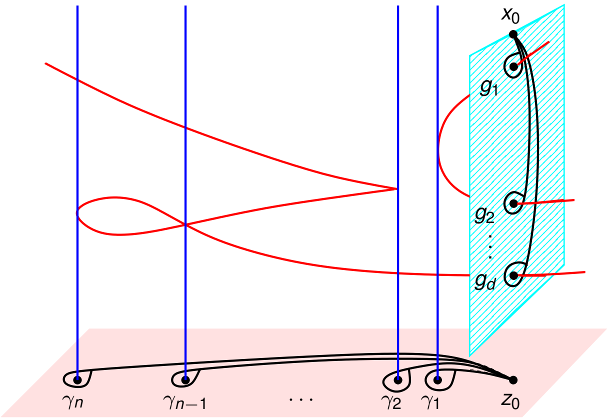


Theorem

$\pi_1(X, x_0)$ admits the following presentation:

$$\langle g_1, ..., g_d, \gamma_1, ..., \gamma_n : g_d g_{d-1} \cdots g_1 = \gamma_n \cdots \gamma_1 = 1, g_i^{\gamma_j} = \gamma_j^{-1} g_i \gamma_j \rangle$$

Zariski-Van Kampen Theorem



Theorem

$\pi_1(\mathbb{P}^2 \setminus \mathcal{C})$ *admits the following presentation:*

$$\langle g_1, \ldots, g_d : g_d g_{d-1} \cdots g_1 = 1, g_i^{\gamma_j} = g_i \rangle$$

Remark

- Let $\mathcal{C} = \mathcal{C}_1 \cup \ldots \cup \mathcal{C}_r$ the decomposition of $\mathcal{C}$ in its irreducible components, then

$$H_1(\mathbb{P}^2 \setminus \mathcal{C}) = \mathbb{Z}^{r-1} \oplus \mathbb{Z}/(d_1, \ldots, d_r),$$

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$$H_1(\mathbb{P}^2 \setminus \mathcal{C}) = \mathbb{Z}^{r-1} \oplus \mathbb{Z}/(d_1, ..., d_r),$$

  where $d_i := \deg \mathcal{C}$.
- It two curves are in a connected family of equisingular curves, then they are isotopic

Zariski-Van Kampen Theorem

Example

$\mathcal{C}$ smooth of degree $d \Rightarrow \pi_1(\mathbb{P}^2 \setminus \mathcal{C}) = \mathbb{Z}/d\mathbb{Z}$.

Zariski-Van Kampen Theorem

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$\mathcal{C}$ smooth of degree $d \Rightarrow \pi_1(\mathbb{P}^2 \setminus \mathcal{C}) = \mathbb{Z}/d\mathbb{Z}$. For computation purposes it is more convenient to use a *non-generic* projection.

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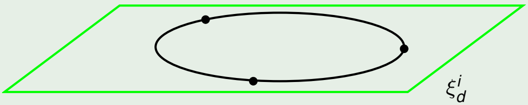
Let us compute the local monodromy of $x = y^d$. Consider $\gamma(t) = e^{2\pi t\sqrt{-1}}$ a loop around $x = 0$. The fiber at $\gamma(t)$ is given by:



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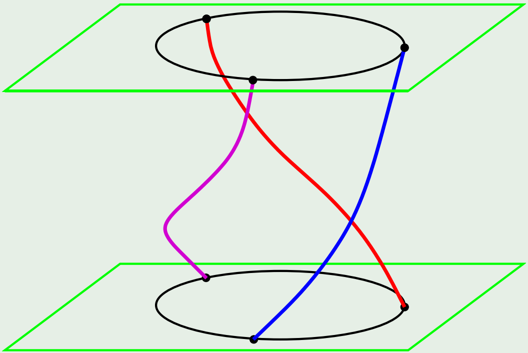
The monodromy around $x = 0$ looks as follows:



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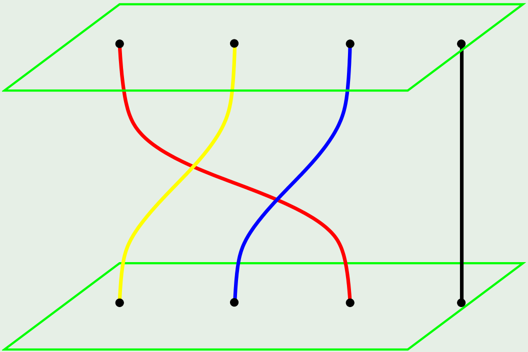
$\mathcal{C}$ smooth of degree $d \Rightarrow \pi_1(\mathbb{P}^2 \setminus \mathcal{C}) = \mathbb{Z}/d\mathbb{Z}$.
Corresponds to the braid $\sigma_1\sigma_2\cdots\sigma_{d-1}$



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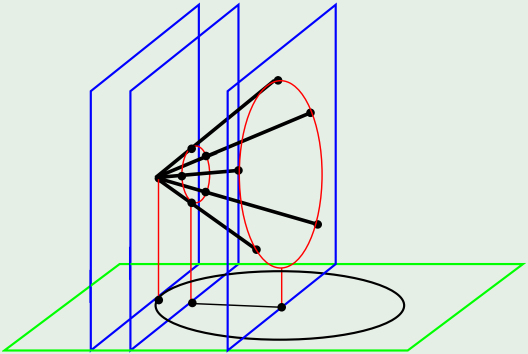
Note that the global part of the monodromy has no contribution:



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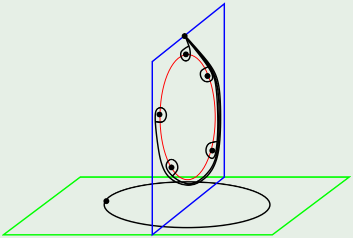

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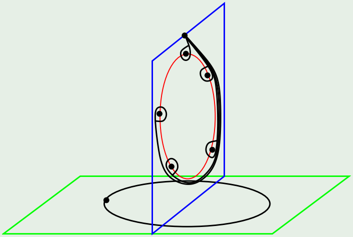

One obtains:

$$g_i = g_i^{(\sigma_1\sigma_2\cdots\sigma_{d-1})} = \begin{cases} g_d & i = 1 \\ g_d^{-1} g_{i-1} g_d & i \neq 1 \end{cases}$$

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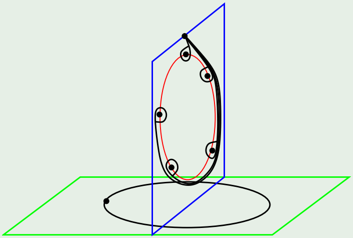

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hence $g_2 = g_d^{-1} g_1 g_d = g_1$, and by induction $g_1 = \ldots = g_d = g$. Finally, $g_1 \cdots g_d = 1$ becomes $g^d = 1$

$$\pi_1(\mathbb{P}^2 \setminus \mathcal{C}) = \langle g : g^d = 1 \rangle = \mathbb{Z}/d\mathbb{Z}.$$

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Example (Zariski)

Let $\mathcal{C}$ be a general nodal rational curve of degree d. Consider $\tilde{\mathcal{C}}$ its dual. Note that $\tilde{\mathcal{C}}$ is a rational curve of degree $2(d-1)$, $2(d-2)(d-3)$ nodes, and $3(d-2)$ cusps. The fundamental group of $\tilde{\mathcal{C}}$ coincides with the fundamental group of the unordered configuration space of d points in $\mathbb{S}^2$, that is,

$$\mathbb{B}_d(\mathbb{S}^2) = \left\langle g_1, \ldots, g_{d-1} : \begin{array}{l} g_i g_j = g_j g_i, \\ g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}, \\ g_1 \cdots g_{d-2} g_{d-1}^2 g_{d-2} \cdots g_1 = 1 \end{array} \right\rangle.$$

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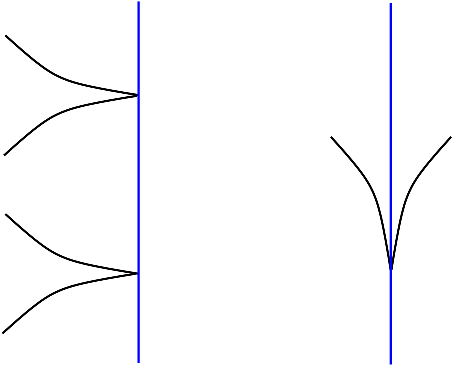

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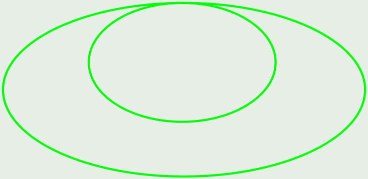
Consider the following quartic:



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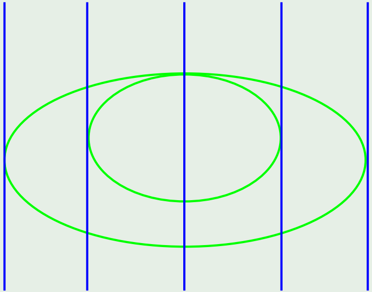
Consider the following quartic and project from $[0 : 1 : 0]$



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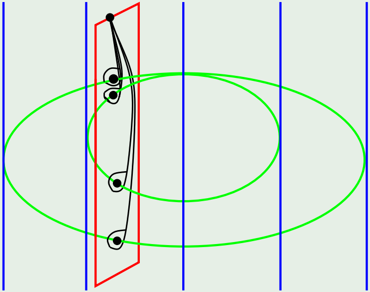
Compute the braid monodromy:



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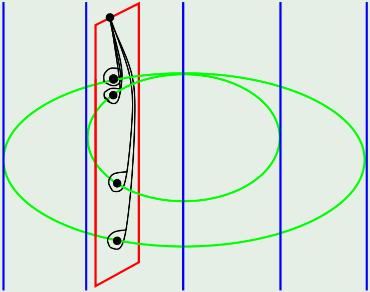

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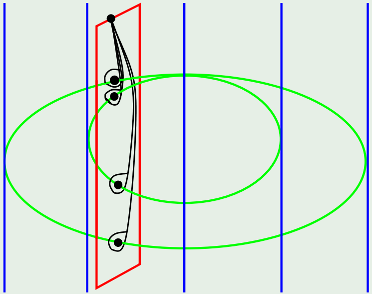

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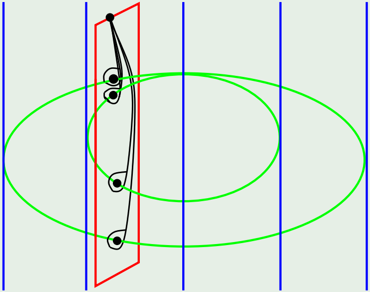

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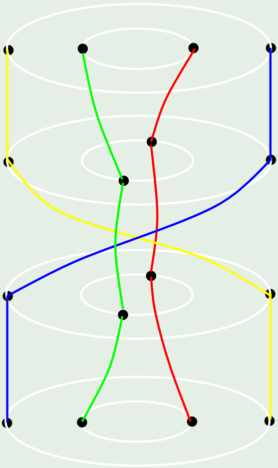

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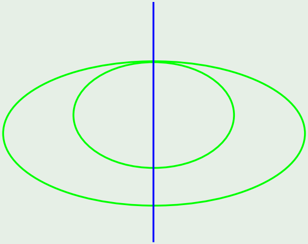

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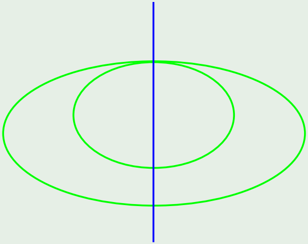

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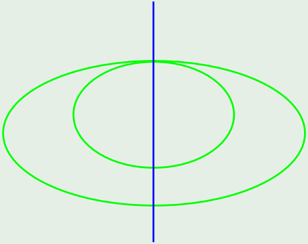

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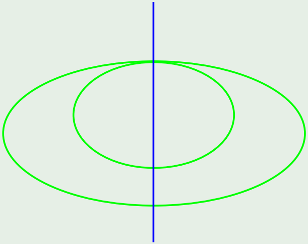

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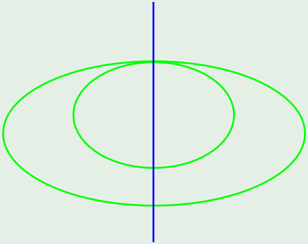

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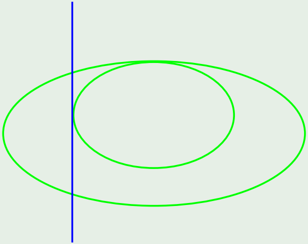

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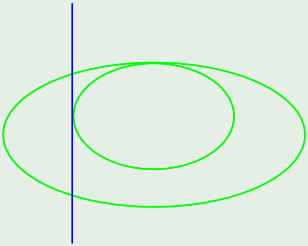

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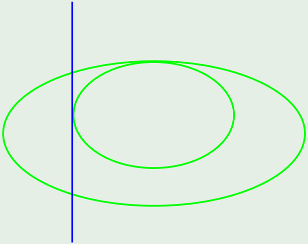

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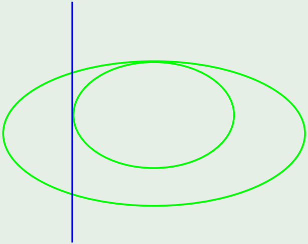

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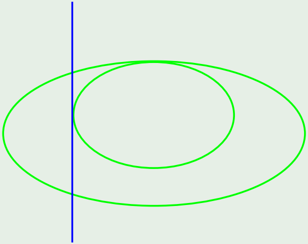

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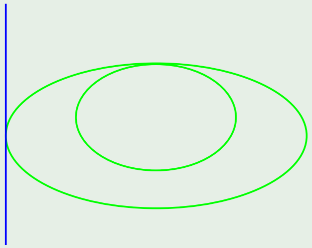

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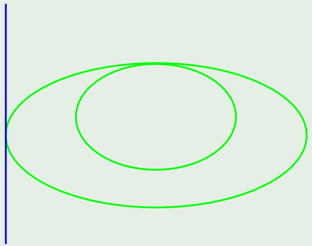

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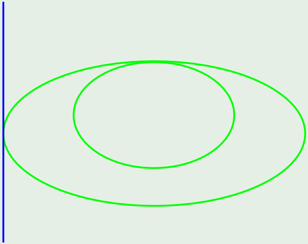

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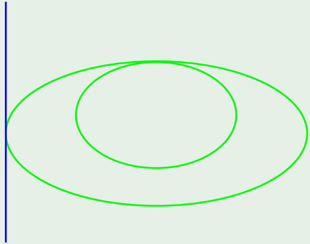

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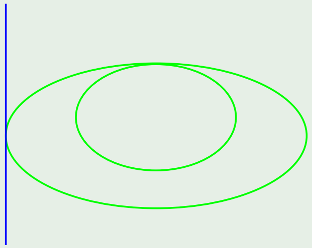

Geometric basis

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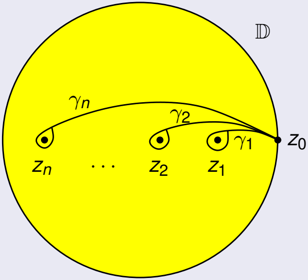

$$\gamma_n \gamma_{n-1} \cdots \gamma_1 = \partial \mathbb{D}$$

Braid Monodromy Representation

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Consider the braid monodromy action:

$$\rho : \pi_1(\mathbb{D} \setminus Z_n, z_0) \longrightarrow Diff^+(F_{z_0}) \cong \mathbb{B}_d.$$

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$$(\rho\gamma_1, ..., \rho\gamma_n) \in \mathbb{B}_d^n$$

is the *Braid Monodromy Representation* of $\mathcal{C}$ relative to (π, Γ, z_0).

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- Both actions commute ($\mathbb{B}_n\times\mathbb{B}_d$). *Hurwitz Moves*.

Theorem

$$\{(\Gamma, z_0)\} \leftrightarrow \{\textit{Hurwitz class}\}$$

Questions

- *Which (positive) factorizations are realizable in the algebraic category?*
- *All theoretical factorizations of a smooth curve are Hurwitz equivalent (Ben Itzak-Teicher), but are there theoretical factorizations of a smooth curve that are not realizable by a smooth curve?*

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The triple $(\mathbb{P}^2, \mathcal{C}, L)$ fully determines the braid monodromy class of $\mathcal{C}$.

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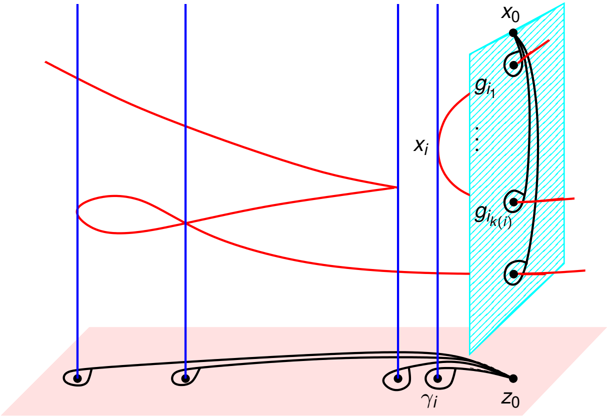
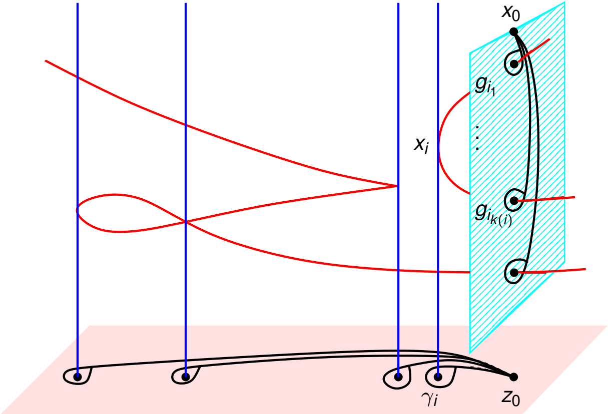
$\pi : \mathbb{C}^2 \setminus \mathcal{C} \to \mathbb{P}^1 \setminus Z_n$ *generic*.
$\mathbb{D}$ a big enough disk containing Z_n



The homotopy type of $(\mathbb{C}^2, \mathcal{C})$



$$(g_{i_k} \cdots g_{i_1})^{\rho\gamma_i} = (g_{i_k} \cdots g_{i_1})$$

Remark

$$\langle g_1, ..., g_{d-1} : g_i^{\gamma_j} = \gamma_j^{-1} g_i \gamma_j, i = 1, ..., d-1, j = 1, ..., i_{k(i)} - 1 \rangle$$

is a presentation of $\pi_1(\mathbb{C}^2 \setminus \mathcal{C})$.

The homotopy type of $(\mathbb{C}^2, \mathcal{C})$

Theorem (Libgober)

The 2-dimensional complex associated with the Zariski presentation has the homotopy type of $\mathbb{C}^2 \setminus \mathcal{C}$.

Proof.

Lemma

The 2-dimensional complex associated with the Wirtinger presentation of a link $K \subset \mathbb{S}^3$ has the homotopy type of $K \setminus \mathbb{S}^3$.

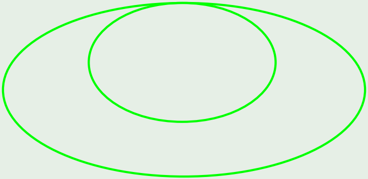
Lemma

The 2-dimensional complex associated with the Artin presentation of a link $K \subset \mathbb{S}^3$ has the homotopy type of $K \setminus \mathbb{S}^3$.

□

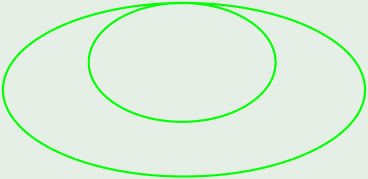
The homotopy type of $(\mathbb{C}^2, \mathcal{C})$

Example



The homotopy type of $(\mathbb{C}^2, \mathcal{C})$

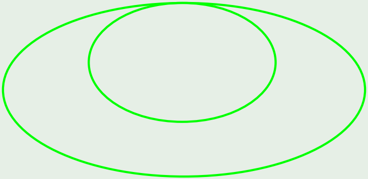
Example



$$\pi_1(\mathbb{C}^2 \setminus \mathcal{C}) = \left\langle g_1, g_2, g_3, g_4 : \begin{array}{l} [(g_2 g_1)^4, g_1] = 1, \\ g_2 = g_3, \\ g_4 = g_2 g_1 g_2^{-1}, \\ g_2 = g_3, \\ g_4 = g_2 g_1 g_2^{-1} \end{array} \right\rangle \equiv$$

The homotopy type of $(\mathbb{C}^2, \mathcal{C})$

Example



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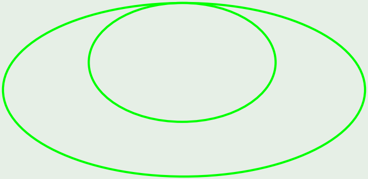
$$\equiv \langle g_1, g_2 : [(g_2 g_1)^4, g_1] = 1, 1 = 1, 1 = 1 \rangle$$

The homotopy type of $(\mathbb{C}^2, \mathcal{C})$

Example



$$\pi_1(\mathbb{C}^2 \setminus \mathcal{C}) = \left\langle g_1, g_2, g_3, g_4 : \begin{array}{l} [(g_2 g_1)^4, g_1] = 1, \\ g_2 = g_3, \\ g_4 = g_2 g_1 g_2^{-1}, \\ g_2 = g_3, \\ g_4 = g_2 g_1 g_2^{-1} \end{array} \right\rangle \equiv$$

$$\equiv \langle g_1, g_2 : [(g_2 g_1)^4, g_1] = 1, 1 = 1, 1 = 1 \rangle$$

Hence $\mathbb{C}^2 \setminus \mathcal{C} \overset{\text{h.t.}}{\cong} (\mathbb{S}^3 \setminus K_{2,8}) \vee \mathbb{S}^2 \vee \mathbb{S}^2$.

The homotopy type of $(\mathbb{C}^2, \mathcal{C})$

Questions

1. *Does the fundamental group and the Euler characteristic determine the homotopy structure of complements to affine curves?*

The homotopy type of $(\mathbb{C}^2, \mathcal{C})$

Questions

1. *Does the fundamental group and the Euler characteristic determine the homotopy structure of complements to affine curves?*
2. *Not true for general 2-dimensional complexes (Dunwoody).*

Braid Action