Torsion of Jacobians and embedding of plane curves

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Conformal Geometry and Low Dimensional Manifolds
A Hybrid Conference in Honour of Antonio F. Costa
Ávila, July 1st 2022

Joint work with S.Bannai, T. Shirane and H. Tokunaga

Papers

______, S. Bannai, T. Shirane, and H. Tokunaga, *Torsion divisors of plane curves and Zariski pairs*, to appear at St. Petersbourg Math. J., available at `arXiv:1910.06490`, 2022.

______, S. Bannai, T. Shirane, and H. Tokunaga, *Torsion divisors of plane curves with maximal flexes and Zariski pairs*, to appear at Math. Nachr., available at `arXiv:2005.12673`, 2020.

Main result

Arrangements of curves

- $\mathcal{D}_0 \subset \mathbb{P}^2$ smooth curve of degree d_0
- $\mathcal{D}_1 \subset \mathbb{P}^2$ reduced (maybe reducible) curve of degree d_1

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Theorem (simplified version)

Let $(\mathcal{D}_0^i, \mathcal{D}_1^i)$ as above, $d_0^1 = d_0^2 \neq d_1^2 = d_2^2$, $n^1 = n^2$. If $G^1 \not\equiv G^2$ then there is no homeomorphism $h : (\mathbb{P}^2, \mathcal{D}_0^1 \cup \mathcal{D}_1^1) \to (\mathbb{P}^2, \mathcal{D}_0^2 \cup \mathcal{D}_1^2)$

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Remarks

- A much more general statement is true

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Arrangements of curves

- $\mathcal{D}_0 \subset \mathbb{P}^2$ smooth curve of degree d_0, $O \in \mathcal{D}_0$ maximal flex.
- $\mathcal{D}_1 \subset \mathbb{P}^2$ reduced (maybe reducible) curve of degree d_1
- $n := \gcd(\{(\mathcal{D}_0 \cdot \mathcal{D}_1)_P \mid P \in \mathcal{D}_0 \cap \mathcal{D}_1\})$.
- $\mathfrak{o} := O \in \operatorname{Pic}(\mathcal{D}_0)$, L tangent line
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Prehistory

Smooth cubics and tangent to flexes

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$$\pi_1(\mathbb{P}^2 \setminus \mathcal{D}) \cong \begin{cases} \mathbb{Z} \times \pi_1(\mathbb{S}^3 \setminus T_{2,6}) & \text{if } P, Q, R \text{ aligned} \\ \mathbb{Z}^3 & \text{if } P, Q, R \text{ not aligned} \end{cases}$$

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- Group $(\mathcal{C}, P)$:

$$\langle Q, R \rangle \cong \begin{cases} \mathbb{Z}/3 & \text{if } P, Q, R \text{ aligned} \\ \mathbb{Z}/3 \times \mathbb{Z}/3 & \text{if } P, Q, R \text{ not aligned} \end{cases}$$

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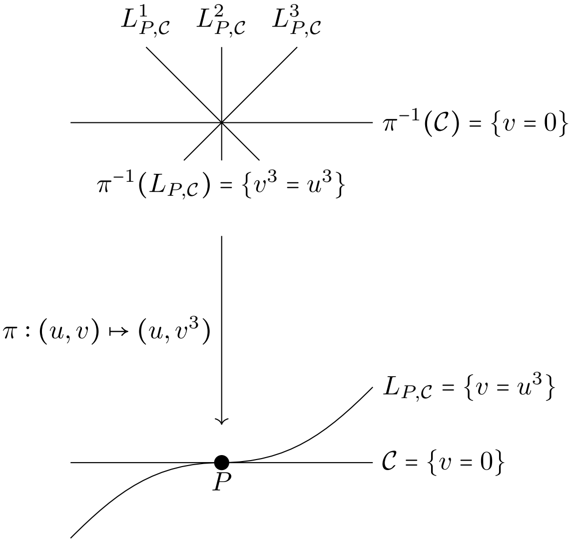
- $\mathcal{D}_0 = \mathcal{C}$, $\mathcal{D}_1 = L_{P,\mathcal{C}} \cup L_{Q,\mathcal{C}} \cup L_{R,\mathcal{C}}$, $\mathfrak{d} = P + Q + R$:

$$\mathbb{Z}\langle P + Q + R - 3O\rangle \cong \begin{cases} 0 & \text{if } P, Q, R \text{ aligned} \\ \mathbb{Z}/3 & \text{if } P, Q, R \text{ not aligned} \end{cases}$$

A 3-fold cyclic cover

$$\{[x:y:z:t] \in \mathbb{P}^3 \mid t^3 = f_3(x,y,z)\} =: S \xrightarrow{\pi} \mathbb{P}^2$$
$$[x:y:z:t] \longmapsto [x:y:z]$$

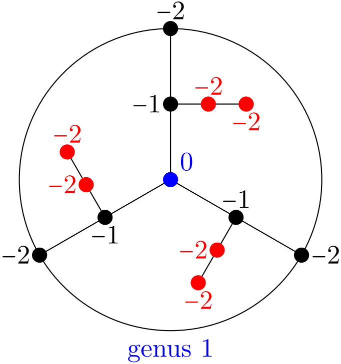
A 3-fold cyclic cover



Ramification divisor or boundary at infinity

Please choose

- Algebraic geometers: read blowing-up and dual graph of ramification divisor
- Topologists: read plumbing graph of a graph manifold



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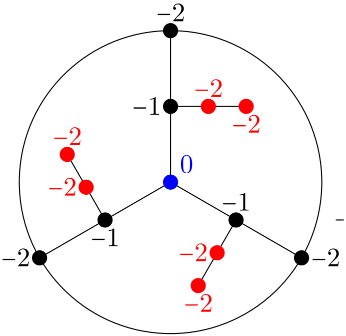
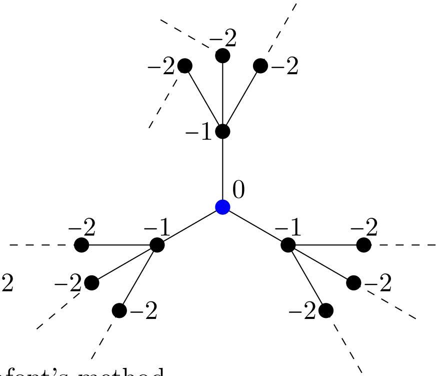
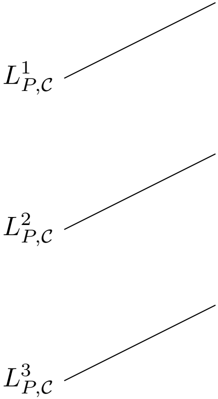


genus 1



Safont’s method

Triangles and 9-gons



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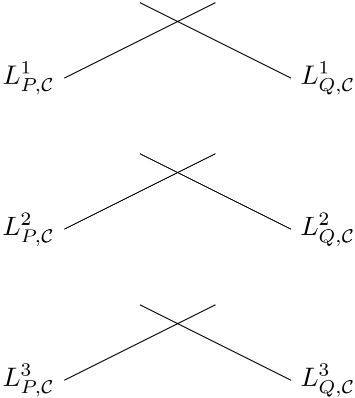

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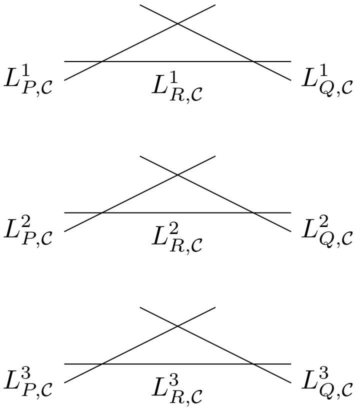

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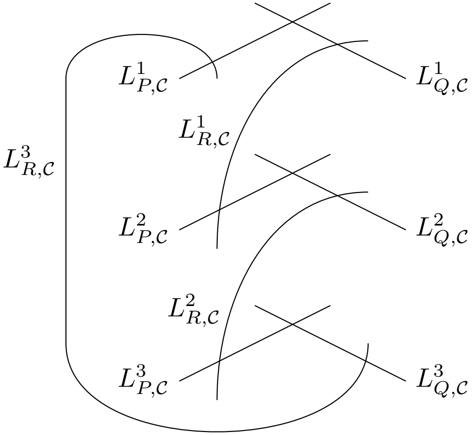
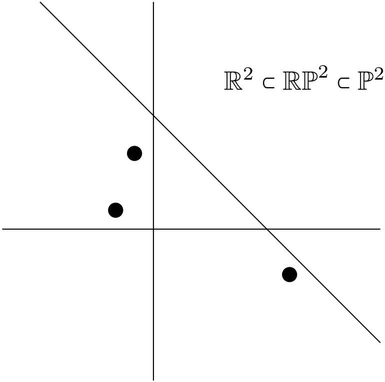


Computing a cycle, aligned flexes

$$F(x,y,z) := xy(x+y+z)F_3(x,y,z) \qquad \zeta = \exp\frac{2i\pi}{3}$$

$$\begin{aligned} F_3(x,y,1) = & x^3 - 3(\zeta+1)(x^2y - 4xy - 9y^2 + x) - 3(3\zeta+2)xy^2 \\ & - 3(2\zeta+1)y^3 + 3\zeta x^2 - 3(\zeta+2)y + 1 \end{aligned}$$



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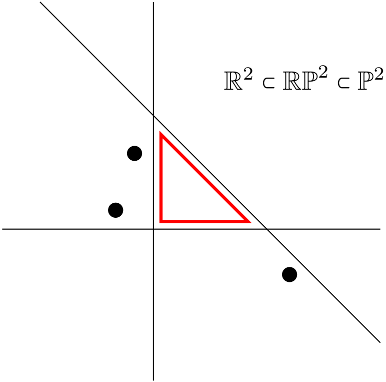

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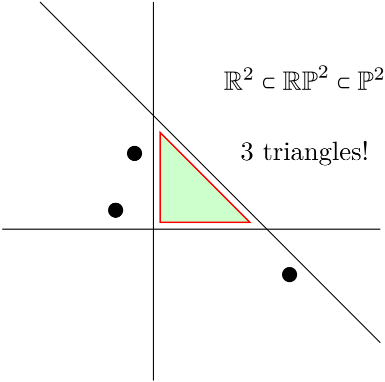

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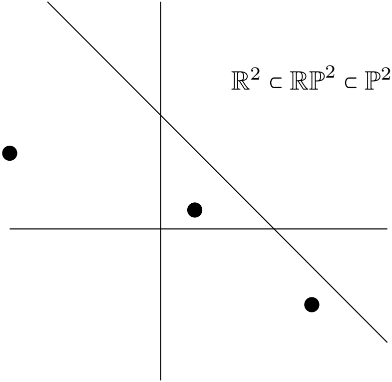

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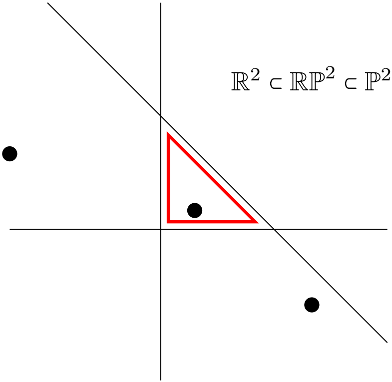

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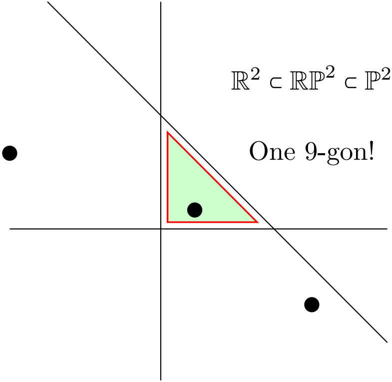


$\mathbb{R}^2 \subset \mathbb{RP}^2 \subset \mathbb{P}^2$

One 9-gon!

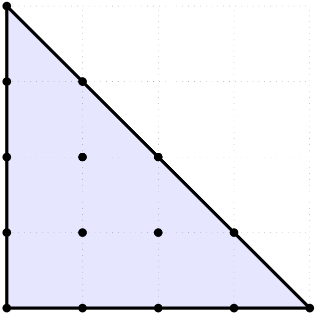
More papers

T. Shirane, *Connected numbers and the embedded topology of plane curves*, Canad. Math. Bull. **61** (2018), no. 3, 650–658.

______, J.I. Cogolludo, and J. Martín. *Triangular Curves and Cyclotomic Zariski Tuples.* **Collect. Math. 71** (3) 2020, 427–41

Quartics with 3 generic maximal flexes

$$xyzF_4(x, y, z) = 0$$



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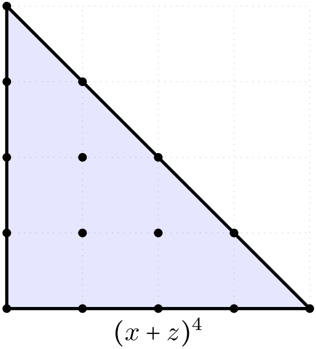

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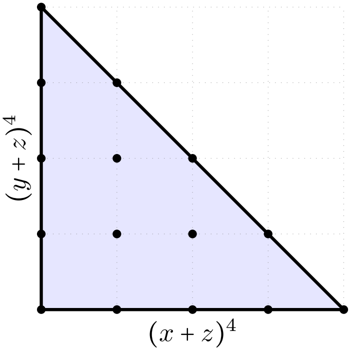

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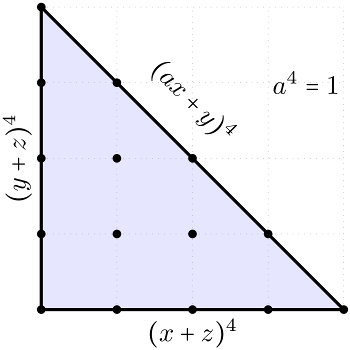

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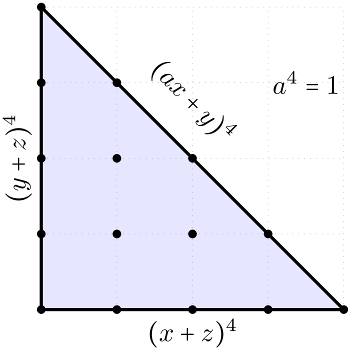

Theorem (Embedding properties of Riemann surfaces)

- $\mathcal{C}_a := \{xyzF_{4,a}(x, y, z) = 0\},\ a = 1, -1, i$

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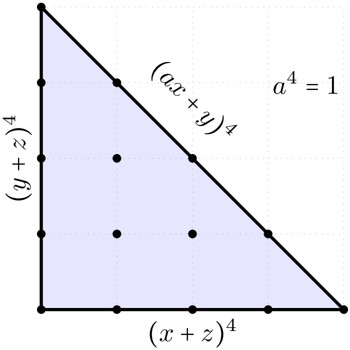

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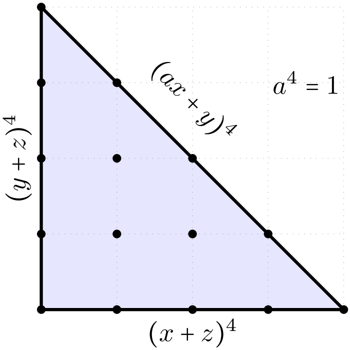

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- *Pairwise non-homeomorphic embeddings*

Open problem

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- Are they topologically equivalent?

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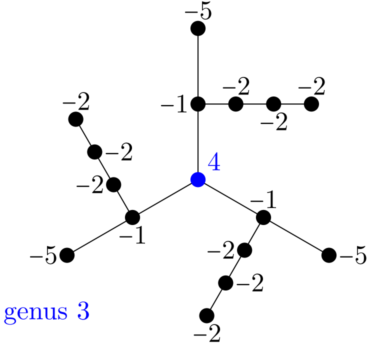
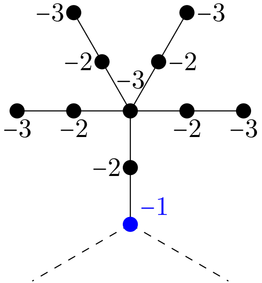
- Are they topologically equivalent?
- X_a exterior of $\mathcal{D}_a$, compact 4-manifold, $\pi_1(X_a) = \mathbb{Z}/8$



Open problem

- $\mathcal{D}_a = \{F_{4,a}(yz, xz, xy) = 0\}$
- $\deg \mathcal{D}_a = 8$
- Three singular points $u^4 = v^5$.
- They are not embedded in the same way if we add $xyz = 0$
- There is no isotopy among projective curves
- Are they topologically equivalent?
- X_a exterior of $\mathcal{D}_a$, compact 4-manifold, $\pi_1(X_a) = \mathbb{Z}/8$
- Y_a 8-fold universal cover



¡Felicidades, Antonio!