# Jacobian quotients of polynomial mappings

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H. Maugendre (1999)

$\Phi := (f, g) : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ finite morphism

- $\mathcal{J}$ Jacobian locus of $\Phi$,
- $\Phi(\mathcal{J}) =: \Delta$ discriminant locus.

Then $A = B$.

$$A := \{\text{slopes of Newton polygon of } \Delta\}.$$

$(\mathbb{S}^3, K_f \cup K_g)$ iterated torus link: $\exists \{T_i\}_{i \in I}$ finite set of 2-tori s.t. $\overline{\mathbb{S}^3 \setminus V(T_i)} = \bigcup_{j \in J} M_j$:

- $M_j$ Seifert manifold,
- $K_f \cup K_g$ union of leaves.

Up to isotopy a unique minimal family exists.

$$B := \left\{ \frac{\mathscr{L}(K_g, v_j)}{\mathscr{L}(K_f, v_j)} \mid j \in J \right\}.$$

## Generalizations

- Lê, Maugendre, Weber (2001), $\Phi : (X, p) \to (\mathbb{C}^2, 0)$, finite morphism, $X$ normal surface. Replace:
  - $\mathbb{S}^3$ by the graphed manifold $M$ (link of $X$).
  - $\mathscr{L}$ by a suitable invariant (not needed if $M$ is a $\mathbb{Q}$-homology sphere).
- C. Reydy (algebraic proof up to one technical case)

$\Phi := (f, g) : \mathbb{C}^2 \to \mathbb{C}^2$ finite-fiber map

- $\mathcal{J}$ Jacobian locus of $\Phi$,
- $\Phi(\mathcal{J}) =: \Delta$ discriminant locus.
- $S$ minimal s.t. $C := \Delta \cup S$,

  $\mathbb{C}^2 \setminus \Phi^{-1}(C) \to \mathbb{C}^2 \setminus C$ unramified covering

Consider Newton polygon of $C$ $\circlearrowleft$.

$$l \text{ edge } (x_0^l, y_0^l) \to (x_1^l, y_1^l)$$

$A := \{[(y_1^l - y_0^l, x_0^l - x_1^l)] \mid l \text{ edge}\}$ under $(n_1, n_2) \sim (m_1, m_2) \Leftrightarrow \exists u \in \mathbb{Q}_{>0}$ s.t $(m_1, m_2) = u(n_1, n_2)$.

$$A = \bigcup_{s,t=\pm,0} A_{s,t}.$$

$(\mathbb{S}^3, L_{f,0} \cup L_{g,0})$ iterated torus link,

$B := \{[(\mathscr{L}(L_{f,0}, v_j), \mathscr{L}(L_{g,0}, v_j))] \mid j \in J\}$,

$$B = \bigcup_{s,t=\pm,0} B_{s,t}.$$

**Theorem.** *$A_{s,t} \supset B_{s,t}$ and $A_{s,t} = B_{s,t}$ if $+ \in \{s,t\}$.*

## Normal resolution of $\Phi$

$\Phi := (F, G) : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$; solve the indetermination by a composition $\pi$ of blow-ups and analytic contractions such that $\hat{\Phi} := (\hat{F}, \hat{G}) : \hat{X} \to \mathbb{P}^1 \times \mathbb{P}^1$ is a finite map, $\hat{X}$ having normal singularities whose links are $\mathbb{Q}$-homology spheres.

$\hat{\Phi} : (\hat{X}, p) \to (\mathbb{P}^1 \times \mathbb{P}^1, (\infty, \infty)) \Rightarrow A_{+,+} = B_{+,+}$.

$\hat{\Phi} : (\hat{X}, p_k) \to (\mathbb{P}^1 \times \mathbb{P}^1, (\infty, 0)) \Rightarrow A_{+,-} = B_{+,-}$.

$\hat{\Phi} : (\hat{X}, q_l) \to (\mathbb{P}^1 \times \mathbb{P}^1, (0, \infty)) \Rightarrow A_{-,+} = B_{-,+}$.

## Consequences

- If $D \subset \hat{X} \setminus \mathbb{C}^2$ irreducible is bad $f$-dicritic and $g$-infinity, $\mathcal{J} \neq \emptyset$.
- $S$ is union of $\Phi(D)$, $D$ $(f, g)$-dicritic $\Rightarrow$ $A_{+,+}$, using multiplicities of $f_{|D}$ and $g_{|D}$. In Moh examples (1983), jacobian pairs are *hidden*.
- If $f$ is a bad field generator, it cannot give a counterexample to the Jacobian Conjecture.