Orbifolds and fundamental groups of plane curves

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Main Problem

Check if a (f.p. non-abelian) group is a quasiprojective or curve group. Relate the properties of G and those of X or H.

Orbifolds

Definition

An *orbifold* is a pair (C, φ) where C is a quasiprojective curve and $\varphi : C \to \mathbb{Z}_{>0}$ is a map such that $\#M_\varphi < \infty$ where $M_\varphi := \{p \in C \mid \varphi(p) > 1\}$.

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Remark

The meridians of D_1 in X form a conjugacy class of $\pi_1(X; p)$. Moreover, the inclusion

$$i : X \hookrightarrow X_1 := \bar{X} \setminus (D - D_1)$$

induces an epimorphism

$$i_* : \pi_1(X; p) \twoheadrightarrow \pi_1(X_1; p)$$

such that $\ker i_*$ is the normal subgroup generated by the meridians of D_1 in X.

Orbifold groups and orbifold maps

Definition

Let (C,φ) be an orbifold and let $p \in C \setminus M_\varphi$. The *orbifold fundamental group* $\pi_1^{\text{orb}}(C,\varphi;p)$ is the quotient of $\pi_1(C \setminus M_\varphi;p)$ by the normal subgroup generated by $\mu_q^{\varphi(q)}$, where μ_q is a meridian of q in $C \setminus M_\varphi$ and $q \in M_\varphi$.

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Proposition

Let $\Psi : X \to C$ be an algebraic morphism inducing an orbifold morphism for (C,φ). Then the mapping on fundamental groups factors through $\Psi_* : \pi_1(X) \to \pi_1^{orb}(C,\varphi)$. Moreover, if Ψ is primitive (connected generic fibers) then Ψ_* is surjective.

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- For the last assertion, note that $(\Psi_1)_*$ is surjective for primitive morphisms.

Examples of orbifold groups

Notation

Since the position of the points in M_φ is not important for $\pi_1^{\text{orb}}(C,\varphi)$ we will denote $(C,\varphi) = C_{m_1,\dots,m_n}$, where $m_1 \geq \ldots \geq m_n$, $M_\varphi = \{p_1,\dots,p_n\}$ and $\varphi(p_i) = m_i$.

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Example

Let $C^{g,n}$ be the complement of n points in a projective curve of genus g. Then, $\pi_1^{\text{orb}}(C^{g,n}_{m_1,\ldots,m_r})$ is generated by $a_1,\ldots,a_g$, $b_1,\ldots,b_g$, $x_1,\ldots,x_n$, $y_1,\ldots,y_r$ with the relations

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Orbifolds and non-abelian groups I

Example

The simplest projective plane curve with non-abelian fundamental group is the union of three concurrent lines L_i, $i = 1, 2, 3$. The pencil of lines through the intersection point defines a mapping $\Psi : \mathbb{P}^2 \setminus \bigcup_{i=1}^{3} L_i \to C^{0,3}$, which induces an isomorphism for fundamental groups (the free group $\mathbb{F}_2$).

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Zariski's hexacuspidal sextic curves are defined by an equation $f_2^3 + f_3^2 = 0$, f_j *generic* homogeneous polynomial of degree j. The pencil generated by the non-reduced curves defined by f_2^3 and f_3^2 define an orbifold morphism of the complement of such a sextic onto $C_{2,3}^{0,1}$. The morphism for the fundamental groups is an isomorphism (the groups are isomorphic to $\mathbb{Z}/2 * \mathbb{Z}/3$). This example generalizes to curves of equation $f_p^q + f_q^p = 0$.

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Example

Up to degree 4, there is only one irreducible curve with non-abelian fundamental group. It is the tricuspidal quartic and Zariski proved that its fundamental group is non-abelian of order 12. Let us assume that its equation is $f_4 = 0$. Take the tangent line to any smooth point and assume its equation is $f_1 = 0$. Then there exists a conic $f_2 = 0$ and a nodal cubic $f_3 = 0$ such that $f_2^3 + f_3^2 = f_1^2 f_4$. The pencil generated by the non-reduced curves defined by f_2^3 and f_3^2 define a morphism of the complement of the quartic onto $C^{0,0} = \mathbb{P}^1$ which induces an orbifold morphism onto $C^{0,0}_{2,2,3}$. In this case, the morphism for the fundamental groups is only an epimorphism since $\pi_1^{\text{orb}}(C^{0,0}_{2,2,3})$ is the dihedral group of order 6.

Orbifolds and non-abelian groups II

Example

The relationship between pencils and non-abelian fundamental groups goes in both directions. If $\pi_1(\mathbb{P}^2 \setminus C)$ admits an epimorphism onto $\mathbb{D}_{2n}$, odd n, then there is a dominant morphism of $\mathbb{P}^2 \setminus C$ into $C^{0,0}_{2,2,n}$. Moreover, if many enough of such morphisms exist (with a compatibility condition) then a morphism of $\mathbb{P}^2 \setminus C$ into $C^{0,1}_{2,2}$ and in particular an epimorphism onto $\mathbb{Z}/2 * \mathbb{Z}/2$.

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Example

In degree 4 there are only four examples of reducible curves (which are not line arrangements) with non-abelian fundamental group. One of them corresponds to the semicubic parabola, with projective equation $(y^2z - x^3)z = 0$. The complement admits an orbifold morphism onto $C^{0,1}_{2,3}$; the pencil is generated by $y^2z = 0$ and $x^3 = 0$.
The other examples are related with pencil of conics with non-reduced fibers: two bi-tangent smooth conics, two smooth conics with one intersection point and a smooth conic with two tangent lines. The pencil generated by $xy = 0$ and $z^2 = 0$ induces an orbifold morphism onto $C^{0,2}_{2}$, giving an epimorphism onto $\mathbb{Z} * \mathbb{Z}/2$ (which is an isomorphism for the two first cases).

Triangle groups

Example

There are only two irreducible quintics with non-abelian fundamental group. Let us consider one of them. It is the dual curve of a quartic with an $\mathbb{A}_6$-point and it has four singular points: one of type $\mathbb{A}_6$ and three ordinary cuspidal points.
Once a presentation of the fundamental group is found, one proves the non-abelianity of this group by the exhibition of an epimorphism onto the *triangle group of type* $(2,3,7)$ which is $\pi_1^{\text{orb}}(C_{2,3,7}^{0,0})$.
Let $f_5 = 0$ be the equation of this curve and let $f_1 = 0$ be the equation of the tangent line to the $\mathbb{A}_6$ point. Then, it is not hard to express $f_5 f_1^7 = f_6^2 + f_4^3$ for suitable f_4, f_6. This fact allows to prove that the fundamental group is non-abelian with no group computation.

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Question

Let C be a curve such that $\pi_1(\mathbb{P}^2 \setminus C)$ is non-abelian. Does there exist an orbifold morphism from $\mathbb{P}^2 \setminus C$ onto an orbifold with non-abelian fundamental group?

Degtyarev's curve

Example

There is a second irreducible quintic with non-abelian fundamental group. Let C_5 be the curve of equation:

$$21x^2y^3\left(5+2\sqrt{5}\right)+22zxy^3+4z^3y^2+4z^3x^2-11z^2y^3-11z^2x^3+11z^2y^2x-44zy^2$$
$$x^2+2zx^3y-8z^3yx+11z^2yx^2+z^2x^3\left(5+2\sqrt{5}\right)+21z^2y^3\left(5+2\sqrt{5}\right)-55\left(5+2\sqrt{5}\right)x^3y^2$$
$$+88zy^2x^2\left(5+2\sqrt{5}\right)-42zxy^3\left(5+2\sqrt{5}\right)-6zx^3y\left(5+2\sqrt{5}\right)-11x^2y^3+29x^3y^2$$
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Proposition

The fundamental group of the complement of this curve is finite of order 320 *(Degtyarev). There is no dominant map into an orbifold with non-abelian fundamental group*

Sketch of the proof

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- There is a small list of orbifolds with finite non-abelian fundamental group: $C^{0,0}_{2,2,n}$ (the dihedral group $\mathbb{D}_{2n}$), $C^{0,0}_{2,3,3}$ (of order 12), $C^{0,0}_{2,3,4}$ (of order 24) and $C^{0,0}_{2,3,5}$ (of order 60).

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- If a morphism onto such an orbifold exists there is an epimorphism $\pi_1(\mathbb{P}^2 \setminus C_5) \twoheadrightarrow \mathbb{D}_{10}$ and it is not the case.

Definitions

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Remark

It is a generalization of Alexander polynomial theory.

Quasiprojective groups

Let G be the fundamental group of a quasiprojective manifold X.

Theorem (Arapura)

Let Σ be an irreducible component of $V_1(G)$. Then,

- *If* $\dim \Sigma > 0$ *then there exists a surjective morphism* $\rho : X \to C$, *C algebraic curve, and a torsion element* σ *such that* $\Sigma = \sigma \rho^*(H^1(C; \mathbb{C}^*))$.
- *If* $\dim \Sigma = 0$ *then* Σ *is unitary.*

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But it is not true…

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Theorem (A., Cogolludo)

Let Σ be an irreducible component of $V_1(G)$. Then one of the two following statements holds:

- *There exists a surjective orbifold morphism $\rho : X \to (C,\varphi)$ and an irreducible component Σ_1 of $V_1(\pi_1^{orb}(C,\varphi))$ such that $\Sigma = \rho^*(\Sigma_1)$.*
- *Σ consists of a torsion point.*

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- In this case $H = \mathbb{Z}$, $T_H = \mathbb{C}^*$ and $V_1(G)$ consists of the primitive 10-roots of unity. In classical language, the Alexander polynomial is $t^4 - t^3 + t^2 - t + 1$.

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Let (C, φ) be an orbifold having the primitive 10-roots of unity as first characteristic variety. Then, (C, φ) has a dominant map in $C^{0,0}_{2,5,10}$.

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The derived short exact sequence is:

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Consequence

Let us consider the derived short exact sequence

$$1 \longrightarrow G' \longrightarrow G \xrightarrow{\sigma} \mathbb{Z} \longrightarrow 0$$

If there exists an orbifold morphism of $G := \mathbb{P}^2 \setminus (C_5 \cup L)$ then there is an epimorphism of $G_{10} := \sigma^{-1}(10\mathbb{Z})$ onto $\mathbb{F}_4$.

Final Facts

Structure of G

Let us consider the derived short exact sequence

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Final result

The curve $C_5 \cup L$ has non abelian fundamental group and non-trivial Alexander polynomial, but these facts are not induced by orbifold morphisms.

Thanks for your attention

Happy Birthday to Andrew