Notes on Riemann Surfaces

(Following *Complex Functions* by Gareth A. Jones and David Singerman)

Universidad de Zaragoza

February 2022

The Topic of the Course

This course is an introduction to Riemann surfaces with an algebraic and geometric viewpoint as the subtitle of the book followed by the coursesays

In 1851 Riemann studies (using Dirichlet Principle) inverses of conformal functions. Riemann shows that inverses of complex analytical functions are (analytical) functions defined on surfaces: the so called **Riemann surfaces**. Which functions are analytical determines the conformal structure of the surface.

(Compact) Riemann Surface: Surface X with an atlas $X = \cup_{x \in X} U_x$ of **charts** $\phi_x : U_x \to V_x$ open in $\mathbb{C}$ homeomorphism s.t. (for charts ϕ_x, ϕ_y with no empty intersection) $\phi_y \circ \phi_x^{-1} : \phi_x(U_x \cap U_y) \to \phi_y(U_x \cap U_y)$ is analytical (as its inverse). Notice that we can think of having different geometries (distances, given by isometric differential structures) on a topological surface X and the corresponding Riemann surface is the class of conformal geometries.

In fact we are not interested in functions in $\mathbb{C}$ but on functions on the Riemann Sphere $\widehat{\mathbb{C}}$ (which is the complex projective line bringing infinity to Earth), so we consider in general meromorphic functions $\phi_y \circ \phi_x^{-1}$.

Now a **Riemann surface is a topological surface with a class of meromorphic functions**.

The Topic of the Course

In the course we will work with this points:

- Analytical functions between Riemann surfaces are branched coverings. In fact one can see a Riemann surface as a branched covering of the Riemann Sphere (earlier work of Schwarz, Hurwitz, Weiertrass, Clebsch, Klein, and many more).
- The group of meromorphic functions on a Riemann surface is (functorially) the group of of fractional functions of a projective (the infinity brought to Earth) smooth curve. So compact Riemann surfaces are (smooth) projective complex curves (earlier work of Schottky, Wiman, Torelli, Fricke, and many more).
- Finally any R. S. is the quotient of either $\widehat{\mathbb{C}}, \mathbb{C}$ or the complex disk $\mathbb{H}$ by a discrete subgroup of the corresponding group of motions (Poincar, Koebe)

Contents

The lectures of the course will be devoted to the following topics:

-1. Prerequisites. Conformal and meromorphic functions: zeros, poles, Cauchy Integral Formula, Liouville's Th., series, Maximum Modulus Principle. Coverings and Fundamental Groups: Surfaces, coverings, universal coverings, fundamental groups and groups of deck-transformations of coverings, monodromies. (Part of the prerequisites of the course)

0. Riemann Sphere and Möbius Transformations. Meromorphic and rational functions. The group $PSL(2, \mathbb{C})$. (1 hour)

1. Elliptic Functions and Tori. Weiertrass *p*-function. Topology and elliptic functions and tori. (2 hours)

2. Riemann Surfaces. Meromorphic functions and germs. Connected components in the space of germs of meromorphic functions. The space of tori, the group $PSL(2, \mathbb{Z})$. (3 hours)

3. Fuchsian Groups. Discrete subgroups and discontinuous actions. Fundamental regions and Riemann surfaces. Uniformization Th. (3 hours)

Literature

1. **Main Reference** G. A. Jones & D. Singerman, *Complex Functions, An Algebraic and Geometric Viewpoint*. Cambridge Univ. Press, Cambridge, 1988
2. A. F. Beardon, *The Geometry of Discrete Groups*. Graduate Texts in Maths. 91, Springer, Berlin, New York, 1983
3. W. S. Massey, *A Basic Course in algebraic Topology*. Graduate Texts in Maths.127, Springer, Berlin, New York, 1991
4. H. M. Farkas & I. Kra *Riemann Surfaces*. Graduate Texts in Maths. 71, Springer, Berlin, New York, 1992
5. R. D. M. Accola, *Topics in the Theory of Remann Surfaces*. Springer, Berlin, New York, 1994
6. R. Miranda, *Algebraic Curves and Riemann Surfaces*. Graduate Studies in Maths. 5, AMS, Providence, 1995
7. J. E. Marsden & M. J. Hoffman *Basic Complex Analysis*. W.H. Freeman Berlin, New York, 1999
8. A. F. Beardon, *Algebra and Topology*. Cambridge Univ. Press, Cambridge, 2005
9. E. Girondo & G. Gonzaález-Diez, *Introduction to Compact Riemann Surfaces and Dessins d'Enfants*. Cambridge Univ. Press, Cambridge, 2012
10. R. Cavalieri & E. Miles, *Riemann Surfaces and Algebraic Curves*. Cambridge Univ. Press, Cambridge, 2016