Galois-conjugate line arrangements with non-isomorphic fundamental group

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Computational Geometric Topology in Arrangement Theory
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Joint work with J.I. Cogolludo, B. Guerville-Ballé and M. Marco

Combinatorics and Topology

Definition

Combinatorics: $\mathscr{C} := (\mathcal{L}, \mathcal{P})$, $\mathcal{L}$ finite set of *lines* and $\mathcal{P} \subset \{P \subset \mathcal{L} \mid \#P = 2\}$ finite set of *points* mimic arrangement of lines and multiple points.

Definition (Realization of $\mathscr{C}$)

$\mathcal{A}$ line arrangement in $\mathbb{P}^2$: $(\mathcal{A}, \{\text{multiple points}\}) \leftrightarrow (\mathcal{L}, \mathcal{P})$

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Combinatorial objects

- $\mathbb{Z}^{\mathcal{L}} = \bigoplus_{L \in \mathcal{L}} \mathbb{Z} x_L$, $\dfrac{\mathbb{Z}^{\mathcal{L}}}{\mathbb{Z}\left(\sum_{L \in \mathcal{L}} x_L\right)} =: H_1^{\mathscr{C}} \cong H_1(\mathbb{P}^2 \setminus \mathcal{A}; \mathbb{Z})$

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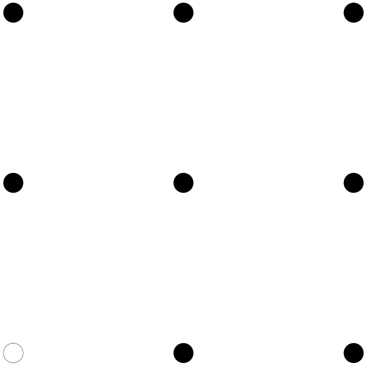
- $H^0_{\mathscr{C}} = \mathbb{Z}$, $H^j_{\mathscr{C}} \cong H^j(\mathbb{P}^2 \setminus \mathcal{A}; \mathbb{Z})$ dual of $H_j^{\mathscr{C}}$, $j = 1, 2$.

McLane arrangements I



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$\operatorname{Aut} \mathscr{ML} = \mathrm{GL}(2, \mathbb{F}_3)$

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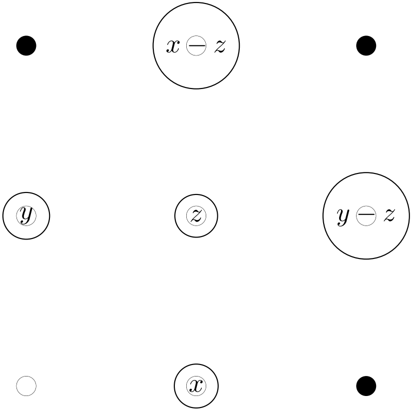

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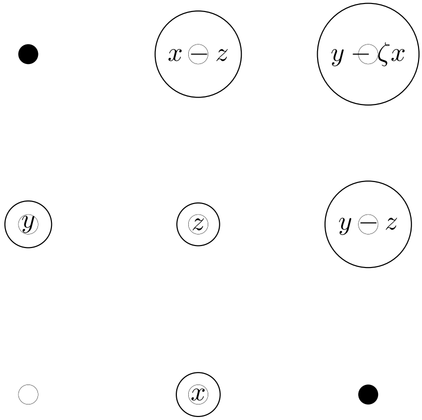

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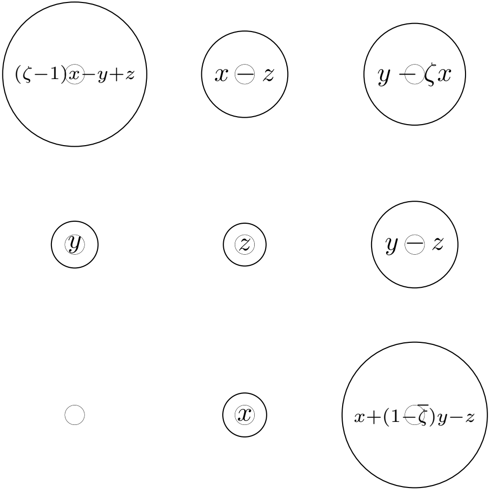

$$\zeta^2+\zeta+1=0$$

$$\mathscr{ML}_{\pm}$$

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McLane arrangements II

Theorem (Rybnikov)

$\nexists \varphi : \pi_1(\mathbb{P}^2 \setminus \mathscr{ML}_+) \to \pi_1(\mathbb{P}^2 \setminus \mathscr{ML}_-)$ group automorphism inducing the identity on homology.

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Order

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Rybnikov's combinatorics

$\mathscr{RB} = \mathscr{ML}_1 \cup_{xz(x-z)=0} \mathscr{ML}_2$ (gluing *in general position*)

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Guidelines of the proof.

Assume they are isomorphic $\Longrightarrow G_{++}/\gamma_4(G_{++}) \cong G_{+-}/\gamma_4(G_{+-})$

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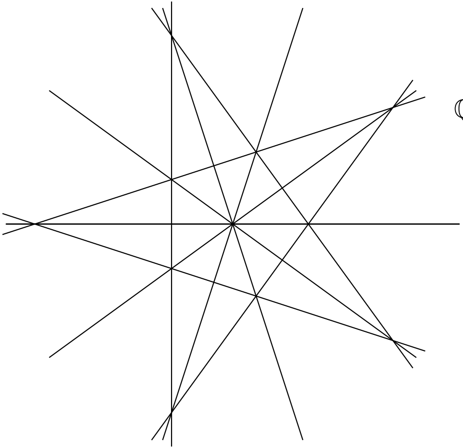
2. It does not happen using *truncated Alexander invariant.*

$\square$

Pentagon and Pentagram



$\mathscr{C}_0$

$\mathbb{Q}\left[\sqrt{5}\right]$

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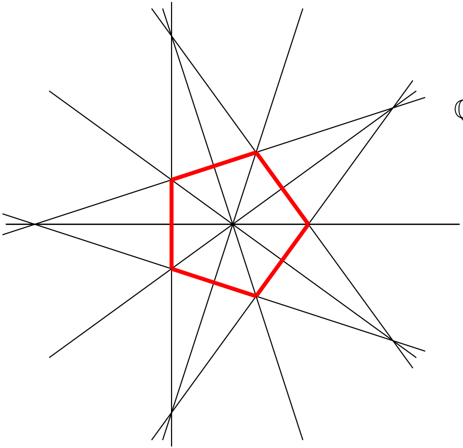

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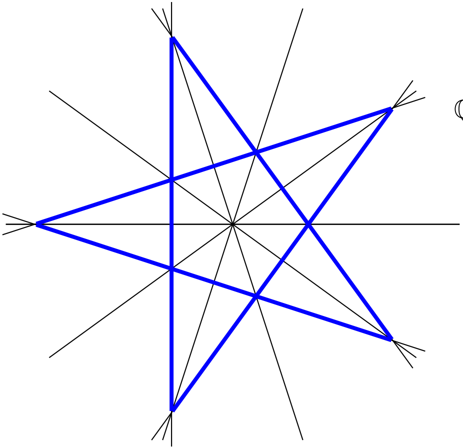

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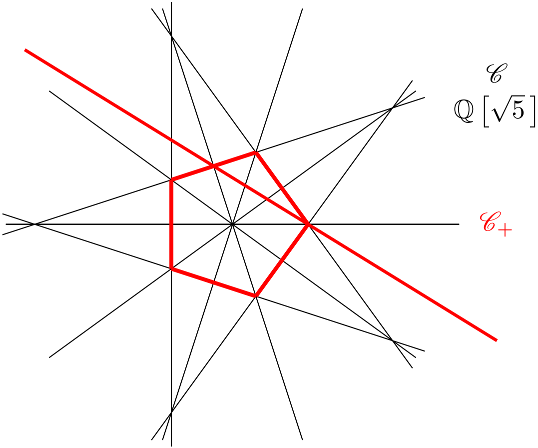

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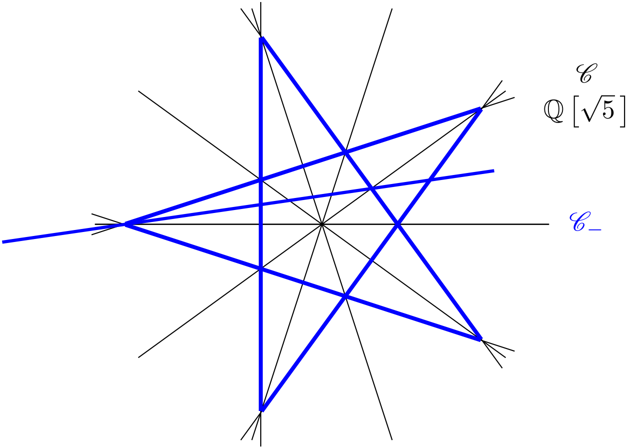

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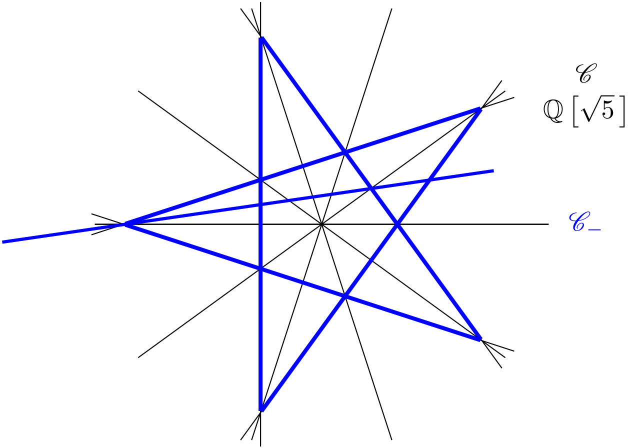

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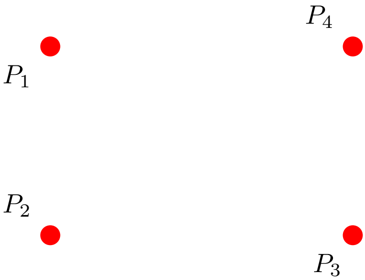
$\mathscr{G}_{91}$ combinatorics



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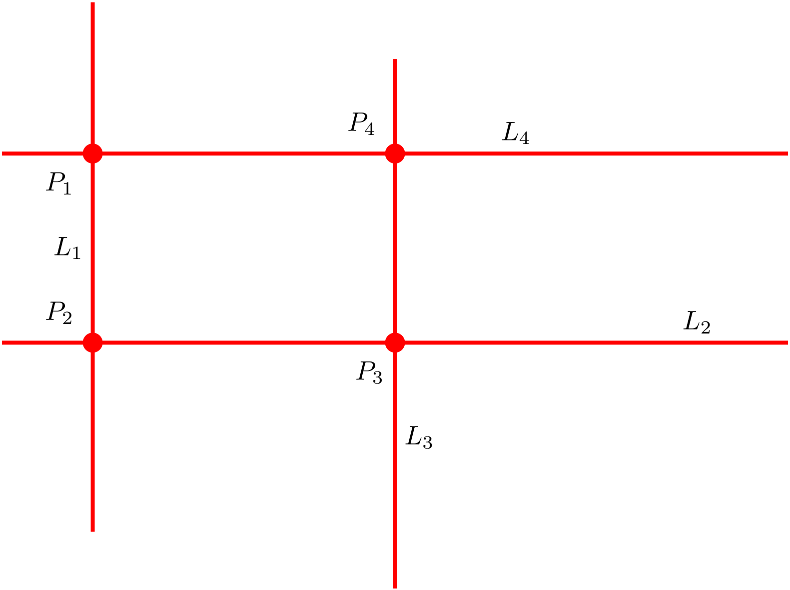

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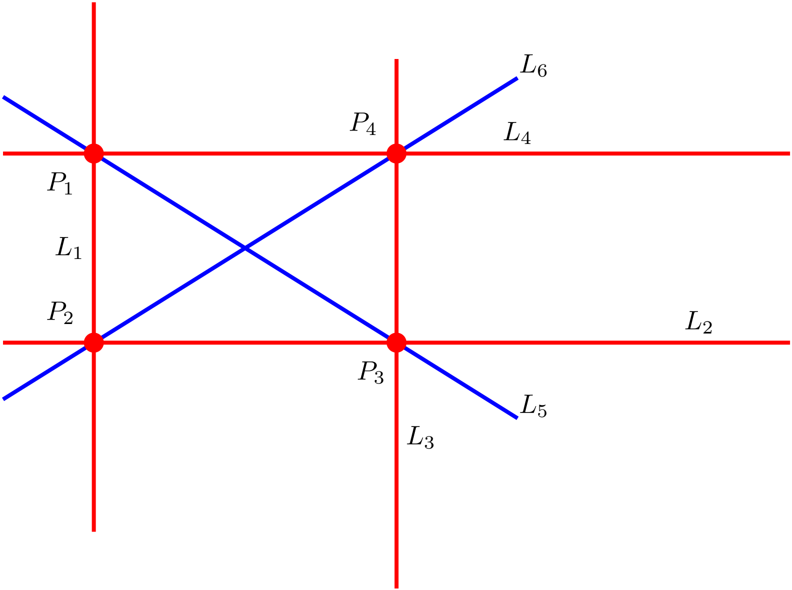

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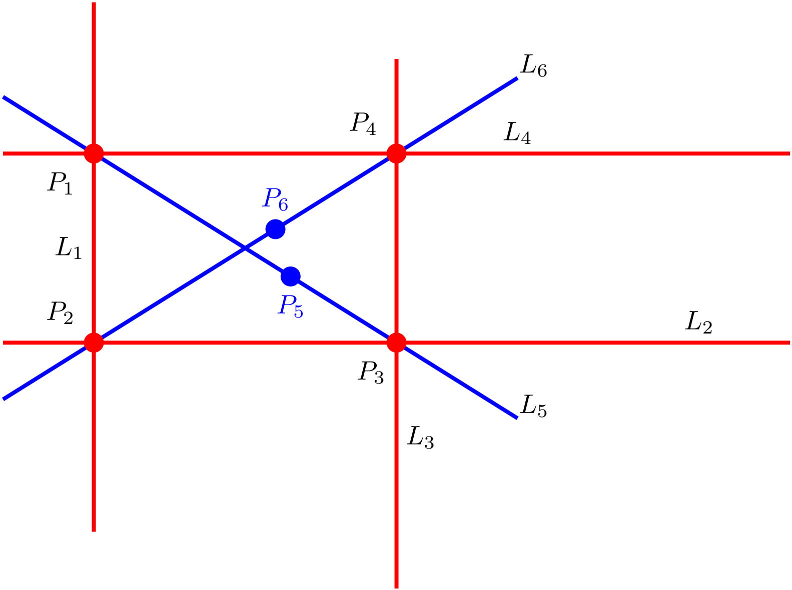

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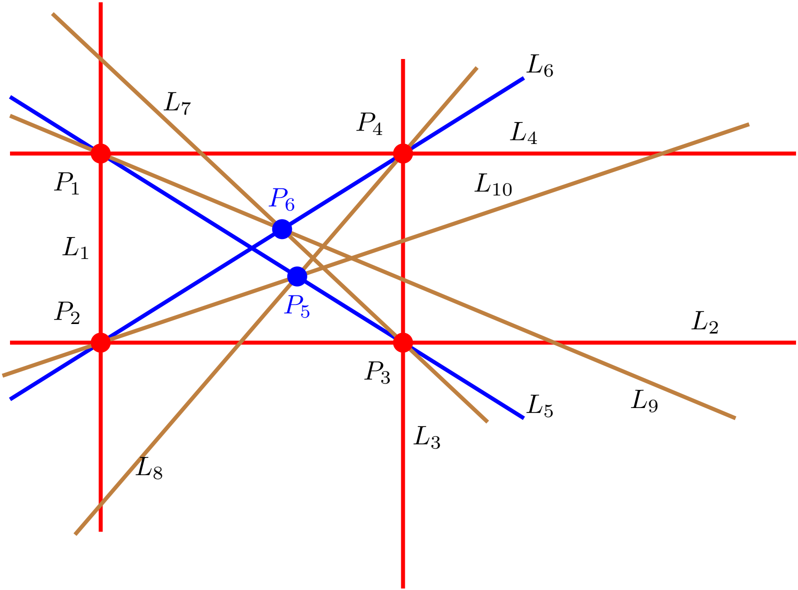

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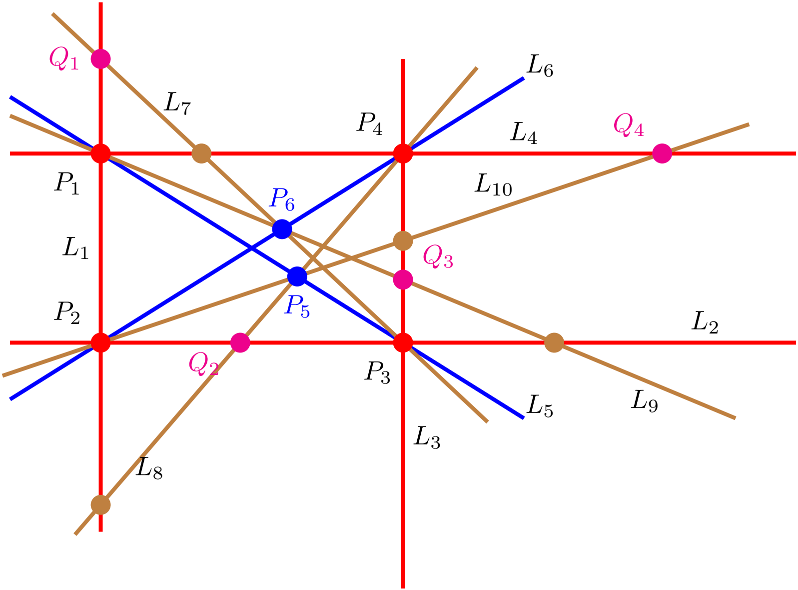

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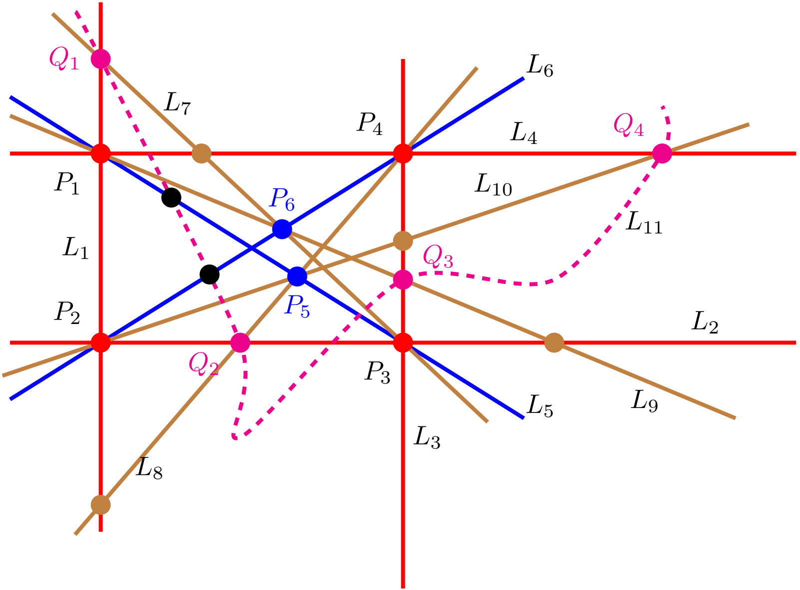

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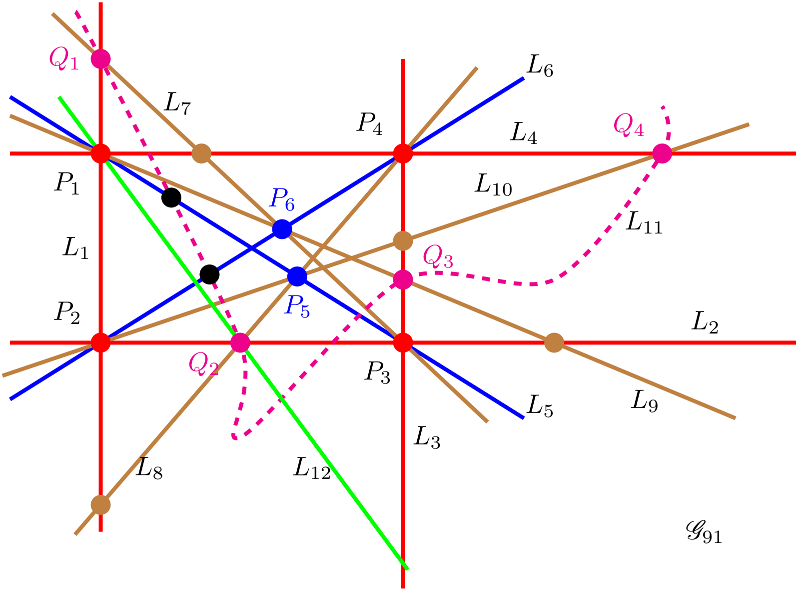

Guerville's example

Theorem

$\mathscr{G}_{91}$ admits four (Galois-conjugate) realizations $\mathcal{A}_\zeta$ with equations in the cyclotomic field $\mathbb{K}_5$, for ζ a primitive fifth root of unity.

There is no oriented homeomorphism $(\mathbb{P}^2, \mathcal{A}_{\zeta_1}) \to (\mathbb{P}^2, \mathcal{A}_{\zeta_2})$ if $\zeta_1 \neq \zeta_2$.

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2. It does not give so much information about the complement (need extra info)

Main result I

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- Purely combinatorial statement.

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Triangle

S_1, S_2, S_3 combinatorial pencils such that

$$\operatorname{codim} \bigcap_i H_{S_i} = \sum_i \operatorname{codim} H_{S_i} - 1.$$

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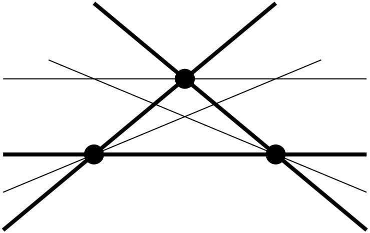

Triangles in $\mathscr{G}_{91}$

i	S_i	dim H_S	Δ_S	Δ_{S,P_1}
1	1, 7, 11	2	18	7
2	3, 9, 11	2	22	8
3	4, 10, 11	2	21	7
4	5, 8, 10	2	24	7
5	6, 9, 7	2	16	6
6	1, 2, 6, 10	3	53	12
7	2, 3, 5, 7	3	49	13
8	2, 8, 11, 12	3	57	15
9	4, 3, 6, 8	3	50	12
10	1, 4, 5, 9, 12	4	91	91
11	1, 2, 3, 4, 5, 6	2	24	8
12	1, 2, 4, 6, 8, 12	2	24	8
13	1, 2, 4, 10, 11, 12	2	20	7
14	1, 2, 5, 6, 7, 9	2	14	7
15	1, 2, 5, 7, 11, 12	2	14	7
16	1, 2, 5, 8, 10, 12	2	20	8
17	1, 3, 5, 7, 9, 11	2	14	7
18	1, 4, 5, 6, 8, 10	2	19	6
19	2, 3, 4, 5, 8, 12	2	20	8
20	2, 3, 5, 6, 8, 10	2	14	0
21	2, 3, 5, 9, 11, 12	2	18	9
22	2, 4, 6, 8, 10, 11	2	15	0
23	3, 4, 5, 6, 7, 9	2	12	6
24	3, 4, 8, 9, 11, 12	2	13	7
25	4, 5, 8, 10, 11, 12	2	15	7

Triangles in $\mathscr{G}_{91}$

i	S_i	dim H_S	Δ_S	Δ_{S,P_1}
1	1, 7, 11	2	18	7
2	3, 9, 11	2	22	8
3	4, 10, 11	2	21	7
4	5, 8, 10	2	24	7
5	6, 9, 7	2	16	6
6	1, 2, 6, 10	3	53	12
7	2, 3, 5, 7	3	49	13
8	2, 8, 11, 12	3	57	15
9	4, 3, 6, 8	3	50	12
10	1, 4, 5, 9, 12	4	91	91
11	1, 2, 3, 4, 5, 6	2	24	8
12	1, 2, 4, 6, 8, 12	2	24	8
13	1, 2, 4, 10, 11, 12	2	20	7
14	1, 2, 5, 6, 7, 9	2	14	7
15	1, 2, 5, 7, 11, 12	2	14	7
16	1, 2, 5, 8, 10, 12	2	20	8
17	1, 3, 5, 7, 9, 11	2	14	7
18	1, 4, 5, 6, 8, 10	2	19	6
19	2, 3, 4, 5, 8, 12	2	20	8
20	2, 3, 5, 6, 8, 10	2	14	0
21	2, 3, 5, 9, 11, 12	2	18	9
22	2, 4, 6, 8, 10, 11	2	15	0
23	3, 4, 5, 6, 7, 9	2	12	6
24	3, 4, 8, 9, 11, 12	2	13	7
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Main result II

Theorem

The groups $\pi_1(\mathbb{P}^2 \setminus \mathcal{A}_\zeta)$ *and* $\pi_1(\mathbb{P}^2 \setminus \mathcal{A}_{\zeta^2})$ *are not isomorphic (while their profinite completions are).*

First step

$\varphi : \pi_1(\mathbb{P}^2 \setminus \mathcal{A}_\zeta) \to \pi_1(\mathbb{P}^2 \setminus \mathcal{A}_{\zeta^2})$ isomorphism $\Longrightarrow \varphi_* = \pm 1_{H_1^{\mathcal{G}_{91}}}$.

Second step

There is no isomorphism such that $\varphi : \pi_1(\mathbb{P}^2 \setminus \mathcal{A}_\zeta) \to \pi_1(\mathbb{P}^2 \setminus \mathcal{A}_{\zeta^2})$ isomorphism $\Longrightarrow \varphi_* = 1_{H_1^{\mathcal{G}_{91}}}$

Truncated Alexander Invariant

- $\mathscr{C}$ combinatorics, $\mathcal{A}$ realization, $G_{\mathcal{A}} := \pi_1(\mathbb{P}^2 \setminus \mathcal{A})$, $\Lambda := \mathbb{Z}[H_1^{\mathscr{C}}]$

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  - Jacobi relations:$(t_i - 1)x_{j,k} + (t_j - 1)x_{k,i} + (t_k - 1)x_{i,j}$

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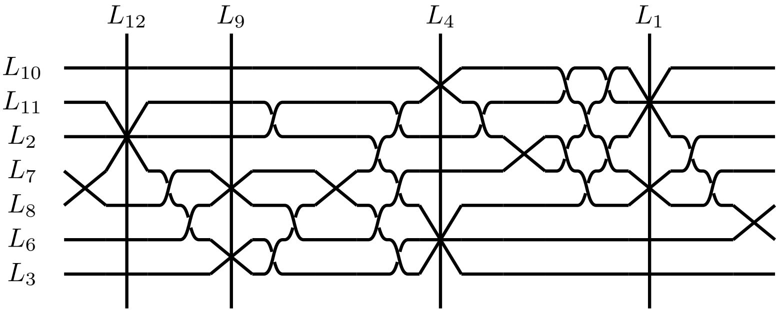
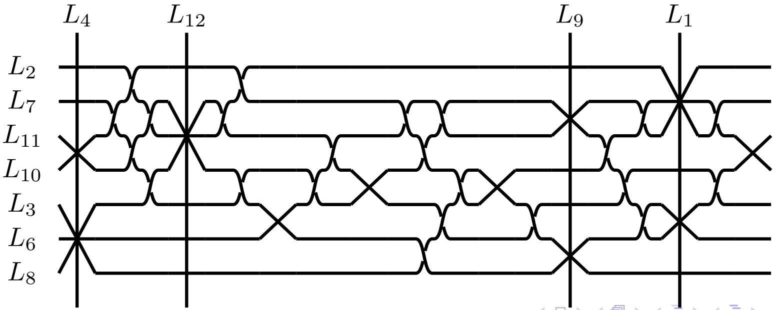
- $g \in H_1$ and $p \in M^k_{\mathcal{A}} \Longrightarrow [g, p] \in M^{k+1}_{\mathcal{A}}$.

Wiring diagrams





Steps of the proof

- Isomorphism $\varphi : G_{\mathcal{A}_\zeta} \to G_{\mathcal{A}_{\zeta^2}}$, $x_i \mapsto x_i g_i$, $g_i \in G'_{\mathcal{A}_{\zeta^2}}$

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$$g_i \equiv \sum_{(j,k)\in\mathcal{B}} n_{i,j,k} x_{j,k} \in M^1_{\mathcal{A}_{\zeta^2}}, \quad n_{i,j,k} \in \mathbb{Z}$$

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- Existence of φ implies integer solutions of a system of $32 \times 91 = 2912$ linear equations in $11 \times 23 = 253$ unknowns.

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- Existence of φ implies integer solutions of a system of $32 \times 91 = 2912$ linear equations in $11 \times 23 = 253$ unknowns.
- Find solutions with `Sagemath`: $\mathbb{Q}$-affine space dim $= 12$, smallest ring of solutions is $\mathbb{Z}\left[\frac{1}{5}\right]$.

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- Whole process 662.49s CPU time.

Main result III

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Third step

There is no isomorphism such that $\varphi : \pi_1(\mathbb{P}^2 \setminus \mathcal{A}_\zeta) \to \pi_1(\mathbb{P}^2 \setminus \mathcal{A}_{\zeta^3})$ isomorphism $\Longrightarrow \varphi_* = 1_{H_1^{\mathscr{G}_{91}}}$