

Travelling around singular conjectures with Alejandro

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Jornada Singular en homenaje a Alejandro Melle
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"Álgebra y Geometría").



Melle and singularities

Melle's work



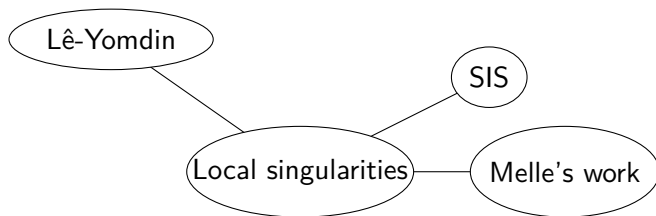
Melle and singularities

Local singularities

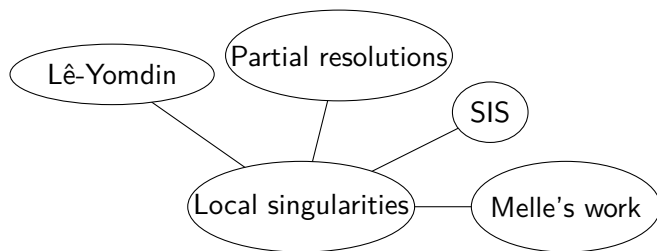
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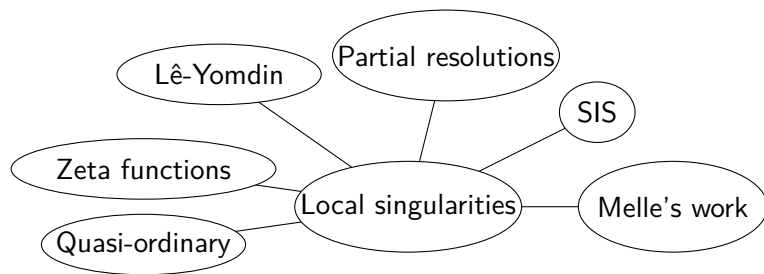
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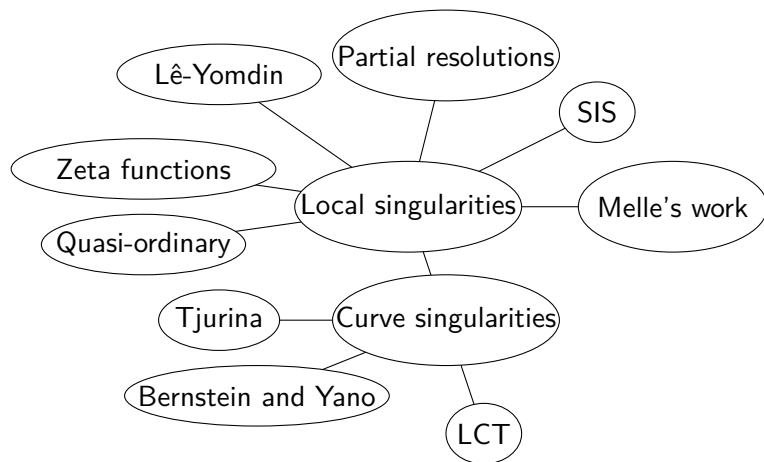
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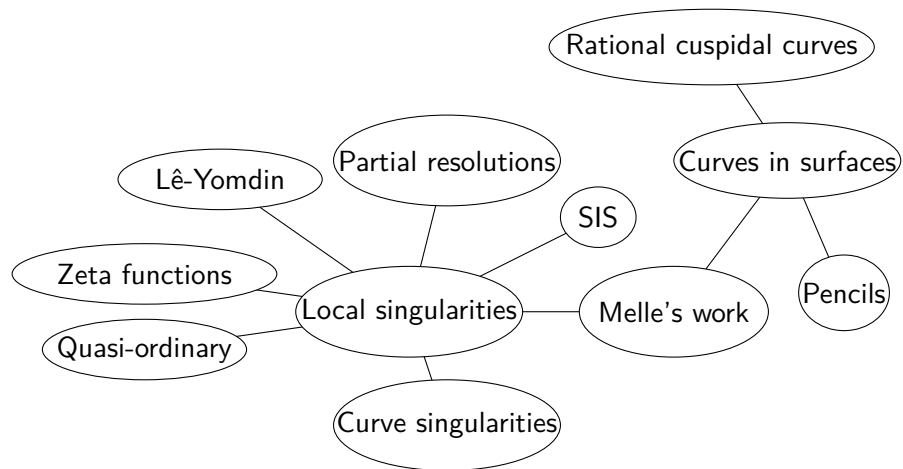
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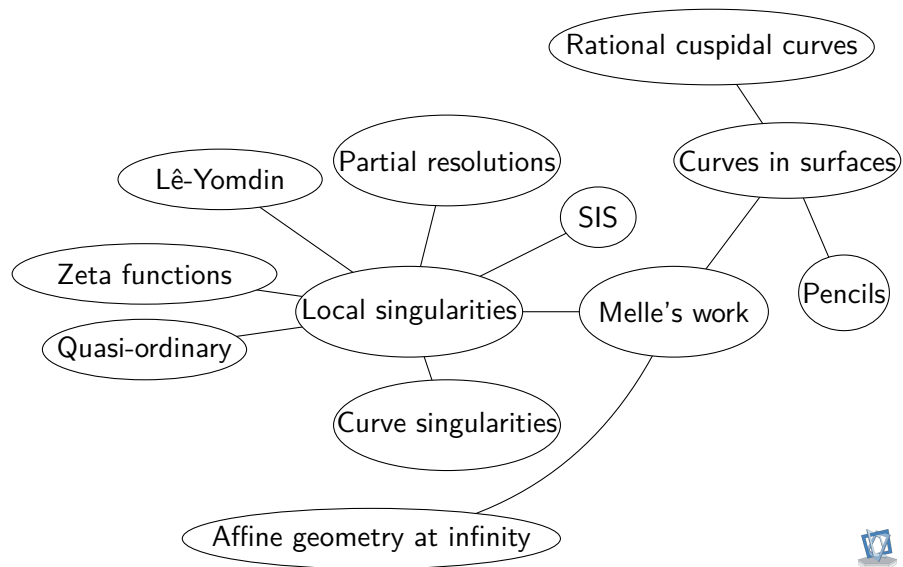
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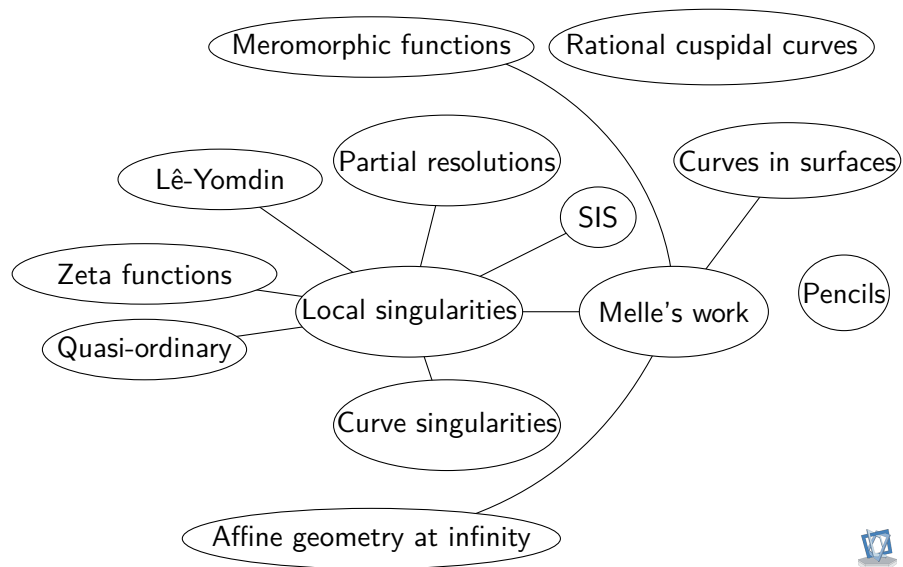
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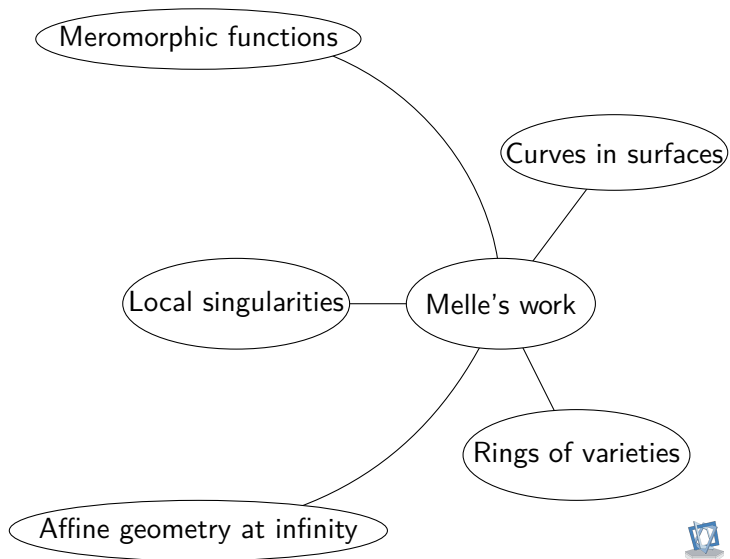
Melle and singularities



Melle and singularities



Melle and singularities



Lê-Yomdin and friends

- ▶ k -Lê-Yomdin singularities: $F = 0$, **reduced tangent cone**.

$$F = f_m + f_{m+k} + \dots, \quad \text{Sing}(V_{\mathbb{P}}(f_m)) \cap V_{\mathbb{P}}(f_{m+k}) = \emptyset$$



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The diagram illustrates the decomposition of a singularity P in $\mathbb{P}^2$. A blue dot labeled P is shown. Two red curves originate from P , representing the reduced tangent cone and the reduced curve. The equations for these curves are given as follows:

$$\mathbb{P}^2 \quad P \quad \begin{array}{l} \dot{\mathcal{S}} : z^m = 0 \\ \mathcal{S}_P : z^m(z^k + f(x, y)) = 0 \\ \dot{\mathcal{S}}_C : z^m x = 0 \end{array}$$



Lê-Yomdin and friends

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- ▶ Superisolated singularities: solved after blowing-up, 1-LY.

$\mathbb{P}^2$ P

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$$V_{\mathbb{P}}(f_m)^{\text{red}} \pitchfork V_{\mathbb{P}}(f_{m+k}).$$

$\mathbb{P}^2$

$\mathcal{S}_Q : z^m(z^k y + x^a) = 0$

$\mathcal{S} : z^m = 0$

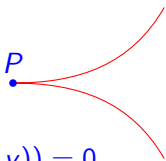
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- ▶ Lê-Yomdin weighted singularities: replace homogeneous by quasi-homogeneous.



The diagram shows a point P (marked with a blue dot) in the projective space $\mathbb{P}^2_\omega$. Two red curves originate from P , representing the reduced tangent cone. The left curve is labeled $\mathcal{S}_P : z^m(z^k + f(x, y)) = 0$ in blue. The right curve is labeled $\mathring{\mathcal{S}}_C : z^m x = 0$ in red. The right curve is also associated with the equation $\mathring{\mathcal{S}} : z^m = 0$ in black.

$$\mathbb{P}^2_\omega \quad P \quad \mathring{\mathcal{S}} : z^m = 0$$
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- ▶ Equisingular family of surfaces with non-constant tangent cone (A_-Bobadilla_-M-Luengo2010).

$$\omega = (2, 3, 1), f_t = \underbrace{z^{12} + xy^3z + x^6 + tx^3y^2}_{f_{12}^\omega} + \underbrace{y^5}_{f_{15}^\omega}$$



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- ▶ Equisingular family of surfaces with non-constant tangent cone (A_-Bobadilla_-M-Luengo2010).
- ▶ M_2000: computation of a bound of the degree of topological sufficiency in terms of polar invariants when the tangent cone has isolated singularities (e.g., SIS or LY).



SIS, rational cuspidal curves: Bobadilla-Luengo-Némethi-M_2006-07

- ▶ Seiberg-Witten, splice type, and geometric genus conjectures for surface singularities whose link is a $\mathbb{Q}$ -rational homology sphere.



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- ▶ Use superisolated singularities with tangent cone rational cuspidal curves.
- ▶ Study rational cuspidal curves: classification with one singularity with one Puiseux pair.



Affine geometry: behaviour of polynomials at infinity

Definition

$f : \mathbb{C}^n \rightarrow \mathbb{C}$. **Bifurcation set** minimal B s.t.

$$f|_{\mathbb{C}^n \setminus f^{-1}(B)} : \mathbb{C}^n \setminus f^{-1}(B) \rightarrow \mathbb{C} \setminus B$$

locally trivial fibration.



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- ▶ Well understood if $n = 2$. In general, B strictly contains the set of singular values.



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- ▶ A_-Luengo-M_2000.



Zeta functions

Theorem (A_, Cassou-Noguès, M_, Luengo $\equiv$ ACLM2002)

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Zeta functions and Jordan blocks (M_-Torrelli-Veys2009-10)

Multiple roots of Bernstein polynomial, Jordan blocks, poles of maximal order: applications to curves and SIS



Quasi-ordinary and ν -quasi-ordinary

ACLM $\rightarrow$ González Pérez and González Villa 2014

Toric resolution and motivic Milnor fiber of a quasi-ordinary singularity

ν -Quasi-ordinary singularities (ACLM2011-13)

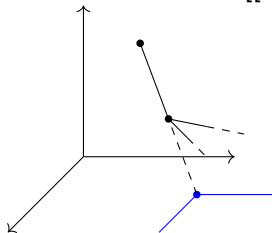
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- ▶ Characterization of quasi-ordinary singularities: chain of ν -quasiordinary series.
- ▶ Consequence for plane curves: computing the log canonical threshold using Newton polygons on special coordinates.



Bernstein, Yano, and Tjurina

- ▶ Yano conjectures the generic b -exponents of an irreducible germ of curve singularity.
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- ▶ Topological roots: b -exponents for each element of the μ -constant stratum.
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 $f(x, y) = (x^4 - y^5)^2 + x^7 y^3 + t x^6 y^6$, $-\frac{39}{94}$ is not a b -exponent for $t = \frac{63}{136}$.



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- ▶ Dimca-Greuel's conjecture for irreducible plane curve singularities (Alberich-Almirón-Blanco-M_2021): $3\mu < 4\tau$.



Global/local theorems

- Veys conjecture.

$$C_i \subset \mathbb{P}^3 \text{ irr. surfaces: } \chi(\mathbb{P}^3 \setminus \bigcup C_i) \geq 0 \implies \kappa(C_i) = -\infty.$$

ALM2000: Double cover of $\mathbb{P}^2$ ramified along two nodal cubics.



Global/local theorems

- ▶ Veys conjecture.
- ▶ Durfee's conjecture: *the homological monodromy is of finite order $\iff$ an abstract resolution has no cycles*

$$(xz - y^2)^3 - ((y - x^2)x^2)^2 + z^7.$$

A_-Carmona-M_2010: Link is a $\mathbb{Q}$ -homology sphere and monodromy is not semisimple



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- ▶ Local pencils and dicritical: effectively compute the number of special fibers of a pencil via Newton polygons.
ALM2015: Positive answer to Gwoździewicz's question: *there exist polynomials f where $f^{-1}(0)$ has n branches at infinity and n non-zero special values (maximal bound).*



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- ▶ Conjectures about free and nearly free divisors (A_-Gorochategui-Luengo-M_2017).

$$(y^2z - tx^2)^2z - x^5 = 0.$$



¡Muchas gracias Alejandro!

