Fundamental group, topology and combinatorics of line arrangements [2, 3]

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MSRI, October 2004

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Contents

1. Results

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Theorem 2 ([2]). $\exists$ *two line arrangements* $\mathscr{A}_1, \mathscr{A}_2$ *in* $\mathbb{RP}^2$ *such that* $\mathscr{C}(\mathscr{A}_1) = \mathscr{C}(\mathscr{A}_2)$ *and their complexifications* $\mathscr{A}_i^{\mathbb{C}} := \mathscr{A}_i \otimes_{\mathbb{R}} \mathbb{C}$, $i = 1, 2$, *have non-homeomorphic embeddings in* $\mathbb{P}^2$.

Theorem 2 ([2]). *$\exists$ a line arrangement $\mathscr{A}$ in $\mathbb{Q}(\sqrt{5})\mathbb{P}^2$ such that the complexifications $\mathscr{A}_1^{\mathbb{C}}, \mathscr{A}_2^{\mathbb{C}}$ induced by the two inclusions $\mathbb{Q}(\sqrt{5}) \hookrightarrow \mathbb{C}$ have non-homeomorphic embeddings in $\mathbb{P}^2$.*

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Open questions Let $M(\mathscr{A}_i^{\mathbb{C}}) := \mathbb{P}^2 \setminus \bigcup \mathscr{A}_i^{\mathbb{C}}$.

Are $M(\mathscr{A}_1^{\mathbb{C}})$ and $M(\mathscr{A}_2^{\mathbb{C}})$ homeomorphic?

Are $\pi_1(M(\mathscr{A}_1^{\mathbb{C}}))$ and $\pi_1(M(\mathscr{A}_2^{\mathbb{C}}))$ isomorphic groups?

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Their profinite completions are isomorphic.

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Proposition 1 ([4]). *Let $\mathbb{K} = \mathbb{C}$ and $\mathscr{C}(\mathscr{A}_1) = \mathscr{C}(\mathscr{A}_2) = \mathscr{C}$, $\mathscr{A}_1, \mathscr{A}_2$ in the same connected component of $\Sigma(\mathscr{C})$. Then, $\mathscr{A}_1$ and $\mathscr{A}_2$ have the same oriented topology.*

3. Arrangements over finite fields

Rybnikov's example is based on McLane combinatorics. By duality, one can produce combinatorics by point arrangements.

Definition 1. $\mathscr{C}_{\mathrm{ML}}$ McLane combinatorics $\mathscr{L}_{\mathrm{ML}} := \mathbb{F}_3^2 \setminus \{0\} \subset \mathbb{F}_3\mathbb{P}^2$.

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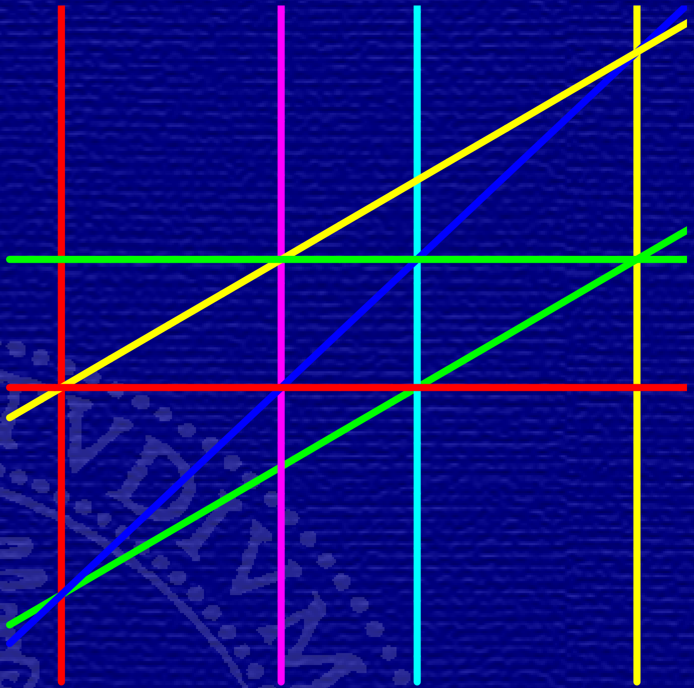
Consider $\mathbb{P}^2(\mathbb{F}_4) = \mathbb{F}_4^2 \coprod L_\infty$.

$P := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $A := \begin{pmatrix} 0 & 1 \\ 1 & \zeta \end{pmatrix} \in GL(2;\mathbb{F}_4)$, $A^5 = I_2$, $\zeta \in \mathbb{F}_4 \setminus \mathbb{F}_2$.

$\mathscr{C}$ given by:

$$\mathscr{L} := L_\infty \coprod \{A^j P \mid 0 \leq j \leq 4\}.$$



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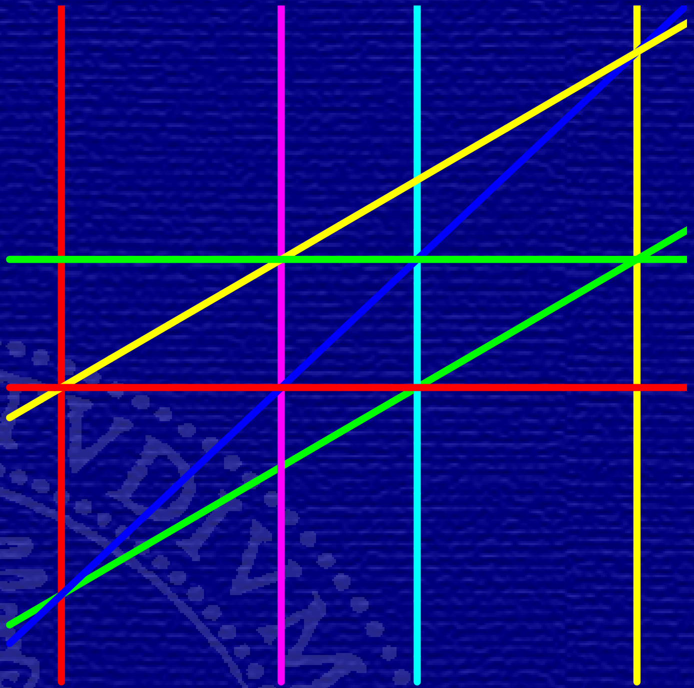

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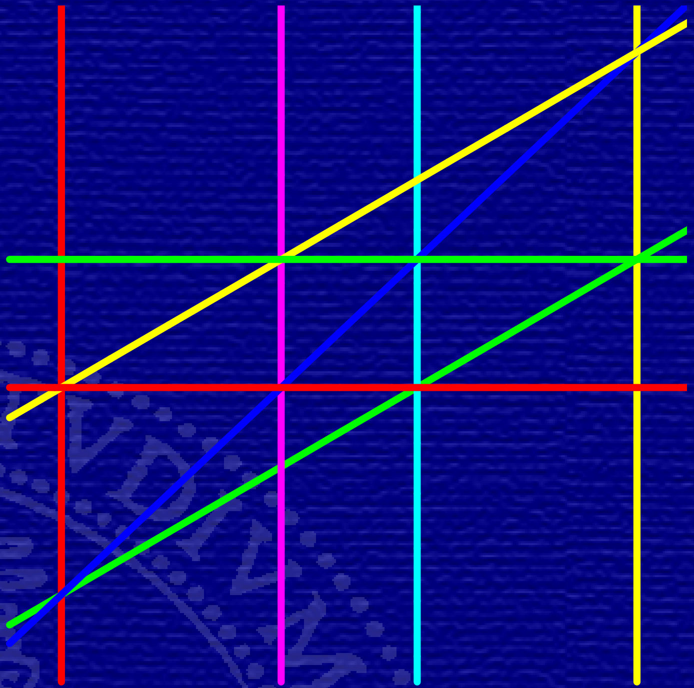

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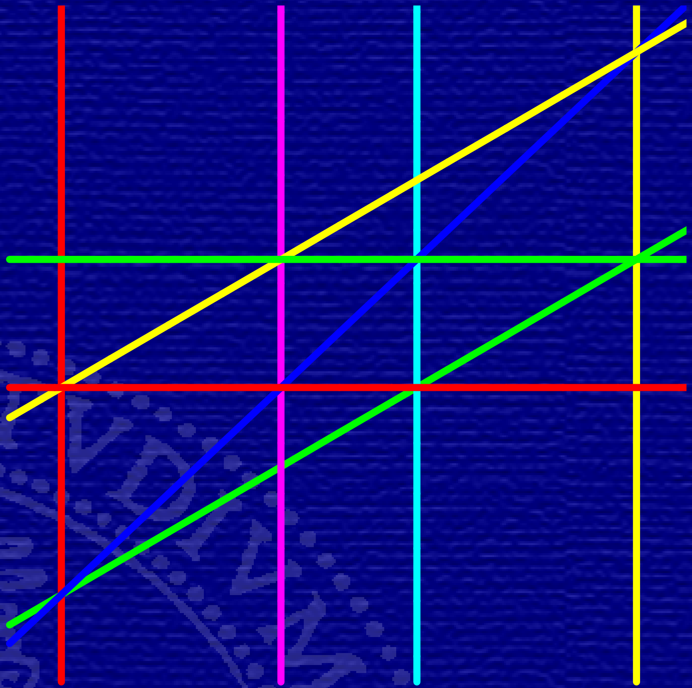

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4. Topology of ordered arrangements

$\mathscr{A}$ line arrangement in $\mathbb{P}^2$. $H := H_1(\mathbb{P}^2 \setminus \bigcup \mathscr{A})$ depends on $\mathscr{C}(\mathscr{A})$.

Theorem 3 ([5, 3]). *$\mathscr{A}_\pm$ representing elements $\mathscr{M}^{ord}_{\mathbb{C}}(\mathscr{C}_{ML})$. There exists no isomorphism $G_+ \to G_-$ inducing the identity on the homology H, $G_\pm := \pi_1(\mathbb{P}^2(\mathbb{C}) \setminus \mathscr{A}_\pm)$.*

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There is no Λ_2-isomorphism $M^2_+ \to M^2_-$ induced by an isomorphism $G_+ \to G_-$. Filtration linearizes the problem.

Theorem 4 ([2]). $\mathscr{A}_\pm$ *representing elements* $\mathscr{M}_{\mathbb{C}}^{\mathrm{ord}}(\mathscr{L})$*; there is no line-order-preserving homeomorphism* $(\mathbb{P}^2, \mathscr{A}_+) \to (\mathbb{P}^2, \mathscr{A}_-)$.

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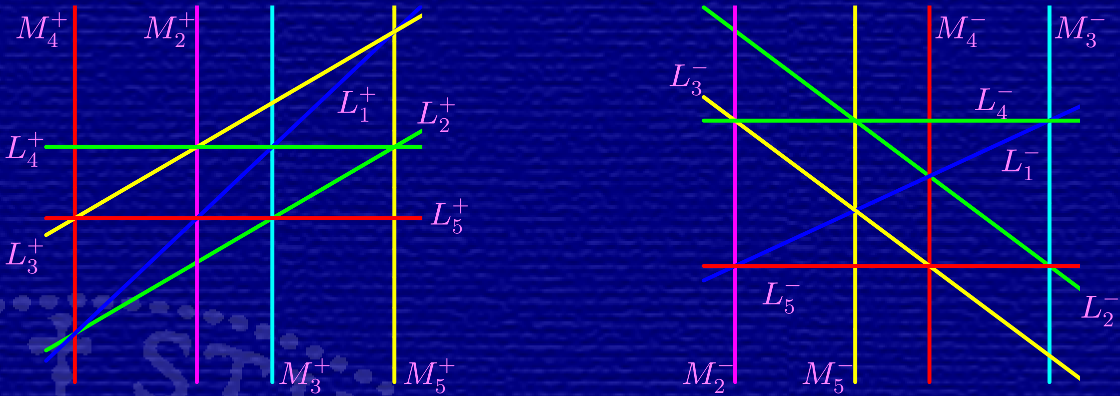
Set $L_\infty \in \mathscr{A}_i^v$ (same ordered line), P_i point at infinity of vertical lines.

Then monodromy groups and pseudo-Coxeter elements of affine $\mathscr{A}_1^h$ and $\mathscr{A}_2^h$ are conjugate by the same element in $\mathbb{P}_m$, $m = \#\mathscr{A}_1^h = \#\mathscr{A}_2^h$.



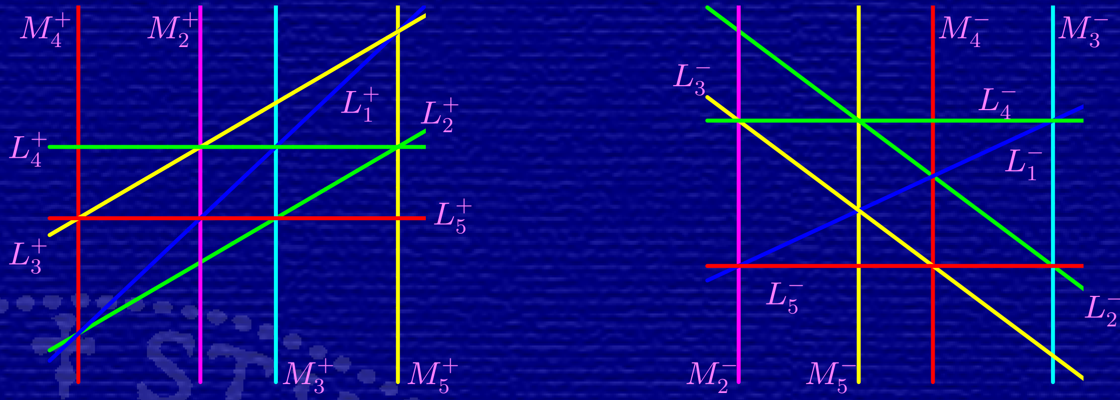
(a) $\mathscr{C}^+$ (b) $\mathscr{C}^-$

Figure 2: $M_1^\pm$ line at infinity

$$M_1 : z = 0, \quad M_2 : x = 0, \quad M_3 : x = z, \quad L_1 : y = x, \quad L_4 : y = z,$$

$$M_4 : x = -(\gamma + 1)z, \quad M_5 : x = (\gamma + 2)z, \quad L_2 : y = \gamma(x - z),$$

$$L_3 : y = \gamma x + z, \quad L_5 : y = 0, \quad \gamma^2 + \gamma - 1 = 0.$$



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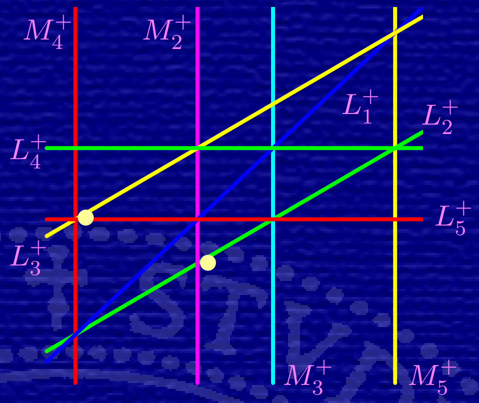
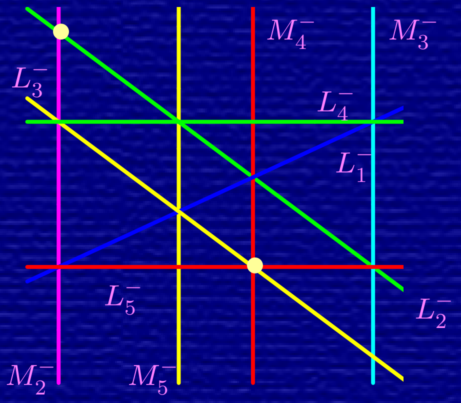
$$L_3 : y = \gamma x + z, \quad L_5 : y = 0, \quad \gamma^2 + \gamma - 1 = 0.$$

We compute braid monodromies and using a finite representation of $\mathbb{P}_5$ we deduce that monodromy groups and pseudo-Coxeter elements are not simultaneously conjugate in $\mathbb{P}_5$.

5. Same combinatorics and different topology





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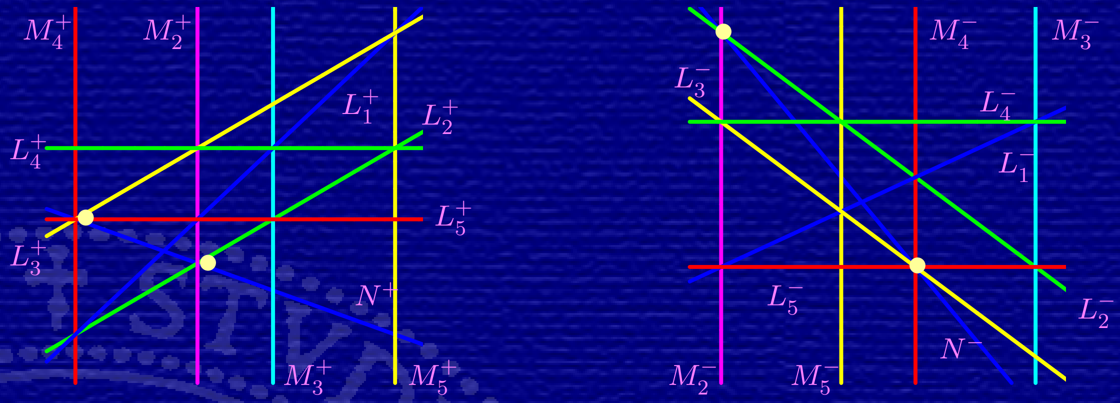

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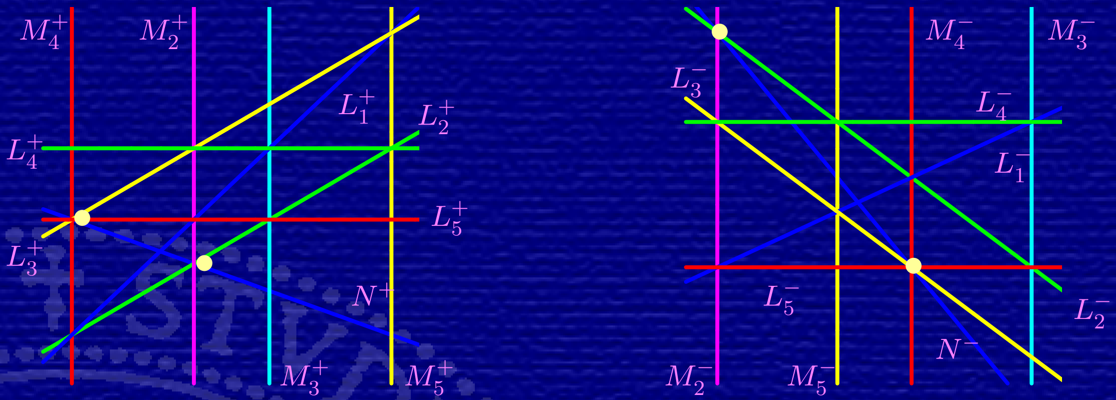

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$\widetilde{\mathscr{C}} = \mathscr{C}(\mathscr{H}^\pm)$, $\operatorname{Aut}\widetilde{\mathscr{C}}$ trivial $\Longrightarrow$ Theorem 2.

$\mathscr{C}_{\mathsf{Ryb}}$ combines two copies of $\mathscr{C}_{\mathsf{ML}}$ having a triple point $\{L_0, L_1, L_2\}$ in $\mathscr{P}_{\mathsf{ML}}$ in common: $\#\mathscr{L}_{\mathsf{Ryb}} = 13$.

The subgroup of $\mathrm{Aut}\,\mathscr{C}_{\mathsf{ML}}$ fixing $\{L_0, L_1, L_2\}$ is isomorphic to Σ_3 (transpositions exchange $\mathscr{A}_\pm$).

Construct $\mathscr{R}_{a,b}$ line arrangements s.t. $\mathscr{C}(\mathscr{R}_{a,b}) = \mathscr{C}_{\mathsf{Ryb}}$, taking an arrangement $\mathscr{A}_a$ and a modified $\mathscr{A}_b$, $a, b \in \pm$.

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Theorem 3 implies that $\mathscr{R}_{+,+}$ and $\mathscr{R}_{+,-}$ do not have the same topology.

Theorem 5 ([5, 3]). *Define* $G_{a,b} := \pi_1(\mathbb{P}^2 \setminus \bigcup \mathscr{R}_{a,b})$. *There exists no isomorphism* $\varphi : G_{+,+} \to G_{-,+}$ *inducing the identity on the homology.*

If so, $\varphi : M^2_{+,+} \to M^2_{-,+}$ $(\Lambda_2)_{\mathsf{Ryb}}$-isomorphism. Annihilating the action of the last five lines, $(\Lambda_2)_{\mathsf{Ryb}} \twoheadrightarrow (\Lambda_2)_{\mathsf{ML}}$.

Define $\widehat{M}^2_a := M^2_{a,+} \otimes (\Lambda_2)_{\mathsf{ML}}$, $a = \pm$, S_+ generated by commutators of the meridians of the first $\mathscr{C}_{\mathsf{ML}}$, excepted the corresponding to the first line.

$$\begin{array}{ccc} \widehat{M}^2_+ & \longrightarrow & \widehat{M}^2_- \\ \uparrow & & \downarrow \\ S_+ & \twoheadrightarrow & M^2_- \\ \downarrow & & \\ M^2_+ & & \end{array}$$

6. Proof of Rybnikov's result

$\mathscr{C}$ combinatorics, $\mathscr{A}$ line arrangement such that $\mathscr{C}(\mathscr{A}) = \mathscr{C}$,

$$H := \frac{\bigoplus_{\ell \in \mathscr{L}} \mathbb{Z} x_\ell}{\langle \sum_{\ell \in \mathscr{L}} x_\ell \rangle}, \qquad M_j^1 = \frac{H \wedge H}{R},$$

R generated by $\sum_{j \in P} x_i \wedge x_j$, $i \in P$, $P \in \mathscr{P}$.

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$$\operatorname{Aut}^1(H) := \{\psi\in\operatorname{Aut} H \mid \psi\wedge\psi \in \operatorname{Aut}(M_j^1)\}.$$

Note that $\operatorname{Aut}\mathscr{C} \times \{\pm 1_H\} \subset \operatorname{Aut}^1(H)$. Let $\psi\in\operatorname{Aut}^1(H)$, $n := \#\mathscr{L}$, $A := (a_{i,j}) \in M(n;\mathbb{Z})$ well-defined up to $\mathbb{1}_n := {}^t(1,\dots,1)\in\mathbb{Z}^n$.

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For simplicity, suppose $\mathscr{P}$ has at most triple points.

Admissibility conditions

$$\begin{vmatrix} a_u^i & a_v^i & 1 \\ a_u^j & a_v^j & 1 \\ a_u^k & a_v^k & 1 \end{vmatrix} = 0, \qquad \begin{vmatrix} a_\bullet^i & a_u^i + a_v^i + a_w^i & 1 \\ a_\bullet^j & a_u^j + a_v^j + a_w^j & 1 \\ a_\bullet^k & a_u^k + a_v^k + a_w^k & 1 \end{vmatrix} = 0$$

$\{\ell_u, \ell_v\} \in \mathscr{P}_2, \{\ell_i, \ell_j, \ell_k\}, \{\ell_u, \ell_v, \ell_w\} \in \mathscr{P}_3, \bullet = u, v, w.$

Given $P \in \mathscr{P}_3$, $A_P^{\psi} \in M(3 \times n, \mathbb{Z})$, involving P-rows. Let $\Sigma_k := \mathbb{Z}^{k+1}/\mathbb{1}_{k+1}$ and $v_1(P), \ldots, v_n(P) \in \Sigma_2$, column vectors $(\mathrm{mod}\ \mathbb{1}_3)$.

Lemma 1.

(1) $v_1(P) \ldots, v_n(P)$ *span* Σ_2 *and* $\sum_{j=0}^{r} v_j(P) = 0 \in \Sigma_2$.

(2) *Given* $Q \in \mathscr{P}$, $\ell_u \in Q$, $v_u(P)$ *and* $\sum_{\ell_i \in Q} v_i(P)$ *are l.d.*

(3) $\exists Q \in \mathscr{P}_3$ *such that* $\{v_i(P) \mid \ell_i \in Q\}$ *spans a rank-two sublattice of* Σ_2 *(*$\Rightarrow \sum_{\ell_i \in Q} v_i(P) = 0$*).*

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Lemma 1 is also true if we forget null v_i's.

$$\mathrm{Adm}_\psi(P) := \{\ell_i \in \mathscr{L} \mid v_i(P) \neq 0\}.$$

Definition 2. A line combinatorics $\mathscr{C}$ is called 3-admissible if it is possible to assign $\ell_i \mapsto v_i \in \mathbb{Z}^2 \setminus \{0\}$ s.t.:

1. $\exists P \in \mathscr{P}_3$, such that $\mathrm{rank}\langle v_j \mid \ell_j \in P\rangle = 2$.
2. $\forall P \in \mathscr{P}, \forall \ell_i \in P$, $\mathrm{rank}\langle v_i, \sum_{\ell_j\in P} v_j\rangle = 1$.
3. $\sum_{\ell_i\in\mathscr{L}} v_i = (0,0)$.

Examples 1.

1. $\mathrm{Adm}_\psi(P)$ by Lemma 1.

2. Triple points.

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Definition 3. A combinatorics $\mathscr{C}$ is pointwise 3-admissible if the only 3-admissible subcombinatorics of $\mathscr{C}$ are triple points.

Proposition 3. *If $\mathscr{C}$ is a pointwise 3-admissible combinatorics then any $\psi \in \mathrm{Aut}^1(H_{\mathscr{C}})$ induces a permutation ψ_3 of $\mathscr{P}_3$, $P \mapsto \mathrm{Adm}_\psi(P)$.*

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Question 1. *Does ψ_3 come from an element of $\mathrm{Aut}\,\mathscr{C}$?*

Definition 4. Three triple points P, Q, R of a line combinatorics are said to be in a triangle if $P \cap Q = \{\ell_1\}$, $P \cap R = \{\ell_2\}$ and $Q \cap R = \{\ell_3\}$ are pairwise different.

Proposition 4. *$\forall\psi \in \mathrm{Aut}^1(H)$, if $\mathscr{C}$ is pointwise 3-admissible, then ψ preserves triangles.*

If there are many triangles, the answer to Question 1 is **YES**.

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Theorem 6. *The fundamental groups of the two complex realizations of Rybnikov's combinatorics are not isomorphic.*

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