

Denef-Loeser zeta functions of Lê-Yomdin and suspension singularities II

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Topological zeta functions

- ▶ $f(x_1, \dots, x_d) \in \mathbb{C}\{x_1, \dots, x_d\}$, $f(0) = 0$, $X = V(f)$, ω **monomial** holomorphic d -form of $\mathbb{C}^d$.



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- ▶ $\pi^* X = \sum_{i \in I} N_i E_i$, $(\pi^* \omega) = \sum_{i \in I} (\nu_i - 1) E_i$
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$$Z_{\text{top}}(f, \omega, s)_0 := \sum_{J \in \mathcal{P}_0} \chi(\mathring{E}_J) \prod_{i \in J} \frac{1}{\nu_i + N_i s}$$

- ▶ This definition does not depend on the choice of π .



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Monodromy and holomorphy conjectures

Notation

- ▶ $\text{Pol}^+(g, \omega) := \{\rho_0 \in \mathbb{Z} \mid -\rho_0 \text{ is a pole of } Z_{\text{top}}(g, s)\}$, $\rho_0 \in \text{Pol}^+(g, \omega)$ is of the form $\frac{\nu_i}{N_i}$.
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Strong monodromy conjecture, Igusa

If $\rho_0 \in \text{Pol}^+(g)$, then $-\rho_0$ is a root of the Bernstein polynomial.



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Question

Is there an isolated singularity $(g^{-1}(0), \mathbf{0}) \subset (\mathbb{C}^{d+1}, \mathbf{0})$
such that if $\ell \in \mathcal{E}_g^{\text{ord}}$, then $\ell \equiv 2 \pmod{4}$?



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Question

Is there an isolated singularity $(g^{-1}(0), \mathbf{0}) \subset (\mathbb{C}^{d+1}, \mathbf{0})$, with multiplicity > 2 , such that if $\ell \in \mathcal{E}_g^{\text{ord}}$, then $\ell \equiv 2 \pmod{4}$?



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- ▶ Holomorphy conjecture for curves (Veys), *most* homogeneous singularities (Rodrigues-Veys), Newton non-degenerate surfaces (Castricky-Ibadula-Lemahieu).



Lê-Yomdin singularities and suspension

Lê-Yomdin singularities

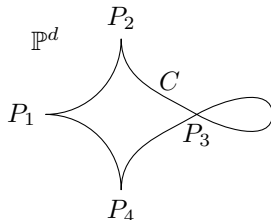
$V := F^{-1}(0)$, $F(\mathbf{x}) = F_m(\mathbf{x}) + F_{m+k}(\mathbf{x}) + \dots$, $\text{Sing } V_{\mathbb{P}}(F_m) \cap V_{\mathbb{P}}(F_{m+k}) = \emptyset$.
In particular, $C_m := V_{\mathbb{P}}(F_m)$ has isolated singularities.

From $(d-1)$ -global to d -local.

Blow-up

$\pi : \hat{\mathbb{C}}^{d+1} \rightarrow \mathbb{C}^{d+1}$, chart $(x_1, \dots, x_d, z) \mapsto (x_1 z, \dots, x_d z, z)$

$$\begin{aligned}\pi \circ F &= z^m (F_m(x_1, \dots, x_d, 1) + z^k (F_{m+k}(x_1, \dots, x_d, 1) + \dots)) \\ &\equiv z^m (f(u_1, \dots, u_d) + z^k)\end{aligned}$$



$$\mathbb{P}^d \setminus C_m : z^m, z^d d\mathbf{x} \wedge dz$$

$$C_m \setminus \text{Sing } C_m : u_1 z^m, z^d d\mathbf{u} \wedge dz$$

$$P_i : z^m (z^k - f_q(\mathbf{u})), z^d d\mathbf{u} \wedge dz$$



Lê-Yomdin singularities and suspension

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Suspensions

$f(x_1, \dots, x_d) \in \mathbb{C}\{x_1, \dots, x_d\}$, $F := z^k + g(x_1, \dots, x_d) \in \mathbb{C}\{x_1, \dots, x_d, z\}$.

From $(d-1)$ -local to d -local.

Goal

Produce formulas for the zeta functions of suspensions and Lê-Yomdin singularities in terms of 1-less-dimensional zeta functions.



Zeta functions for suspensions

- ▶ $g = z^k - f$, $1 = k_1 < k_2 < \cdots < k_r = k$, divisors of k , $t = s + \frac{1}{k}$

$$\begin{pmatrix} \frac{s+1}{s} Z_{\text{top}}(F, s) \\ Z_{\text{top}}^{(k_2)}(F, s) \\ \vdots \\ Z_{\text{top}}^{(k_r)}(F, s) \end{pmatrix} =$$

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$$\sum_{j=1}^r J_2(k_j) = k^2 \quad A = k\mathbf{1}_r - \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \begin{pmatrix} J_2(k_1) & \cdots & J_2(k_r) \end{pmatrix}$$

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- $\ell \nmid k, \ell_1 := \frac{\ell}{\gcd(\ell, q)}, \mathfrak{m}(k, \ell, q) = \ell_1 \gcd(k, \ell_1^a), a \gg 1:$

$$Z_{\text{top}}^{(\ell)}(F, s)_{\mathbf{0}} = Z_{\text{top}}^{(\ell)}(f, t)_{\mathbf{0}} - \sum_{e|k} \frac{J_2(e)}{k} Z_{\text{top}}^{(\text{lcm}(e, \mathbf{m}(k, \ell, k)))}(f, t)_{\mathbf{0}}.$$



Untwisted topological zeta function for k -LYS and SIS

$$r = \frac{d+1+(m+k)s}{k}, \quad r-s = \frac{d+1+ms}{k}, \quad r=0 \text{ not a pole}$$

$$\begin{aligned} Z_{\text{top}}(F, s)_{\mathbf{0}} &= \frac{\chi(\mathbb{P}^d \setminus C_m)}{k(r-s)} + \frac{\chi(C \setminus \text{Sing } C_m)}{k(r-s)(s+1)} \\ &\quad + \frac{1}{k} \sum_{q \in \text{Sing } C_m} \left(\frac{1}{r} + \frac{s(r+1)(s-r+1)}{r(s+1)(r-s)} Z_{\text{top}}(f_q, r)_{\mathbf{0}} \right. \\ &\quad \left. - \frac{s}{s+1} \sum_{1 \neq e|k} J_2(e) Z_{\text{top}}^{(e)}(f_q, r)_{\mathbf{0}} \right). \end{aligned}$$

$$\Delta(\tau) = \frac{(\tau^m - 1)^{\chi(\mathbb{P}^2 \setminus C_m)}}{\tau - 1} \prod_{q \in \text{Sing } C_m} \Delta_q^{(k)}(\tau^{m+k})$$

$\Delta_q^{(k)}$ characteristic polynomial of $\Psi_q^k(f_q)_{\tilde{H}^*}$ of (C_m, q) .



Untwisted topological zeta function for k -LYS and SIS

$$t = d + 1 + (m + 1)s$$

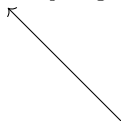
$$Z_{\text{top}}(F, s)_{\mathbf{0}} = \frac{\chi(\mathbb{P}^d \setminus C_m)}{t - s} + \frac{\chi(C \setminus \text{Sing } C_m)}{(t - s)(s + 1)} \\ + \sum_{q \in \text{Sing } C_m} \left(\frac{1}{t} + \frac{s(t + 1)(s - t + 1)}{t(s + 1)(t - s)} Z_{\text{top}}(f_q, t)_{\mathbf{0}} \right).$$

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$$\mathrm{Pol}^+(F) \subset \left\{1, \frac{n+1}{m}\right\} \cup \bigcup_{q \in \mathrm{Sing} C_m} \left\{ \frac{k\rho_0 + n + 1}{m+k} \mid \rho_0 \in \mathrm{Pol}^+(f_q) \right\}$$



Sign of $\chi(\mathbb{P}^2 \setminus C_m)$

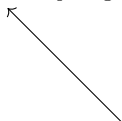


$$\bigcup_{q \in \mathrm{Sing} C_m} \overline{\left\{ \frac{\ell(m+k)}{\gcd(\ell, k)} \mid \ell \in \overline{\mathcal{E}}_{f_q}^{\mathrm{ord}} \right\}} \subset \overline{\mathcal{E}}_F^{\mathrm{ord}} \subset \overline{\{m\}} \cup \bigcup_{q \in \mathrm{Sing} C_m} \overline{\left\{ \frac{\ell(m+k)}{\gcd(\ell, k)} \mid \ell \in \overline{\mathcal{E}}_{f_q}^{\mathrm{ord}} \right\}}$$



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Twisted topological zeta functions for k -LYS and SIS

k -LYS: $1 \neq \ell \mid \gcd(m+k, m)$

$$\begin{aligned} Z_{\text{top}}^{(\ell)}(F, s)_{\mathbf{0}} &= \frac{\chi(\mathbb{P}^n \setminus C_m)}{k(r-s)} \\ &+ \sum_{q \in \text{Sing } C_m} \left(\frac{1 - (r+1)Z_{\text{top}}(f_q, r)_{\mathbf{0}}}{kr} + \frac{Z_{\text{top}}^{(\ell)}(f_q, r)_{\mathbf{0}}}{k(r-s)} \right. \\ &\left. - \sum_{1 \neq e \mid k} \frac{J_2(e)}{k} Z_{\text{top}}^{(e)}(f_q, r)_{\mathbf{0}} \right). \end{aligned}$$



Twisted topological zeta functions for k -LYS and SIS

k -LYS: $\ell \mid m+k$, $\ell \nmid m$

$$Z_{\text{top}}^{(\ell)}(F, s)_{\mathbf{0}} = \sum_{q \in \text{Sing } C_m} \left(\frac{1 - (r+1)Z_{\text{top}}(f_q, r)_{\mathbf{0}}}{kr} - \sum_{1 \neq e \mid k} \frac{J_2(e)}{k} Z_{\text{top}}^{(e)}(f_q, r)_{\mathbf{0}} \right).$$

SIS: $\ell \mid m+1$

$$Z_{\text{top}}^{(\ell)}(F, s)_{\mathbf{0}} = \sum_{q \in \text{Sing } C_m} \frac{1 - (t+1)Z_{\text{top}}(f_q, t)_{\mathbf{0}}}{t}.$$



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$$\ell_1 := \frac{\ell}{\gcd(\ell, m+k)}, \mathbf{m}(k, \ell, m+k) = \ell_1 \gcd(k, \ell_1^a), a \gg 1$$

$$Z_{\text{top}}^{(\ell)}(F, s)_{\mathbf{0}} = \frac{\chi(\mathbb{P}^n \setminus C_m)}{k(r-s)} + \sum_{q \in \text{Sing } C_m} \left(\frac{Z_{\text{top}}^{(\ell)}(f_q, r)_{\mathbf{0}}}{k(r-s)} - \sum_{e \mid k} \frac{J_2(e)}{k} Z_{\text{top}}^{(\ell)}(f_q, r)_{\mathbf{0}}^{(\text{lcm}(e, \mathbf{m}(k, \ell, m+k)))} \right).$$

SYS: $\ell \mid m$

$$Z_{\text{top}}^{(\ell)}(F, s)_{\mathbf{0}} = \frac{\chi(\mathbb{P}^n \setminus C_m)}{t-s} + \sum_{q \in \text{Sing } C_m} \frac{s-t+1}{t-s} Z_{\text{top}}^{(\ell)}(f_q, t)_{\mathbf{0}}.$$



Twisted topological zeta functions for k -LYS and SIS

k -LYS: $\ell \nmid m+k, \ell \nmid m$:

$$Z_{\text{top}}^{(\ell)}(F, s)_{\mathbf{0}} = -\frac{1}{k} \sum_{q \in \text{Sing } C_m} \sum_{e|k} J_2(e) Z_{\text{top}}^{(\text{lcm}(e, \mathbf{m}(k, \ell, m+k)))}(f_q, r)_{\mathbf{0}}.$$

SIS: $\ell \nmid m+1, \ell \nmid m$:

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Monodromy conjecture for k -LYS, $d = 2$

- ▶ Same structure as SIS (with some arithmetics).



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$$\mathrm{Pol}^+(F) \subset \left\{1, \frac{n+1}{m}\right\} \cup \bigcup_{q \in \mathrm{Sing} C_m} \left\{ \frac{k\rho_0 + n+1}{m+k} \mid \rho_0 \in \mathrm{Pol}^+(f_q) \right\}$$

Sign of $\chi(\mathbb{P}^2 \setminus C_m)$

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- Veys' structure for curves does not work.



Holomorphy conjecture for SIS, $d = 2$

- ▶ True if C_m smooth.

$$\bigcup_{q \in \text{Sing } C_m} \overline{\left\{ \ell(m+1) \mid \ell \in \overline{\mathcal{E}}_{f_q}^{\text{ord}} \right\}} \subset \overline{\mathcal{E}}_F^{\text{ord}} \subset \overline{\{m\}} \cup \bigcup_{q \in \text{Sing } C_m} \overline{\left\{ \ell(m+1) \mid \ell \in \overline{\mathcal{E}}_{f_q}^{\text{ord}} \right\}}$$



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- ▶ If $\ell \mid m$ and $\chi(\mathbb{P}^2 \setminus C_m) < 0$ impossible by the structure of Kashiwara's pencils.



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Holomorphy conjecture for suspensions

Thom-Sebastiani formula for poles in zeta functions (Denef-Veys)

$f(\mathbf{x}) \in \mathbb{C}[\mathbf{x}]$, $g(\mathbf{y}) \in \mathbb{C}[\mathbf{y}]$, $F := f(\mathbf{x}) + g(\mathbf{y})$, $\ell \in \mathbb{Z}_{>0}$, $\rho_0 \in \text{Pol}^{+(\ell)}(F)$.

- ▶ $\exists \ell_1, \ell_2 \in \mathbb{Z}_{>0}$, with $\ell_1 \ell_2 = \ell$,
- ▶ $\exists c \in \mathbb{C}$,
- ▶ $\exists \rho_f \in \text{Pol}^{+(\ell_1)}(f + c)$, $\exists \rho_g \in \text{Pol}^{+(\ell_2)}(g - c)$,
- ▶ $\rho_0 = \rho_f + \rho_g$.
- ▶ (Thom-Sebastiani) $\mathcal{E}_F = \{\zeta_f \cdot \zeta_g \mid \zeta_f \in \mathcal{E}_f, \zeta_g \in \mathcal{E}_g\}$.



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- ▶ $\exists \ell_1 \in \mathbb{Z}_{>0}$, with $\ell_1 \mid \ell$,
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Theorem (Denef-Veys)

f satisfies the holomorphy conjecture

- ▶ $f + z^k$ satisfies the holomorphy conjecture, $k > 2$.
- ▶ $f + z_1^2 + z_2^2$ satisfies the holomorphy conjecture.
- ▶ Brieskorn-Pham singularities satisfy the holomorphy conjecture.



Holomorphy conjecture for suspensions by two points

Eigenvalues behavior

- ▶ If $k > 2$, $\overline{\mathcal{E}}_f^{\text{ord}} \subset \overline{\mathcal{E}}_{f+z^k}^{\text{ord}}$.
- ▶ Cyclotomic polynomials: $\Phi_{4a}(-t) = \Phi_{4a}(t)$, $\Phi_{4b+2}(-t) = \Phi_{2b+1}(t)$.



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Theorem (Denef-Veys)

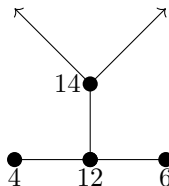
If f satisfies the holomorphy conjecture, and $\nexists a$ such that

- ▶ $2(2a+1) \in \mathcal{E}_f^{\text{ord}}$
- ▶ $4(2a+1) \notin \overline{\mathcal{E}}_f^{\text{ord}}$

then the holomorphy conjecture holds for $f + z^2$.

$$f = (y^2 - x^3)^2 - x^4 y^2$$

$$\Delta(\tau) = \frac{(\tau-1)(\tau^{12}-1)(\tau^{14}-1)}{(\tau^4-1)(\tau^6-1)}$$



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Theorem (Denef-Veys+AGGL)

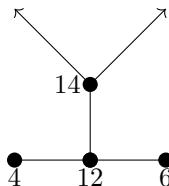
If f satisfies the holomorphy conjecture, $\ell \notin \overline{\mathcal{E}}_{f+z^2}^{\text{ord}}$, and

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then $Z_{\text{top}}^{(\ell)}(f + z^2, s) = 0$.

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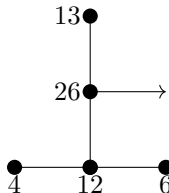
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$$Z_{\text{top}}^{(\ell)}(f + z^2, s), \ell \equiv 2 \pmod{4}, \frac{\ell}{2} \notin \overline{\mathcal{E}}_f^{\text{ord}}$$

Question

Is the vanishing of $Z_{\text{top}}^{(2)}(F, s)_{\mathbf{0}} = \frac{1}{2} \left(\frac{1}{t} - Z_{\text{top}}^{(2)}(f, t)_{\mathbf{0}} - \frac{t+1}{t} Z_{\text{top}}(f, t)_{\mathbf{0}} \right)$ relevant?

It is relevant if all the eigenvalues of f are of order $\equiv 2 \pmod{4}$, i.e., all the eigenvalues of $f + z^2$ are of odd order.



$$Z_{\text{top}}^{(\ell)}(f + z^2, s), \ell \equiv 2 \pmod{4}, \frac{\ell}{2} \notin \overline{\mathcal{E}}_f^{\text{ord}}$$

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Lemma

$$\ell = 2\ell_2, \ell_2 \text{ odd}$$

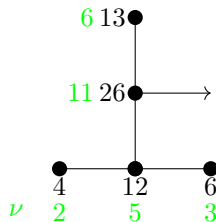
$$Z_{\text{top}}^{(\ell)}(F, s) = -\frac{1}{2} \left(Z_{\text{top}}^{(\frac{\ell}{2})}(f, t) + Z_{\text{top}}^{(\ell)}(f, t) \right).$$



Example and theorem

$$f = (y^2 - x^3)^2 - x^5 y$$

$$\Delta(\tau) = \frac{(\tau-1)(\tau^{12}-1)(\tau^{13}+1)}{(\tau^4-1)(\tau^6-1)}$$



Example and theorem

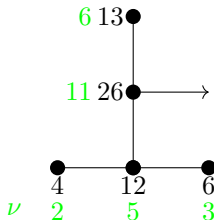
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Example

$$f = (y^2 - x^3)^2 - x^5 y:$$

$$Z_{\text{top}}^{(26)}(f, s) = \frac{-1}{11 + 26s}$$



Example and theorem

$$f = (y^2 - x^3)^2 - x^5 y$$

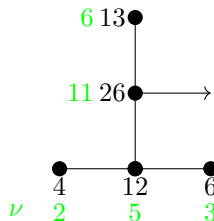
$$\Delta(\tau) = \frac{(\tau-1)(\tau^{12}-1)(\tau^{13}+1)}{(\tau^4-1)(\tau^6-1)}$$

Example

$$f = (y^2 - x^3)^2 - x^5 y:$$

$$Z_{\text{top}}^{(26)}(f, s) = \frac{-1}{11 + 26s}$$

$$Z_{\text{top}}^{(13)}(f, s) = \frac{-1}{11 + 26s} + \frac{1}{(11 + 26s)(6 + 13s)} + \frac{1}{6 + 13s}$$



Example and theorem

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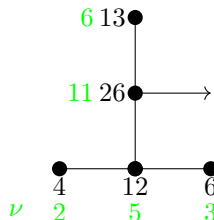
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$$Z_{\text{top}}^{(26)}(f, s) = \frac{-1}{11 + 26s}$$

$$\begin{aligned} Z_{\text{top}}^{(13)}(f, s) &= \frac{-1}{11 + 26s} + \frac{1}{(11 + 26s)(6 + 13s)} + \frac{1}{6 + 13s} \\ &= \frac{-6 - 13s + 1 + 11 + 26s}{(11 + 26s)(6 + 13s)} \end{aligned}$$



Example and theorem

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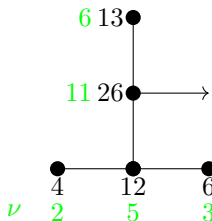
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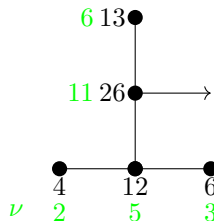
$$\begin{aligned} Z_{\text{top}}^{(13)}(f, s) &= \frac{-1}{11 + 26s} + \frac{1}{(11 + 26s)(6 + 13s)} + \frac{1}{6 + 13s} \\ &= \frac{-6 - 13s + 1 + 11 + 26s}{(11 + 26s)(6 + 13s)} \\ &= \frac{1}{11 + 26s}. \end{aligned}$$



Example and theorem

$$f = (y^2 - x^3)^2 - x^5 y$$

$$\Delta(\tau) = \frac{(\tau-1)(\tau^{12}-1)(\tau^{13}+1)}{(\tau^4-1)(\tau^6-1)}$$



Theorem

The holomorphy conjecture holds for suspensions of curves by two points.

Gracias por su atención
Muito obrigado
Thank you very much

