Topology of arrangements and position of singularities

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Arrangements in Pyrénées
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Complement of $\bigcup \mathcal{A}$ in either K^n or $\mathbb{P}^n(K)$:
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Topology

$K = \mathbb{C}$ in order to deal with codimension 2 objects.

Topological properties

Cohomological properties

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Line arrangements in $\mathbb{P}^2$

The fundamental group of an arrangement is also the fundamental group of a line arrangement.

Fundamental Group I

From projective to affine case

$\mathcal{A} := \{\bar{L}_0, \bar{L}_1, \ldots, \bar{L}_n\}$ line arrangement in $\mathbb{P}^2$. Consider $\mathbb{C}^2 = \mathbb{P}^2 \setminus \bar{L}_\infty$; $L_j := \bar{L}_j \cap \mathbb{C}^2$.

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- Choose coordinates x, y such that no line is vertical and if $(x, y_1), (x, y_2) \in \mathcal{P}$ then $y_1 = y_2$.
- Later on we may relax these conditions.

Zariski-van Kampen I

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Long exact homotopy sequence $\Longrightarrow$ Short exact sequence

$$1 = \pi_2(\mathbb{C} \setminus \mathcal{B}) \to \pi_1(F) \to \pi_1(M(\mathcal{A}^\varphi)) \to \pi_1(\mathbb{C} \setminus \mathcal{B}) \to 1.$$

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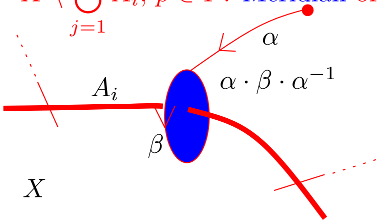
Definition

X quasi-projective smooth variety, $A_1, \ldots, A_r \subset X$ irreducible hypersurfaces, $Y := X \setminus \bigcup_{j=1}^{r} A_i$, $p \in Y$. *Meridian* of A_i in $\pi_1(Y; p)$:



Geometric basis and suitable polydisks

Definition

A *geometric basis* of the free group $\pi_1(\mathbb{C} \setminus \{t_1, \ldots, t_r\}; t_0)$ is a basis of meridians $\mu_1, \ldots, \mu_r$ such that $(\mu_r \cdot \ldots \cdot \mu_1)^{-1}$ is a meridian of ∞.

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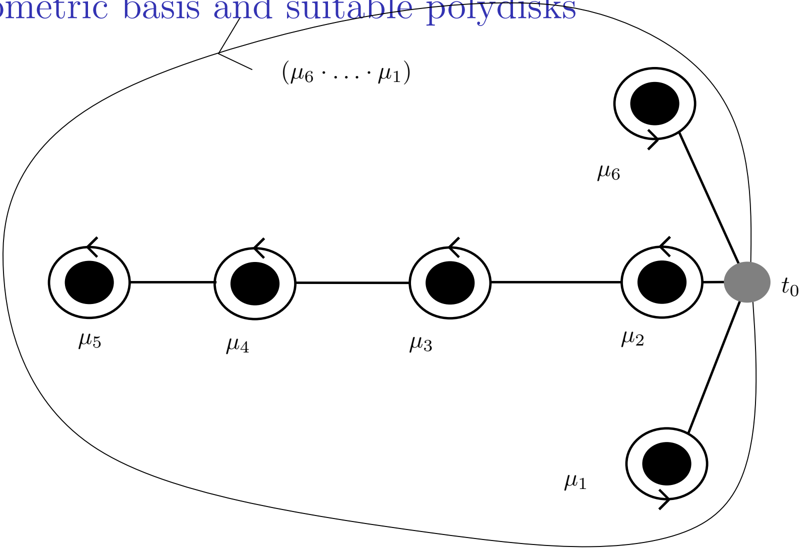

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- Lift $\alpha_1, \ldots, \alpha_r$ to $\tilde{\alpha}_1, \ldots, \tilde{\alpha}_r$ in the line $y = t_y$.

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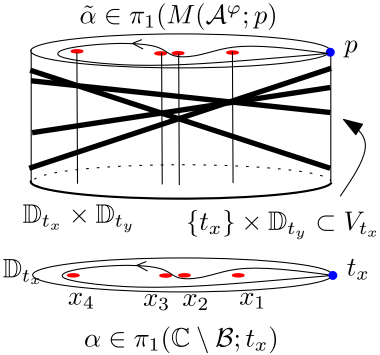

Geometric basis and suitable polydisks

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Polydisk

- $t_x \gg 0$ such that $\mathcal{B} \subset \mathring{\mathbb{D}}_{t_x}$.
- $|y_j| \ll t_y$ and $\bigcup \mathcal{A} \cap (\mathbb{D}_{t_x} \times \mathbb{D}_{t_y}) \subset \partial \mathbb{D}_{t_x} \times \mathring{\mathbb{D}}_{t_y}$.
- $(\mathbb{D}_{t_x} \times \mathbb{D}_{t_y}) \setminus \bigcup \mathcal{A}^\varphi \hookrightarrow M(\mathcal{A}^\varphi)$ is a homotopy equivalence.
- $p := (t_x, t_y)$; $F = V_{t_x} \setminus \bigcup \mathcal{A}$
- $\mu_1, \ldots, \mu_n$ geometric basis of $\pi_1(F; p)$,
- $\alpha_1, \ldots, \alpha_r$ geometric basis of $\pi_1(\mathbb{C} \setminus \mathcal{B}; t_x)$.
- Lift $\alpha_1, \ldots, \alpha_r$ to $\tilde{\alpha}_1, \ldots, \tilde{\alpha}_r$ in the line $y = t_y$.

Lemma

$\mu_1, \ldots, \mu_n$ and $\tilde{\alpha}_1, \ldots, \tilde{\alpha}_r$ generate $\pi_1(M(\mathcal{A}^\varphi); p)$.

Adding components

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Proof.

It is a generalization of the same statement in dimension 1.

- $\pi_1 = \pi_1^{\mathcal{C}^\infty}$.

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- Description of the kernel follows from transversality of mappings $\mathbb{D}^2 \to Z$ with respect to A_j, $1 \leq j \leq s$.

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1. Direct consequence of the lemma: surjectivity statement of Zariski-Lefschetz.

□

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2. Genericity conditions.

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3. - $E := \mathbb{C}^2 \setminus (\mathring{\mathbb{D}}_{t_x} \times \mathring{\mathbb{D}}_{t_y})$ $\check{E} := E \setminus \bigcup \mathcal{A}$, $\pi_1(\check{E}; p) \twoheadrightarrow \pi_1(M(\mathcal{A}); p)$.

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- $\check{E} \cong (\mathbb{C} \setminus \{n-1 \text{ points}\}) \times \mathbb{D}^*$, $\mathbb{D}^* \to \mu_\infty$.

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Goal

Determine the conjugation action of $\tilde{\alpha}_j$ on $\pi_1(F; p) \subset \pi_1(M(\mathcal{A}); p)$.

Zariski-van Kampen II

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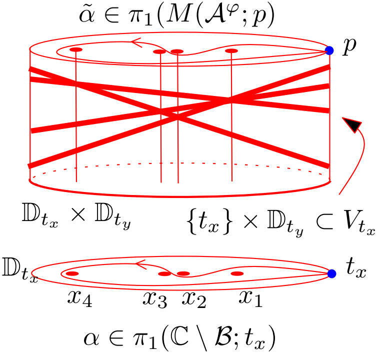
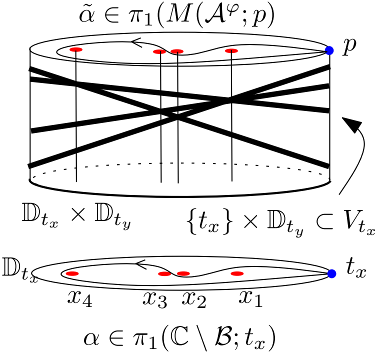
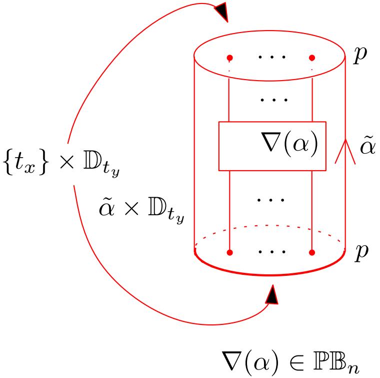


Zariski-van Kampen II





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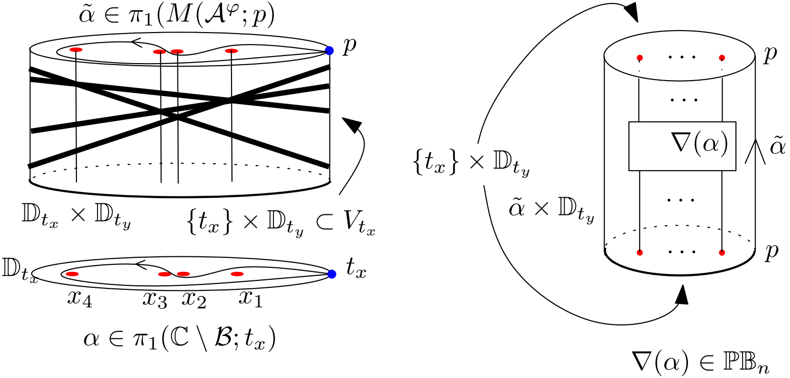

Braid monodromy

$\nabla : \pi_1(\mathbb{C} \setminus \mathcal{B}; t_x) \to \mathbb{PB}_n \subset \mathbb{B}_n$ the pure braid group.

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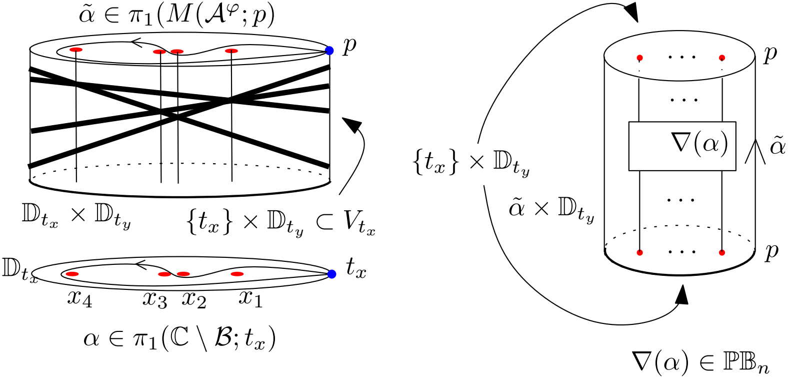

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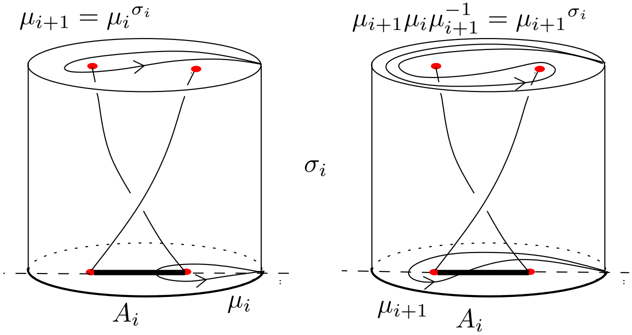
Artin presentation

$$\mathbb{B}_n := \left\langle \sigma_1, \ldots, \sigma_{n-1} \,\middle|\, \underset{1<i+1<j<n}{[\sigma_i, \sigma_j] = 1}, \quad \underset{1 \leq i < n-1}{\sigma_i \cdot \sigma_{i+1} \cdot \sigma_i = \sigma_{i+1} \cdot \sigma_i \cdot \sigma_{i+1}} \right\rangle.$$

Braid actions on free groups



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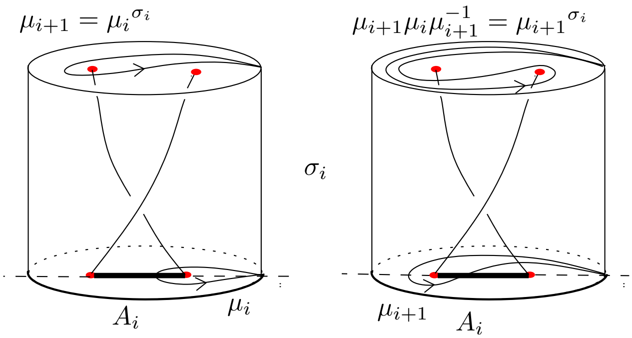

$$\begin{array}{ccc} \pi_1(F) \times \mathbb{B}_n & \to & \pi_1(F) \\ (\mu, \tau) & \mapsto & \mu^\tau. \end{array}$$

$$\mu_i^{\sigma_j} := \begin{cases} \mu_{i+1} & \text{if } i = j \\ \mu_{i+1}\mu_i\mu_{i+1}^{-1} =: \mu_{i+1} * \mu_i & \text{if } i = j+1 \\ \mu_i & \text{if } i \neq j, j+1. \end{cases}$$

Zariski-van Kampen IV

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1. $\pi_1(M(\mathcal{A}^{\varphi}); p) = \langle \mu_1, \ldots, \mu_n, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_r \mid \mu_i^{\tilde{\alpha}_j} = \mu_i^{\nabla(\alpha_j)} \rangle.$

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- In general, $\forall P \in \mathcal{P}$, this is the local behavior.

Puiseux braid monodromy

Proposition

$\forall j \in \{1, \ldots, r\}$, $\nabla(\alpha_j) = \tau_j^{-1} \cdot \Delta^2_{a_j, b_j} \cdot \tau_j$, where $1 \leq a_j < b_j \leq n$ and $\Delta^2_{a_j, b_j}$ is the full-twist involving the $b_j - a_j + 1$ strands from a_j to b_j.

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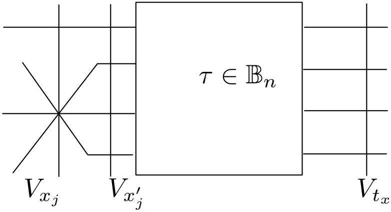

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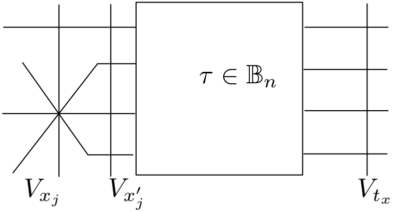
Let $\mu_{i,j} := \mu_i^{\tau_j}$. Then, the set of relations $\mu_i = \mu_i^{\nabla(\alpha_j)}$ for fixed j, can be replaced by $[\mu_{1,j}, \ldots, \mu_{n,j}] = 1$.



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Proof.

- $\mu_i = \mu_i^{\nabla(\alpha_j)} \Longleftrightarrow \mu = \mu^{\nabla(\alpha_j)}, \forall \mu \in \pi_1(F)$

□

Puiseux braid monodromy

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$\forall j \in \{1, \ldots, r\}$, $\nabla(\alpha_j) = \tau_j^{-1} \cdot \Delta^2_{a_j,b_j} \cdot \tau_j$, where $1 \leq a_j < b_j \leq n$ and $\Delta^2_{a_j,b_j}$ is the full-twist involving the $b_j - a_j + 1$ strands from a_j to b_j.

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Let $\mu_{i,j} := \mu_i^{\tau_j}$. Then, the set of relations $\mu_i = \mu_i^{\nabla(\alpha_j)}$ for fixed j, can be replaced by $[\mu_{1,j}, \ldots, \mu_{n,j}] = 1$.

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- $\mu_i = \mu_i^{\nabla(\alpha_j)} \iff \mu = \mu^{\nabla(\alpha_j)}, \forall \mu \in \pi_1(F)$
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$\square$

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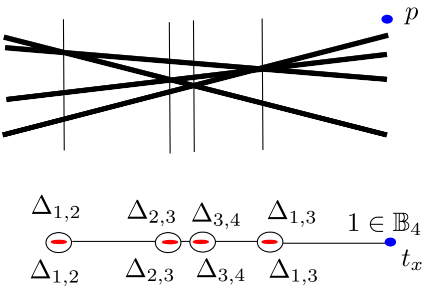
- $\mu_{i,j}$ correspond to a basis of $\pi_1(F)$ in $V_{x'_j}$.

□

Complexified real arrangements



Braid monodromy:$(\Delta_{1,3}^2, \Delta_{1,3} * \sigma_3^2, (\Delta_{1,3}\sigma_3) * \sigma_2^2, (\Delta_{1,3}\sigma_3\sigma_2) * \sigma_1^2)$

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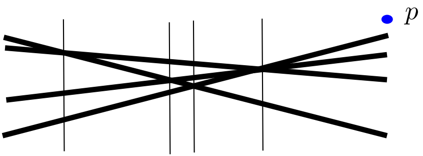

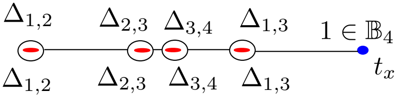

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$$
\begin{aligned}
\mu_5 \equiv \mu_5^{\mu_4\mu_3\mu_2\mu_1} &= \mu_1^{(\Delta_5^2)^{-1}} \\
\mu_5 * \mu_4 \equiv \mu_4^{\mu_3\mu_2\mu_1} &= \mu_2^{(\Delta_5^2)^{-1}} \\
(\mu_5\mu_4) * \mu_3 \equiv \mu_3^{\mu_2\mu_1} &= \mu_3^{(\Delta_5^2)^{-1}} \\
\mu_2^{\mu_1} &= \mu_4^{(\Delta_5^2)^{-1}} \\
\mu_1 &= \mu_5^{(\Delta_5^2)^{-1}}
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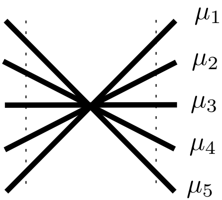

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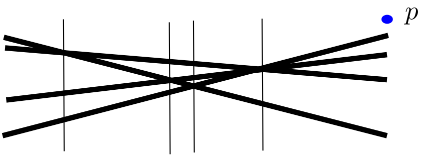

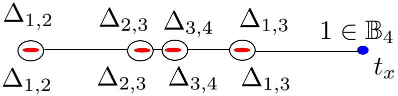

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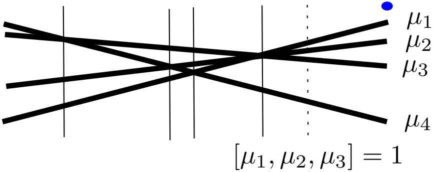

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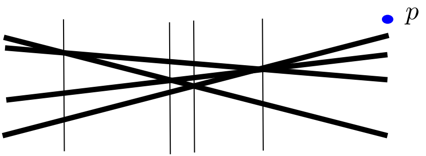

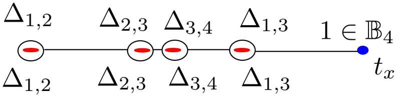

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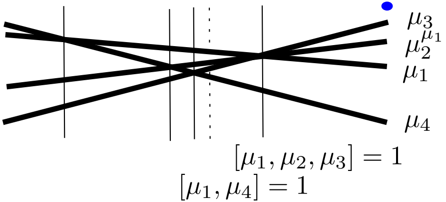

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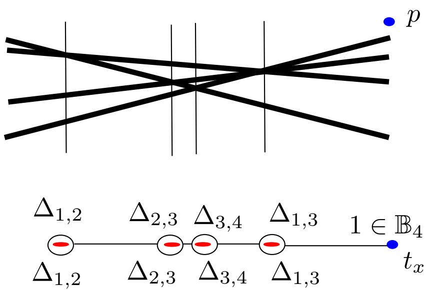

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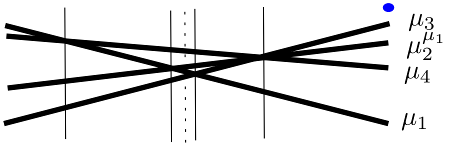

$[\mu_1^{\mu_1}, \mu_4] = 1$ $[\mu_1, \mu_2, \mu_3] = 1$
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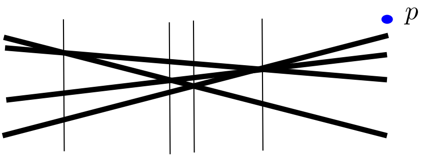

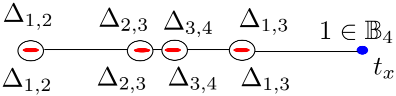

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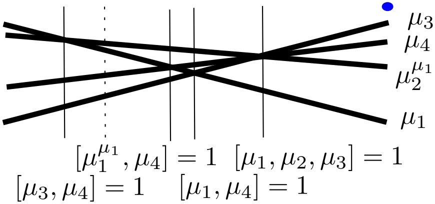

Generic arrangements and Arvola's wiring diagram

Theorem

$(\mathbb{P}^2, \mathcal{A})$ generic arrangement (all points in $\mathcal{P}$ are of multiplicity 2). Then, $\pi_1(M(\mathcal{A}))$ is abelian.

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- From the real picture, all commutators of generators appear.
- $\mathcal{A}$ can be chosen as a generic plane section of the coordinate arrangement in $\mathbb{P}^n$ and the complement is homeomorphic to $(\mathbb{C}^*)^n$: the homology of the group is recovered.

□

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- Equivalent to braid monodromy with Puiseux braid decomposition.

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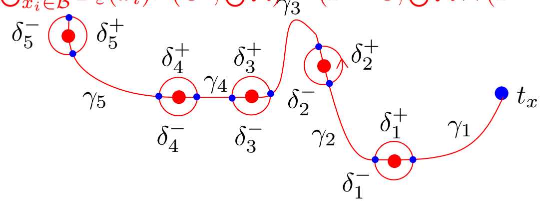

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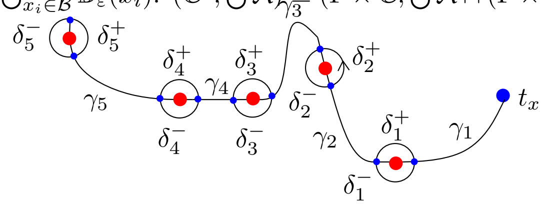

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- $\gamma_j \mapsto \eta_j \in \mathbb{B}_n$, $\delta_j^\pm \mapsto \Delta_{a_j,b_j}$ determine braid monodromy and generators at each fiber.

Generic braid monodromy

Definition

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- $\nabla(\mathcal{A})$ allows to compute invariants via braid representations (Libgober).

From non-generic to generic

Fact

Sometimes is easier to compute non-generic braid monodromies.
What to do?

From non-generic to generic

- If $\bar{L}_\infty \not\pitchfork \mathcal{A}$: $(\tau_1, \ldots, \tau_r) \to (\tau_1, \ldots, \tau_r, (\tau_r \cdot \ldots \cdot \tau_1)^{-1}\Delta_n^2)$.

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- If there are some vertical lines in $\mathcal{A}_\infty$:
  - Each braid for a vertical line V_t produces as many braids as the number of multiple points in V_t.
  - Add a braid for the multiple at infinity.
  - Turn slightly a wiring diagram.

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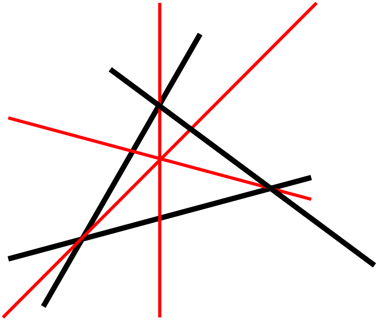

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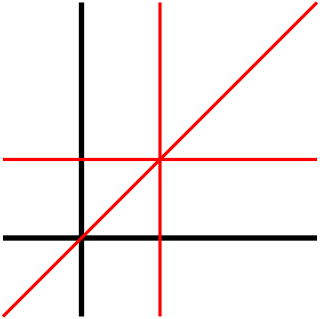

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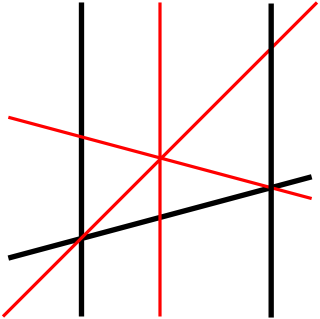

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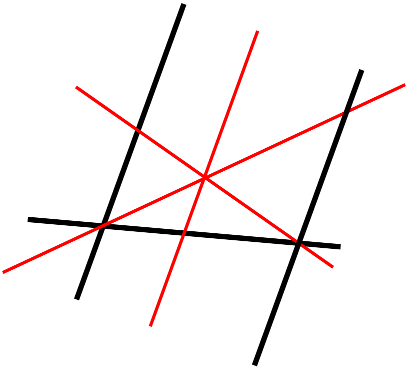

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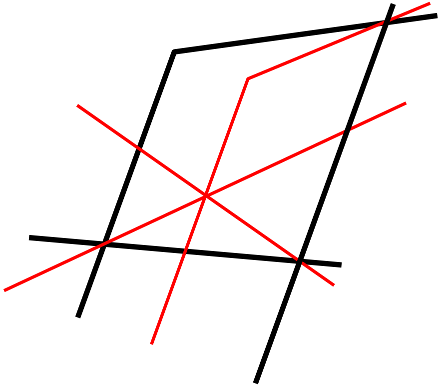

Universal abelian covering

$\mathcal{A} = \{\bar{L}_0, \bar{L}_1, \ldots, \bar{L}_n\}$ line arrangement in $\mathbb{P}^2$,
$\mathcal{P} := \{\bar{L}_i \cap \bar{L}_j \mid 0 \leq i < j \leq n\}$. Use $\mathcal{A}_0$ in order to compute $\pi_1(M(\mathcal{A}))$.

- It is difficult to get properties from a presentation of $\pi_1(M(\mathcal{A}); p)$

Universal abelian covering

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Twisted cohomology

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- $H^1(M(\mathcal{A}); \mathbb{C}_\xi)$ is obtained from the complex $\tilde{C}^*(\mathcal{A}) \otimes_{\Lambda_{\mathbb{C}}} \mathbb{C}_\xi$: evaluate $\tilde{A}_1, \tilde{A}_2$ using ev_ξ.

Characteristic varieties

Definition

$\mathcal{V}_k(\mathcal{A}) := \{\xi \in \mathbb{T}(\mathcal{A}) \mid \dim_{\mathbb{C}} H^1(M(\mathcal{A}); \mathbb{C}_\xi) \geq k\}.$

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Corollary

It is enough to know $H^1(M(\mathcal{A}); \mathbb{C}_\xi)$ for ξ unitary (or torsion).

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Corollary

$\mathcal{V}_k(\mathcal{A})$ depends on the Betti numbers of some quasi-projective smooth varieties.

Quasi-projective and quasi-projective varieties

Settings

$M_\xi(\mathcal{A}) \subset X_\xi(\mathcal{A})$ smooth projective completion such that $D_\xi(\mathcal{A}) := X_\xi(\mathcal{A}) \setminus M_\xi(\mathcal{A})$ is a normal crossing divisor.

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  $0 \to \Omega^1_{X_\xi(\mathcal{A})} \to \Omega^1_{X_\xi(\mathcal{A})} \log(D_\xi(\mathcal{A})) \to \bigoplus_{D \subset D_\xi(\mathcal{A})} i_* \mathcal{O}_D \to 0$
  implies
  $0 \to H^0(X_\xi; \Omega^1_{X_\xi}) \to H^0(X_\xi; \Omega^1_{X_\xi} \log(D_\xi)) \to H(\mathcal{A}) \to 0$
  $0 \to H(\mathcal{A}) \to \bigoplus_{D \subset D_\xi} H^0(D; \mathcal{O}_D) \to H^1(X_\xi; \Omega^1_{X_\xi}) \subset H^2(X_\xi; \mathbb{C})$.

Quasi-projective and quasi-projective varieties II

$$\dim H^1(X_\xi; \mathbb{C}) = 2 \dim H^1(X_\xi; \mathcal{O}_{X_\xi})$$

Quasi-projective and quasi-projective varieties II

$$\dim H^1(M_\xi;\mathbb{C}) = 2\dim H^1(X_\xi;\mathcal{O}_{X_\xi}) + \dim\ker\left(\bigoplus_{D\subset D_\xi}\mathbb{C}\langle D\rangle \to H^2(X_\xi;\mathbb{C})\right).$$

Quasi-projective and quasi-projective varieties II

Proposition

$$\dim H^1(M(\mathcal{A});\mathbb{C}_\xi) = \dim H^1(X_\xi;\mathcal{O}_{X_\xi})^{\xi} + \dim H^1(X_\xi;\mathcal{O}_{X_\xi})^{\bar{\xi}} + \dim\ker\left(\bigoplus_{D\subset D_\xi}\mathbb{C}\langle D\rangle \to H^2(X_\xi;\mathbb{C})\right)^{\xi}$$

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Definition

ξ is *fully ramified* if $\xi(t_j)\neq 1$, $\forall j\in\{0,\dots,n\}$; in general, $\mathcal{A}^\xi := \{L_j \mid \xi(t_j)\neq 1\}$ (ramification locus) and $\mathcal{A}_0^\xi := \{L_j \mid \xi(t_j) = 1\}$ (unramification locus).

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Theorem (Libgober, 1987)

If ξ is fully ramified then $H^1(X_\xi;\mathbb{C})^\xi = H^1(M_\xi;\mathbb{C})^\xi$.

Quasi-adjunction ideals

Real representatives

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- $\xi \leftrightarrow (r_0, r_1, \ldots, r_n) \in [0,1)^{n+1}$ such that $\xi(t_j) = \exp(2i\pi r_j)$.

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Theorem (Libgober (2001))

$\dim H^1(X_\xi; \mathcal{O}_{X_\xi})^\xi =$

$$\dim \operatorname{coker}\left(\sigma_\xi : H^0(\mathbb{P}^2; \mathcal{O}_{\mathbb{P}^2}(\ell(\xi)-3)) \to \bigoplus_{P\in\mathcal{P}} \mathcal{O}_{\mathbb{P}^2,P}/\mathcal{J}_{P,\xi}\right)$$

Quasi-adjunction ideals II

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1. If $m_P = 2$ then $\mathcal{J}_{P,\xi} = \mathcal{O}_{\mathbb{P}^2,P}$. We can restrict our attention to $\mathcal{P}_{>2} := \{P \in \mathcal{P} \mid m_P > 2\}$.

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Example

$\mathcal{A} = \{\bar{L}_0, \ldots, \bar{L}_n\}$, $P \in \bar{L}_i$, ξ fully ramfied of level ℓ, $s_P = \ell - 1$:

$$\dim\operatorname{coker}\left(\sigma_\xi : H^0(\mathbb{P}^2; \mathcal{O}_{\mathbb{P}^2}(\ell - 3)) \to \mathcal{O}_{\mathbb{P}^2,P}/\mathcal{M}_P^{\ell-1}\right) = \ell - 1.$$

$\ell(\bar{\xi}) = n + 1 - \ell$.

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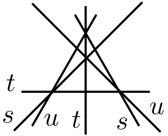

$tsu = 1$, $\ell(\xi) = 4$, $s_{P_i} = 1$,

$$\dim \operatorname{coker}\left(\sigma_\xi : H^0(\mathbb{P}^2; \mathcal{O}_{\mathbb{P}^2}(1)) \to \mathbb{C}^4\right) = 1.$$

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Example (9-Ceva)

$\bigcup \mathcal{A} = \{(x^3 - y^3)(y^3 - z^3)(x^3 - z^3) = 0\}$,
$\mathcal{P} = \{[1:0:0], [0:1:0], [0:0:1]\} \cup \{[1:\zeta:\omega] \mid \zeta^3 = \omega^3 = 1\}$, 12 triple points; ξ defined by $\xi(\mu_i) := \exp(2i\pi\frac{2}{3})$. Then $\ell = 6$, $s_P = 1$.

$$\dim \operatorname{coker} \left(\sigma_\xi : H^0(\mathbb{P}^2; \mathcal{O}_{\mathbb{P}^2}(3)) \to \mathbb{C}^{12}\right) = 2.$$

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Example (________, 1994)

$\bigcup\mathcal{A} = \{xz(4x - y + 2z) = 0\} \cup \{y(x - y - z)(2x + y + 2z) = 0\} \cup \{(x - y)(2x + y + z)(y + 2z) = 0\}$, 9 triple points, 9 double points; ξ defined by $\xi(\mu_i) := \exp(2i\pi\frac{2}{3})$. Then $\ell = 6$, $s_P = 1$.

$$\dim\operatorname{coker}\left(\sigma_\xi : H^0(\mathbb{P}^2; \mathcal{O}_{\mathbb{P}^2}(3)) \to \mathbb{C}^9\right) = 1.$$

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Example (Hesse arrangement)

$\bigcup \mathcal{A} = \{xyz(x^3y^3z^3 - 27(x^3+y^3+z^3)^3) = 0\}$, 9 quadruple points, 12 double points; ξ defined by $\xi(\mu_i) := -1$. Then $\ell = 6$, $s_P = 1$.

$$\dim \operatorname{coker} \left(\sigma_\xi : H^0(\mathbb{P}^2; \mathcal{O}_{\mathbb{P}^2}(3)) \to \mathbb{C}^9 \right) = 1.$$

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Cyclic covers: Esnault-Viehweg, Libgober, Loeser-Vaquié, ______
$H^1(X_\xi; \mathcal{O}_{X_\xi})^\xi \cong H^1(\hat{\mathbb{P}}^2; \mathcal{L}^\xi)$ for some line bundle $\mathcal{L}^\xi$ over a blown-up $\hat{\mathbb{P}}^2$.

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$B \in \tilde{\mathcal{A}}$ is said *unramified* if $\xi(t_B) = 1$; it is *inner unramified* if it is unramified and it is also the case for all its neighbors. $\mathcal{U}_\xi \subset \tilde{\mathcal{A}}$ is the set of the inner unramified components.

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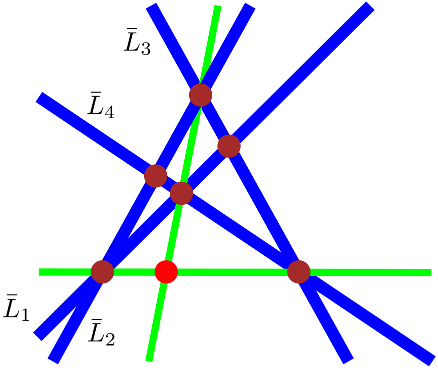
Proposition

If $\Gamma_{\mathcal{U}_\xi}$ is a tree, then $\cdot_\xi = \cdot$.

Ceva arrangement



For which characters ξ, is $\mathcal{U}_\xi$ formed by the green lines?

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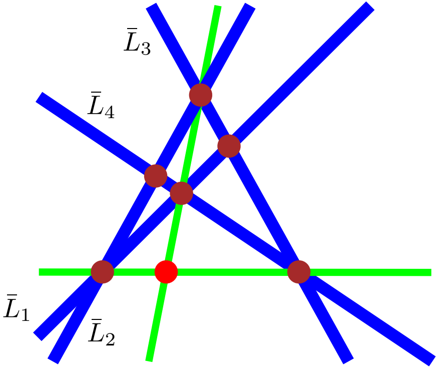

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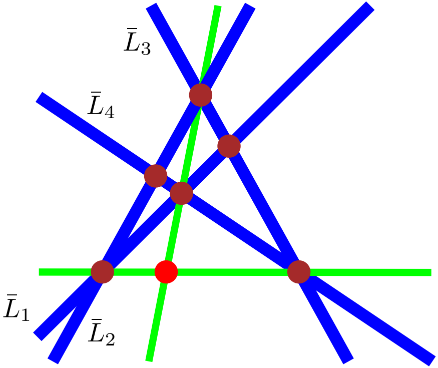

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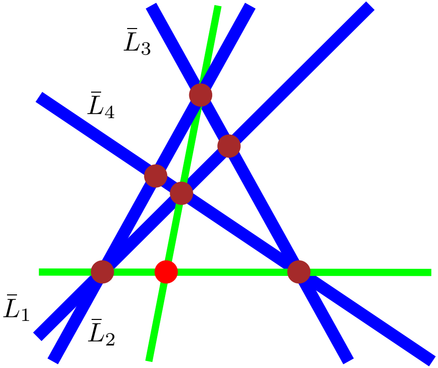

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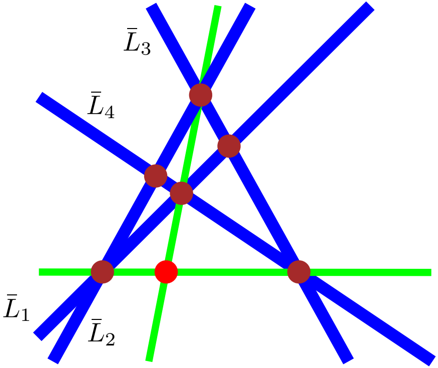

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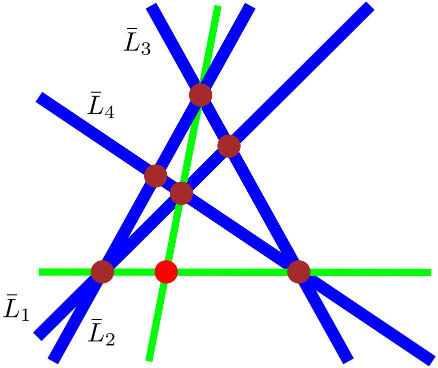

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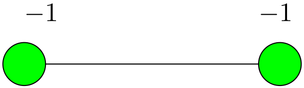

$$\cdot_\xi : \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

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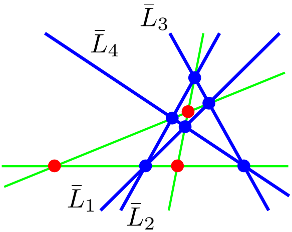
- For an order in $\mathcal{U}_\xi$ consider the matrix $A(\mathcal{U}_\xi)$ of this twisted hermitian product on $\mathcal{U}_\xi$.

Ceva-extended arrangement



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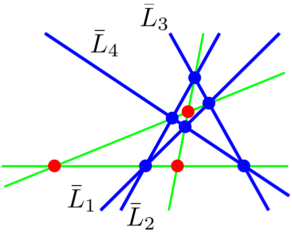

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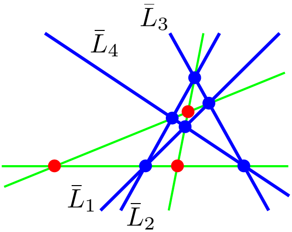

For which characters ξ, is $\mathcal{U}_\xi$ formed by the green lines?

$\xi(t_1) = x \in \mathbb{C}^* \Longrightarrow \xi(t_2) = x^{-1}$

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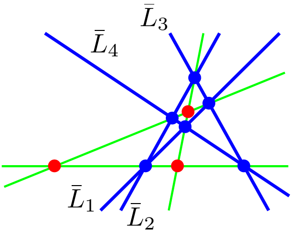

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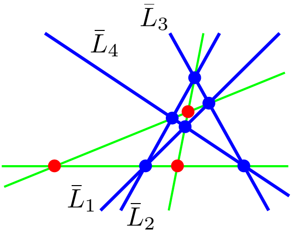

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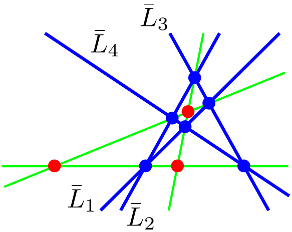

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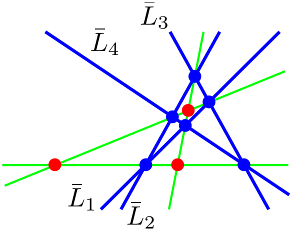

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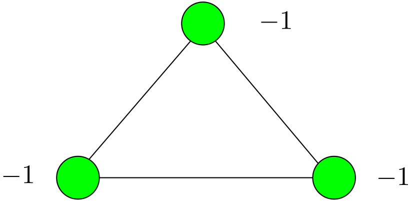

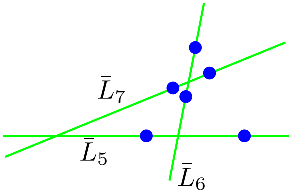

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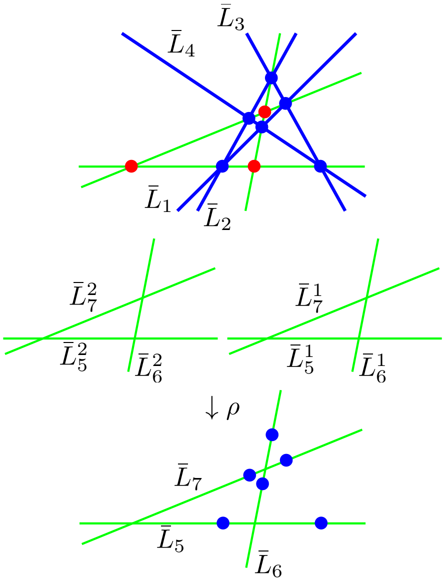

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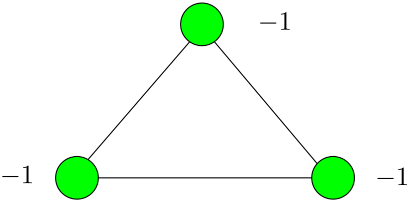

$$\cdot_\xi : \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

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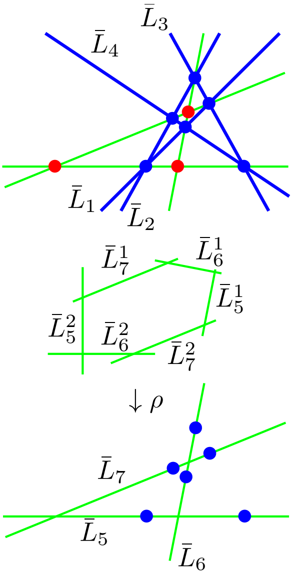

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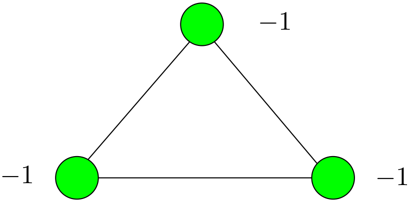

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Seeking for essential coordinate components

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$$\operatorname{corank} A(\mathcal{U}_\xi) = \dim\ker\left(\bigoplus_{D\subset D_\xi}\mathbb{C}\langle D\rangle \to H^2(X_\xi,\mathbb{C})\right)^\xi.$$

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- Corank 2 for extended Ceva Arrangement (Cohen-Suciu).

Thank you!