Bernstein-Sato polynomial and Yano's conjecture

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Topics in Singularities and Valuations
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Joint work with P. Cassou-Noguès, I. Luengo y A. Melle, arxiv:1602.07248

One-variable improper integrals

Hypothesis

$f : [0,1] \to \mathbb{R}$ $\mathcal{C}^\infty$ function, $f > 0$ in $[0,1]$, $a, b \in \mathbb{N}$

$$s \in U \subset \mathbb{C} \xrightarrow{\mathcal{I}} \int_0^1 f^s(x)\, x^{as+b} \frac{dx}{x}$$

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$$\Re s > -\frac{b}{a}$$

Holomorphic for $\Re s > -\frac{b}{a}$

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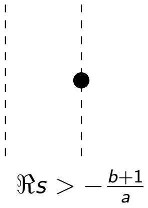
Meromorphic continuation to $\Re s > -\frac{b+1}{a}$ with eventual pole in $-\frac{b}{a}$



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Summary

$\mathcal{I}(s)$ meromophic with order-1 negative rational poles

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$f \in \overline{\mathbb{Q}}_{\mathbb{R}}[x]$, $f > 0$ in $(0,1]$, $a, b \in \mathbb{N}$

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$\mathcal{I}(s)$ meromophic with order-1 negative rational poles with algebraic residues

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f, g $\overline{\mathbb{Q}}_{\mathbb{R}}$-analytic for $x^{\frac{1}{N}}$ en $[0,1]$, $f > 0$ in $(0,1]$, $a, b \in \mathbb{N}$

$$s \in U \subset \mathbb{C} \xrightarrow{\mathcal{I}} \int_0^1 f^s(x) g(x)\, x^{as+b} \frac{dx}{x}$$

$$f^s(x)g(x) = f^s(0)g(0) + x g_s(x) \Rightarrow \mathcal{I}(s) = \frac{f^s(0)g(0)}{as+b} + \int_0^1 g_s(x)\, x^{as+b+1} \frac{dx}{x}$$

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Question

What happens if $f > 0$ in $[0,1]^2 \setminus \{(0,0)\}$?

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- $\mathcal{I}(s)$ meromophic with poles at negative rational numbers
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Question

What happens if $f > 0$ in $[0,1]^2 \setminus \{(0,0)\}$? Singularity properties matter.

Example

$$\int_0^1\int_0^1 \left(x^4 + y^5 + tx^2y^3\right)^s x^{\beta_1} y^{\beta_2} \frac{dx}{x}\frac{dy}{y}$$

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$$20\int_0^1\int_0^1 \left(x^{20}+y^{20}+tx^{10}y^{12}\right)^s x^{5\beta_1}y^{4\beta_2}\,\frac{dx}{x}\,\frac{dy}{y}$$

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$$\mathcal{I}_1 \overset{y\mapsto xy}{=} \int_0^1\int_0^1 \left(1+y^{20}+tx^2y^{12}\right)^s x^{20s+5\beta_1+4\beta_2} y^{4\beta_2}\,\frac{dx}{x}\,\frac{dy}{y}$$

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Poles:

$$-\frac{5\beta_1+4\beta_2+k}{20}$$

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Residues

$$\begin{cases} \dfrac{\mathbf{B}\left(\frac{\beta_1}{4},\frac{\beta_2}{5}\right)}{20} & \text{if } k=0 \\ A+t\dfrac{11\mathbf{B}\left(\frac{3}{4},\frac{4}{5}\right)}{400} & \text{if } \beta_i=1, k=2 \end{cases}$$

Poles

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Theorem

$\alpha = \frac{\beta_1 m + \beta_2 n_1}{m n_1 n_2} \in B_{11}$, $\forall f \in \overline{\mathbb{Q}}_{\mathbb{R}}[x,y]$ *positive, i.e.,*

- $f(x,y) = (x^{n_1} + y^m + \dots)^{n_2} + x^a y^b + \dots$ $(mn_1 n_2 + q = am + bn_1)$
- $f > 0$ *en* $[0,1]^2 \setminus \{(0,0)\}$

Then,

$$\int_0^1 \int_0^1 f^s(x,y) x^{\beta_1} y^{\beta_2} \frac{dx}{x} \frac{dy}{y}$$

has a pole of order 1 *at* $-\alpha$ *with transcendental residue.*

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Theorem

$\alpha = \frac{\beta_1 m + \beta_2 n_1 + k}{m n_1 n_2} \in A_{12}$, $k > 0$, $\exists f \in \overline{\mathbb{Q}}_{\mathbb{R}}[x,y]$ *positive such that*

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Strategy

$$f^s = \frac{1}{b_f(s)} \qquad P \cdot f^{s+1}$$

Strategy

$$f^s x^{\beta_1 - 1} y^{\beta_2 - 1} \qquad = \frac{1}{b_f(s)} \qquad P \cdot f^{s+1} x^{\beta_1 - 1} y^{\beta_2 - 1}$$

Strategy

$$\int_0^1\int_0^1 f^s x^{\beta_1-1} y^{\beta_2-1} dxdy = \frac{1}{b_f(s)} \int_0^1\int_0^1 P \cdot f^{s+1} x^{\beta_1-1} y^{\beta_2-1} dxdy$$

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$-\alpha$ LHS pole with trascendental residue

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Second integral: linear combination lineal of

$$\int_0^1\int_0^1 \frac{\partial^{i+j} f^{s+1}(x,y)}{\partial x^i \partial y^j} x^{\beta_1'} y^{\beta_2'} dxdy$$

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Integration by parts:

$$\int_0^1 \left[\frac{\partial^{i+j-1} f^{s+1}(x,y)}{\partial x^{i-1}\partial y^j} x^{\beta_1'} y^{\beta_2'}\right]_{x=0}^{x=1} dy$$
$$-\beta_1' \int_0^1\int_0^1 \frac{\partial^{i+j-1} f^{s+1}(x,y)}{\partial x^{i-1}\partial y^j} x^{\beta_1'-1} y^{\beta_2'} dxdy$$

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$-\alpha$ LHS pole with trascendental residue
$\mathcal{J}$ linear combination of 1-variable integrals ($\Rightarrow$ algebraic residues) and

$$\underbrace{\int_0^1\int_0^1 f^{s+1}(x,y) x^{\beta_1'} y^{\beta_2'} dxdy}_{\text{maximal pole } -\text{lct}-1}$$

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Theorem

$-\alpha$ is a root of the Bernstein polynomial of f.

Strategy for B_2

$$\iint_{\mathcal{D}} f^s x^{\beta_1 - 1} y^{\beta_2 - 1} g(x,y)^{\beta_3} dxdy$$
$$= \frac{1}{b_f(s)} \iint_{\mathcal{D}} P \cdot f^{s+1} x^{\beta_1 - 1} y^{\beta_2 - 1} g(x,y)^{\beta_3} dxdy$$

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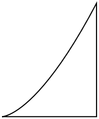
$$f(x,y) \textit{ negative } \mapsto x = \cdots * y^{\frac{m_1}{n_1}} + \ldots + * y^{\frac{m_2}{n_1 n_2}} + \ldots$$

$$g(x,y) \mapsto x = \cdots * y^{\frac{m_1}{n_1}} + \ldots$$

$g(x,y) = 0$



$\mathcal{D}$

Maximal pole of

$$\iint_{\mathcal{D}} f^{s+1}(x,y) x^{\beta_1'} y^{\beta_2'} g(x,y)^{\beta_3'} dxdy \quad \equiv \alpha \bmod \mathbb{Z} \text{ is } -\min B_2 - 1$$