Zariski pairs and line arrangements

Enrique ARTAL BARTOLO

Departamento de Matemáticas
Facultad de Ciencias
Instituto Universitario de Matemáticas y sus Aplicaciones
Universidad de Zaragoza

GEOMETRY, TOPOLOGY AND COMBINATORICS OF HYPERPLANE ARRANGEMENTS AND RELATED PROBLEMS
Zaragoza, September 14th-18th 2015

Works

- Florens, Guerville-Ballé, Marco-Buzunáriz
- Guerville-Ballé
- __, Florens, Guerville-Ballé
- __, Cogolludo-Agustín, Guerville-Ballé, Marco-Buzunáriz

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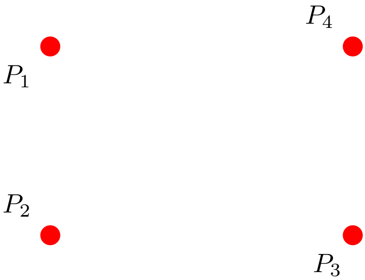
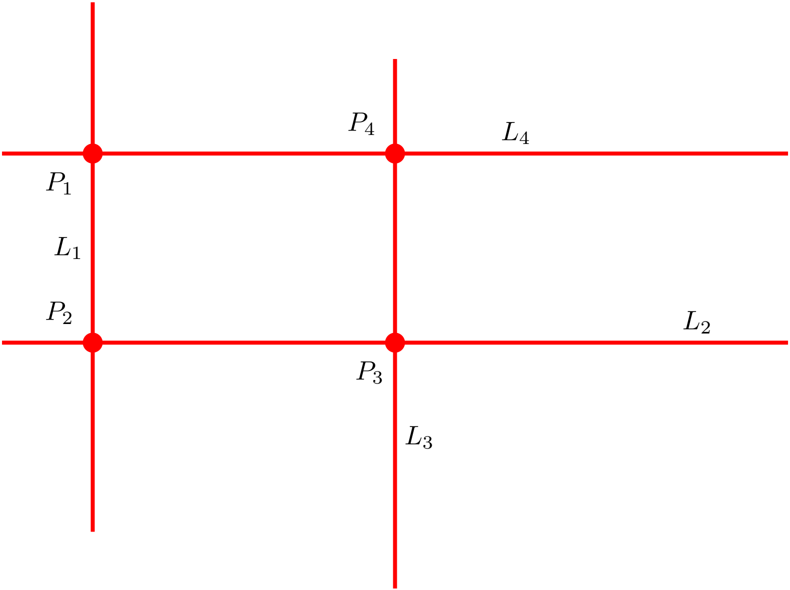
Notions of Zariski pairs.

$\mathscr{G}_{91}$ combinatorics



$\mathscr{G}_{91}$ combinatorics



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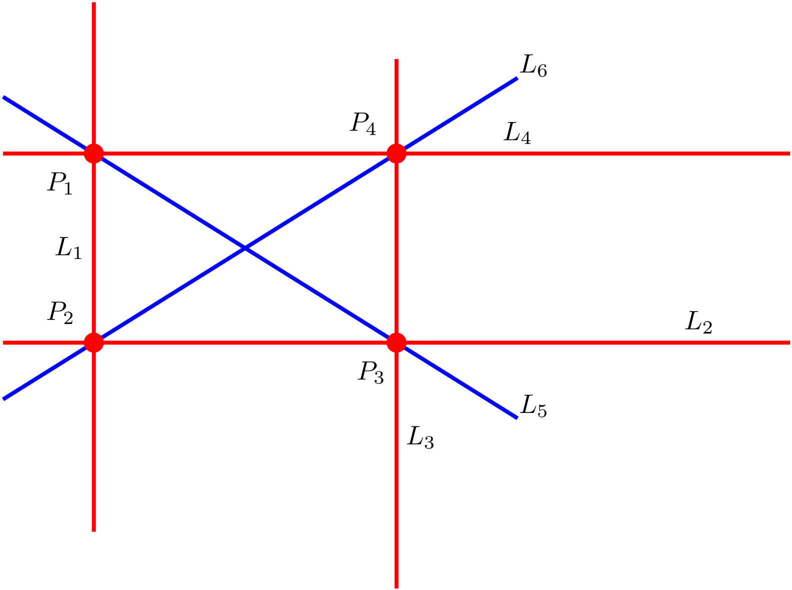

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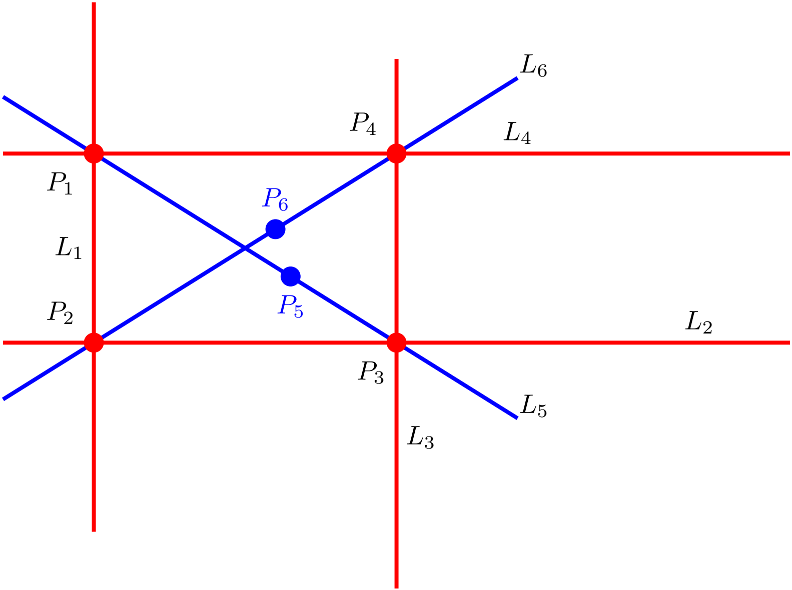
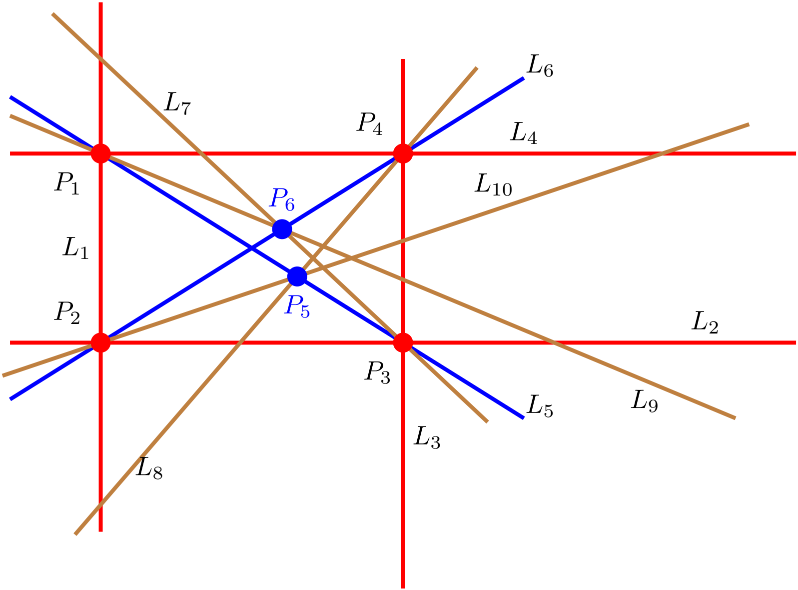


$\mathscr{G}_{91}$ combinatorics



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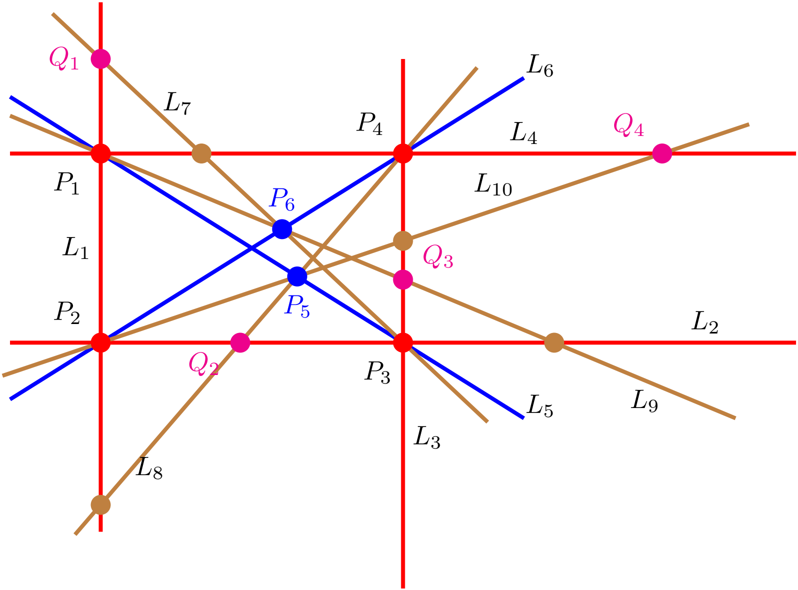

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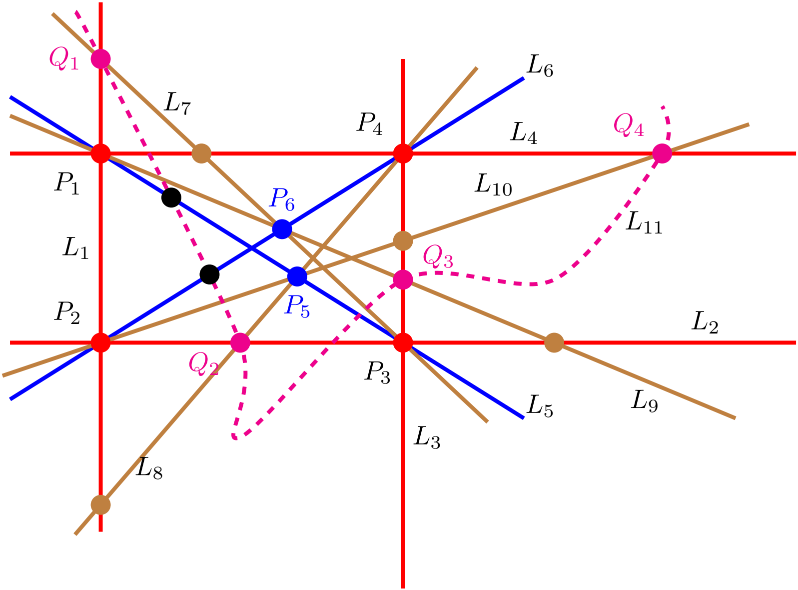

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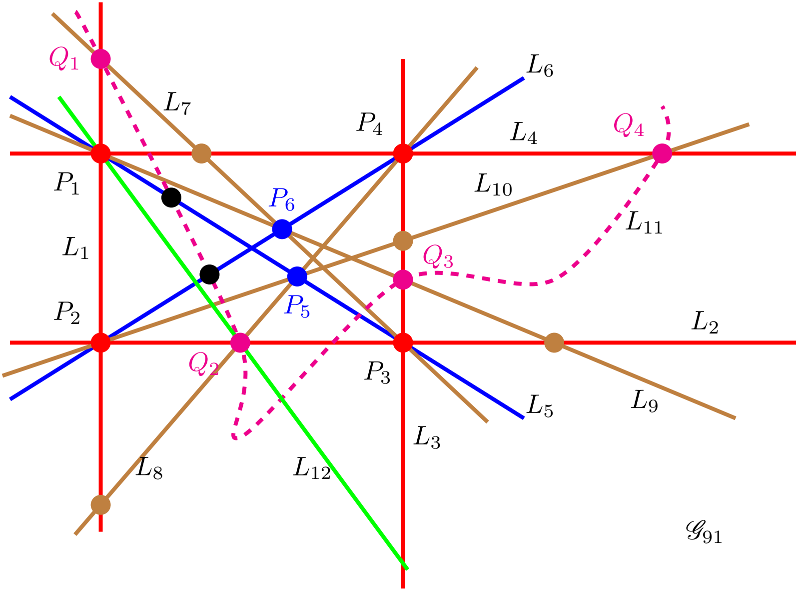
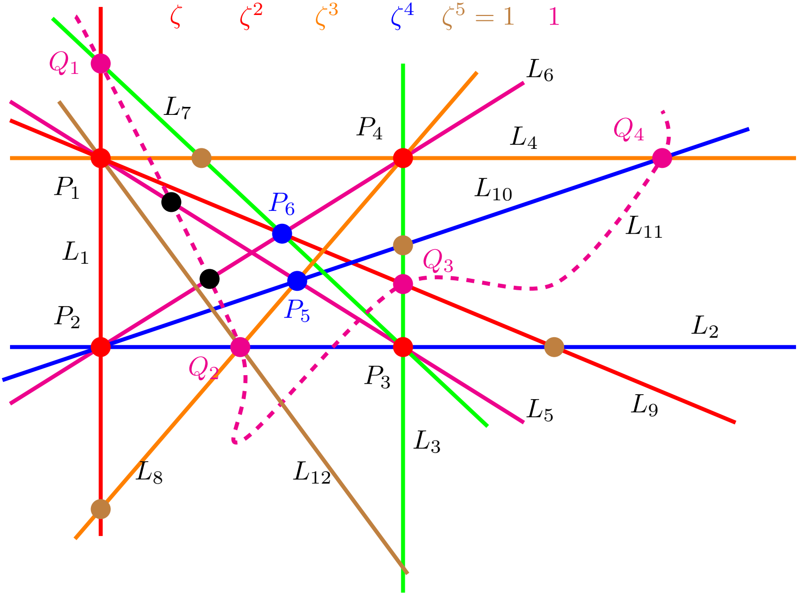
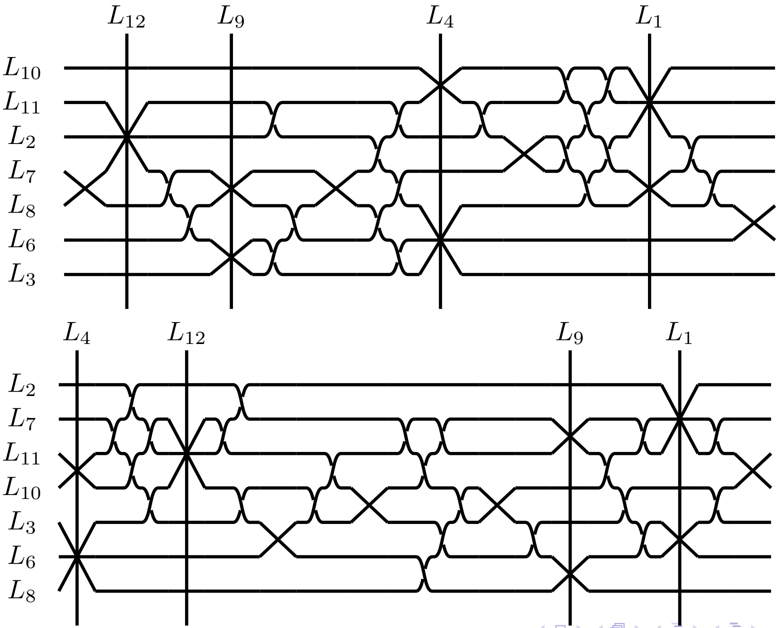


Character



ζ
ζ^2
ζ^3
ζ^4
$\zeta^5 = 1$
1
Q_1
L_6
L_7
P_4
L_4
Q_4
P_1
L_{10}
P_6
L_{11}
L_1
Q_3
P_5
P_2
L_2
Q_2
P_3
L_5
L_9
L_3
L_8
L_{12}

Wiring diagrams



Guerville

Theorem

$\mathcal{G}_{91}$ admits four (Galois-conjugate) realizations $\mathcal{A}_\zeta$ with equations in the cyclotomic field $\mathbb{K}_5$, for $\zeta^5 = 1$, $\zeta \neq 1$.
There is no oriented homeomorphism $(\mathbb{P}^2, \mathcal{A}_{\zeta_1}) \to (\mathbb{P}^2, \mathcal{A}_{\zeta_2})$ if $\zeta_1 \neq \zeta_2$.

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Corollary

There is no homeomorphism $(\mathbb{P}^2, \mathcal{A}_\zeta) \to (\mathbb{P}^2, \mathcal{A}_{\zeta^2})$.

Fundamental group

Theorem

The groups $\pi_1(\mathbb{P}^2 \setminus \mathcal{A}_\zeta)$ *and* $\pi_1(\mathbb{P}^2 \setminus \mathcal{A}_{\zeta^2})$ *are not isomorphic (while their profinite completions are).*

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First step

$\varphi : \pi_1(\mathbb{P}^2 \setminus \mathcal{A}_\zeta) \to \pi_1(\mathbb{P}^2 \setminus \mathcal{A}_{\zeta^2})$ isomorphism $\Longrightarrow \varphi_* = \pm 1_{H_1^{\mathscr{G}_{91}}}$.

Combinatorial pencils

i	S_i	$\dim H_S$	Δ_S	Δ_{S,P_1}
1	1, 7, 11	2	18	7
2	3, 9, 11	2	22	8
3	4, 10, 11	2	21	7
4	5, 8, 10	2	24	7
5	6, 9, 7	2	16	6
6	1, 2, 6, 10	3	53	12
7	2, 3, 5, 7	3	49	13
8	2, 8, 11, 12	3	57	15
9	4, 3, 6, 8	3	50	12
10	1, 4, 5, 9, 12	4	91	91
11	1, 2, 3, 4, 5, 6	2	24	8
12	1, 2, 4, 6, 8, 12	2	24	8
13	1, 2, 4, 10, 11, 12	2	20	7
14	1, 2, 5, 6, 7, 9	2	14	7
15	1, 2, 5, 7, 11, 12	2	14	7
16	1, 2, 5, 8, 10, 12	2	20	8
17	1, 3, 5, 7, 9, 11	2	14	7
18	1, 4, 5, 6, 8, 10	2	19	6
19	2, 3, 4, 5, 8, 12	2	20	8
20	2, 3, 5, 6, 8, 10	2	14	0
21	2, 3, 5, 9, 11, 12	2	18	9
22	2, 4, 6, 8, 10, 11	2	15	0
23	3, 4, 5, 6, 7, 9	2	12	6
24	3, 4, 8, 9, 11, 12	2	13	7
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Main result II

Theorem

The groups $\pi_1(\mathbb{P}^2 \setminus \mathcal{A}_\zeta)$ *and* $\pi_1(\mathbb{P}^2 \setminus \mathcal{A}_{\zeta^2})$ *are not isomorphic (while their profinite completions are).*

First step

$\varphi : \pi_1(\mathbb{P}^2 \setminus \mathcal{A}_\zeta) \to \pi_1(\mathbb{P}^2 \setminus \mathcal{A}_{\zeta^2})$ isomorphism $\Longrightarrow \varphi_* = \pm 1_{H_1^{\mathscr{G}_{91}}}$.

Second step

There is no isomorphism such that $\varphi : \pi_1(\mathbb{P}^2 \setminus \mathcal{A}_\zeta) \to \pi_1(\mathbb{P}^2 \setminus \mathcal{A}_{\zeta^2})$ isomorphism $\Longrightarrow \varphi_* = 1_{H_1^{\mathscr{G}_{91}}}$

Truncated Alexander Invariant

- $\mathscr{C}$ combinatorics, $\mathcal{A}$ realization, $G_{\mathcal{A}} := \pi_1(\mathbb{P}^2 \setminus \mathcal{A})$, $\Lambda := \mathbb{Z}[H_1^{\mathscr{C}}]$

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- $\mathscr{C}$ combinatorics, $\mathcal{A}$ realization, $G_{\mathcal{A}} := \pi_1(\mathbb{P}^2 \setminus \mathcal{A})$, $\Lambda := \mathbb{Z}[H_1^{\mathscr{C}}]$
- $M_{\mathcal{A}} := G'_{\mathcal{A}}/G''_{\mathcal{A}}$ as Λ-module is the *Alexander invariant*.

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- $\mathcal{A} = \{L_0, L_1, \ldots, L_\ell\}$, $G_{\mathcal{A}} = \langle x_1, \ldots, x_\ell \mid R_1, \ldots, R_s \rangle$, $\Lambda = \mathbb{Z}\left[t_i^{\pm 1}\right]$

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- $g \in H_1$ and $p \in M_{\mathcal{A}}^k \Longrightarrow [g, p] \in M_{\mathcal{A}}^{k+1}$.

Steps of the proof

- Isomorphism $\varphi : G_{\mathcal{A}_\zeta} \to G_{\mathcal{A}_{\zeta^2}}$, $x_i \mapsto x_i g_i$, $g_i \in G'_{\mathcal{A}_{\zeta^2}}$

Steps of the proof

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- $\operatorname{rk} M^1_{\mathcal{A}_\zeta} = \operatorname{rk} M^1_{\mathcal{A}_{\zeta^2}} = \operatorname{rk} \operatorname{gr}^0 M_2^{\mathscr{G}_{91}} = 23$, basis $\{x_{i,j} \mid (i,j) \in \mathcal{B}\}$.

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$$\varphi_*(R_i) \in \operatorname{gr}^1 M_{\mathcal{A}_{\zeta^2}} \otimes \mathbb{Z}[n_{i,j,k}], \quad \operatorname{rk} \operatorname{gr}^1 M_2^{\mathcal{G}_{91}} \cong \mathbb{Z}^{91}$$

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- Existence of φ implies integer solutions of a system of $32 \times 91 = 2912$ linear equations in $11 \times 23 = 253$ unknowns.

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- Existence of φ implies integer solutions of a system of $32 \times 91 = 2912$ linear equations in $11 \times 23 = 253$ unknowns.
- Find solutions with `Sagemath`: $\mathbb{Q}$-affine space dim $= 12$, smallest ring of solutions is $\mathbb{Z}\left[\frac{1}{5}\right]$.

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- R_i, $i = 1, \dots, 32$ relation of $G_{\mathcal{A}_\zeta}$ rewritten in $M^2_{\mathcal{A}_\zeta}$.
- $\varphi_*(R_i) \in M^2_{\mathcal{A}_{\zeta^2}} \otimes \mathbb{Z}[n_{i,j,k}]$, more precisely

$$\varphi_*(R_i) \in \operatorname{gr}^1 M_{\mathcal{A}_{\zeta^2}} \otimes \mathbb{Z}[n_{i,j,k}], \quad \operatorname{rk}\operatorname{gr}^1 M_2^{\mathscr{G}_{91}} \cong \mathbb{Z}^{91}$$

- Existence of φ implies integer solutions of a system of $32 \times 91 = 2912$ linear equations in $11 \times 23 = 253$ unknowns.
- Find solutions with `Sagemath`: $\mathbb{Q}$-affine space dim $= 12$, smallest ring of solutions is $\mathbb{Z}\left[\frac{1}{5}\right]$.
- Whole process 662.49s CPU time.

Main result III

Theorem

The groups $\pi_1(\mathbb{P}^2 \setminus \mathcal{A}_\zeta)$ *and* $\pi_1(\mathbb{P}^2 \setminus \mathcal{A}_{\zeta^2})$ *are not isomorphic (while their profinite completions are).*

First step

$\varphi : \pi_1(\mathbb{P}^2 \setminus \mathcal{A}_\zeta) \to \pi_1(\mathbb{P}^2 \setminus \mathcal{A}_{\zeta^2})$ isomorphism $\Longrightarrow \varphi_* = \pm 1_{H_1^{\mathscr{G}_{91}}}$.

Second step

There is no isomorphism such that $\varphi : \pi_1(\mathbb{P}^2 \setminus \mathcal{A}_\zeta) \to \pi_1(\mathbb{P}^2 \setminus \mathcal{A}_{\zeta^2})$ isomorphism $\Longrightarrow \varphi_* = 1_{H_1^{\mathscr{G}_{91}}}$

Third step

There is no isomorphism such that $\varphi : \pi_1(\mathbb{P}^2 \setminus \mathcal{A}_\zeta) \to \pi_1(\mathbb{P}^2 \setminus \mathcal{A}_{\zeta^3})$ isomorphism $\Longrightarrow \varphi_* = 1_{H_1^{\mathscr{G}_{91}}}$