Intersections of ellipsoids and singularities I

E. ARTAL BARTOLO[1] and S. LÓPEZ DE MEDRANO[2]

[1] Instituto Universitario de Matemáticas y sus Aplicaciones
Universidad de Zaragoza
[2] Instituto de Matemáticas
Universidad Nacional Autónoma de México

International Congress on
Complex Geometry, Singularities and Dynamics:
In honor of José Seade
Cuernavaca, June 4th 2024

Joint work with M.T. Lozano Imízcoz. Funded by MCIN/AEI/10.13039/501100011033 (PID2020-114750GB-C31) and Gobierno de Aragón (E22_20R: "Álgebra y Geometría").





Origin of the problem

Dynamical Systems (Camacho-Kuiper-Palis)

Topological classification of generic system of ordinary differential equations. What is the topology of the space of Siegel leaves? Determined by the intersection of a sphere and two coaxial quadrics. Alberto Verjovski asked me to solve the problem in one week. It took years.





Topological results I

- The intersection Z of $m+1$ coaxial ellipsoids:

$$\begin{aligned} x_1^2 + \cdots + x_n^2 &= 1 \\ A_1 x_1^2 + \cdots + A_n x_n^2 &= 0 \\ \mathcal{A} := (A_1, \ldots, A_n) &\subset (\mathbb{R}^m)^n \end{aligned}$$

- Topologically, one can restrict the attention to the first orthant and take square roots: a polytope P:

$$\begin{aligned} r_1 + \cdots + r_n &= 1 \\ r_1 A_1 + \cdots + r_n A_n &= 0 \\ r_1, \ldots, r_n &\geq 0 \end{aligned}$$

- Homology of Z can be expressed using relative homology of P with respect to some subcomplexes.

Instituto de Matemáticas

Topological results II

- Study for C_1, C_2 coaxial ellipsoids, $Z = \mathbb{S}^n \cap C_1 \cap C_2 \subset \mathbb{R}^{n+1}$
- Some of the cases are product of spheres.
- MacGavran studied toric varieties and connected sums appear.
- Using h-cobordism: Z is a connected sum of product of spheres. What about low dimensions?
- Some of these results were obtained in IHES invited by Dennis Sullivan.



Smooth/Singular

- I have related smoothness with a condition of tranversality called *weaky hyperbolicity* (WH).
- Pepe's question: Is it smoothness equivalent to (WH)?
- Solved in a paper in PepeFest volume: roughly speaking, the answer is yes under some non-degeneracy conditions: independent equations, Z not contained in a coordinate hyperplane.
- State singular theorem
- Examples of singularities: cones and suspensions of products of spheres
- Simple polytopes.

