Intersections of ellipsoids and singularities II

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Joint work with M.T. Lozano Imízcoz

Intersections of ellipsoids

$$\begin{aligned}
\text{Intersections} : Z \\
\alpha_1^0 x_1^2 + \cdots + \alpha_n^0 x_n^2 &= 1 \\
\alpha_1^1 x_1^2 + \cdots + \alpha_n^1 x_n^2 &= 1 \\
\cdots \\
\alpha_1^m x_1^2 + \cdots + \alpha_n^m x_n^2 &= 1 \\
\alpha_i^j &> 0
\end{aligned}$$

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- P: the coefficients of the convex linear combination of $\mathcal{A}$ whose result is $\mathbf{0}$.

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Non-degenerate (from Pepe's question):

- $\mathcal{A} \subset (\mathbb{R}^m)^n$ generator system of $\mathbb{R}^m$
- $\mathcal{A} \not\subset$ half closed subspace $\mathbb{R}^m$

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Smooth Z:

- (WH): $\mathbf{0}$ is not a convex linear combination of m elements in $\mathcal{A}$
- Under non-degenerate: Z smooth $\Longleftrightarrow$ (WH)
- Vertices of P are simple and belong to exactly $d = \dim Z = \dim P$ coordinate hyperplanes.

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Goal

Understand intersections with generic singularities.

Examples

$m = 1, n = 2$

- $\mathcal{A} = (-1, 1)$:

$$x_1^2 + x_2^2 = 1$$
$$-x_1^2 + x_2^2 = 0$$

- $P = \{(\frac{1}{2}, \frac{1}{2})\}$
- $Z = \mathbb{S}^0 \times \mathbb{S}^0$ (smooth)

Examples

Proposition

$\mathcal{A}_1 = \mathcal{A} \cup \{\mathbf{0}\} \Longrightarrow Z_1 = \Sigma(Z)$, *suspension.*

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Examples

Proposition

$\mathcal{A}_1 = \mathcal{A} \cup \{\mathbf{0}\} \Longrightarrow Z_1 = \Sigma(Z)$, *suspension.*

$m = 0$

$\mathcal{A} = (\mathbf{0}^2)$: $Z = \mathbb{S}^1$

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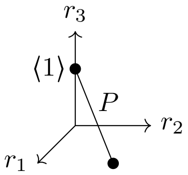
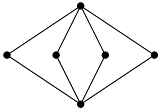
$\mathcal{A} = \{-1, 1, 0\}$

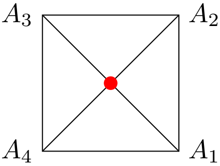


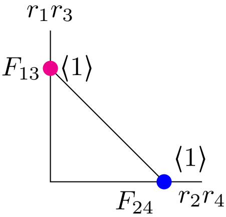


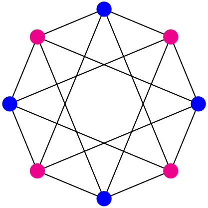
$K_{2,4} \cong Z$

Examples

1-dimensional intersections with two generic singularities







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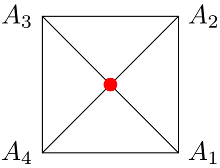
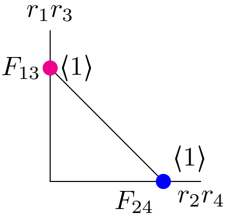
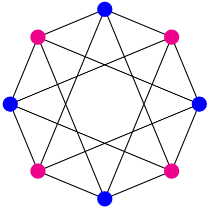
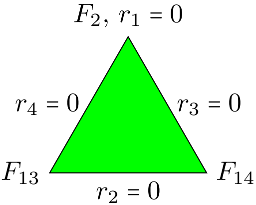
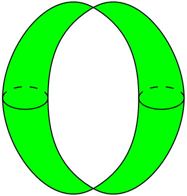






2-dimensional intersections with one generic singularity





$\mathcal{A} = \{-1, -1, 0, 1\}$

Examples

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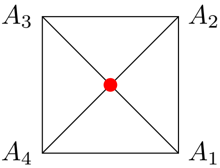

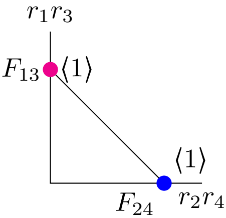

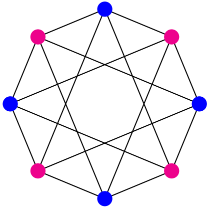

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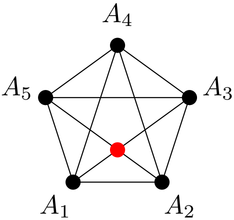

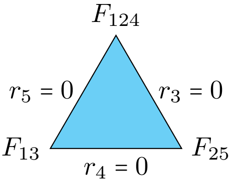

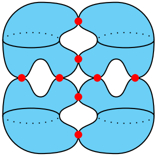

Deformations

Smooth deformations

- $\mathcal{A} = (A_1, \ldots, A_n)$ (Z smooth)
- $\mathcal{A}^t = (A_1^t, \ldots, A_n^t)$ *in a small neighbourhood*
- Z^t and Z are diffeomorphic.

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- Z^t and Z are homeomorphic.

Smoothings of Z with generic singularities

- From $\mathcal{A} = (A_1, \ldots, A_n) \subset (\mathbb{R}^m)^n$, $d = n - m - 1$:
  - $\mathcal{A}^t = (A_1^t, \ldots, A_n^t)$ *in a small neighbourhood and smooth.*
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- From P, $\mathcal{V} := \{V_1, \dots, V_r\}$, *singular* vertices.
  - $v_i : \mathbf{0} = \sum_{j=1}^{m} t_{k_j} A_{k_j}$, $t_{k_j} > 0$, convex linear combination
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  - 2^m copies of $\mathrm{Cone}(\mathbb{S}^{p_i - 1} \times \mathbb{S}^{q_i - 1})$
  - For each i, two possible smoothings:
    - $\mathrm{Cone}(\mathbb{S}^{p_i - 1}) \times \mathbb{S}^{q_i - 1} = \overline{\mathbb{B}}^{p_i} \times \mathbb{S}^{q_i - 1}$
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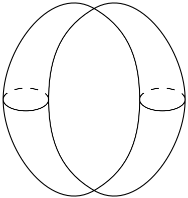
  - 2^r topological types of smoothings.

Examples of smoothings

One generic singularity in dimension 2



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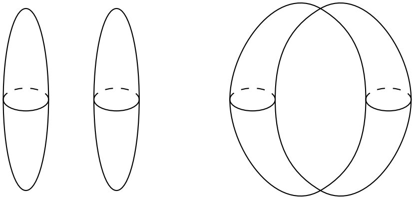

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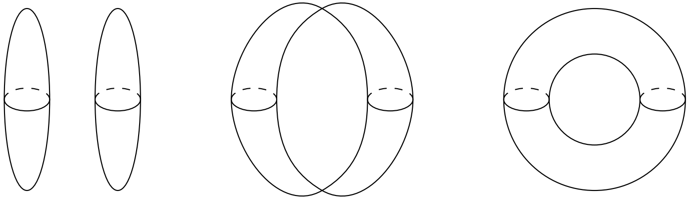

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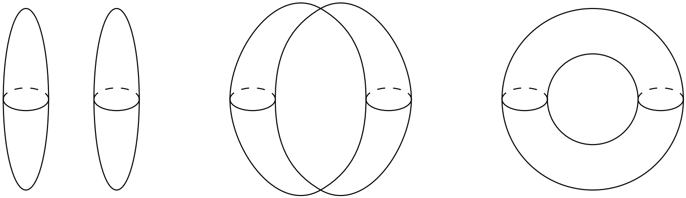
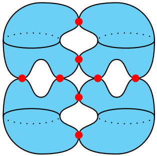


Two generic singularities in dimension 2



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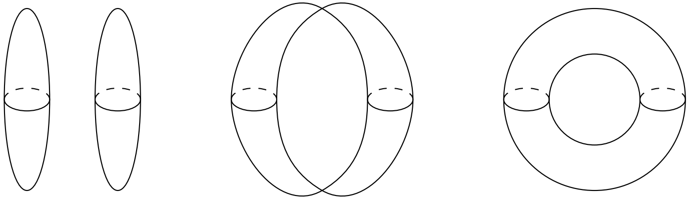

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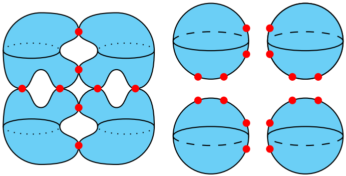

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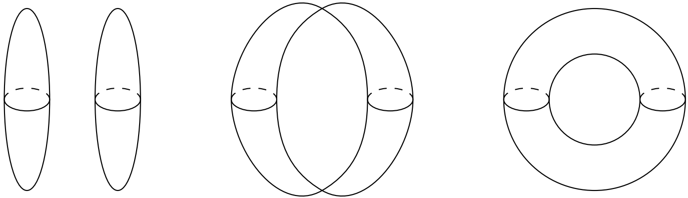

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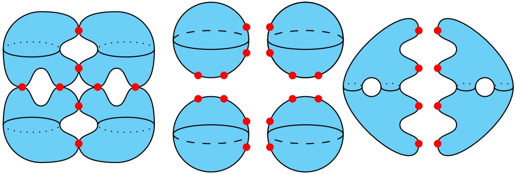

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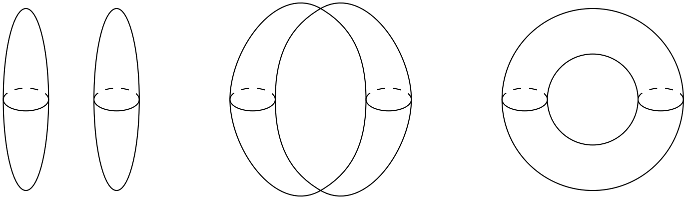

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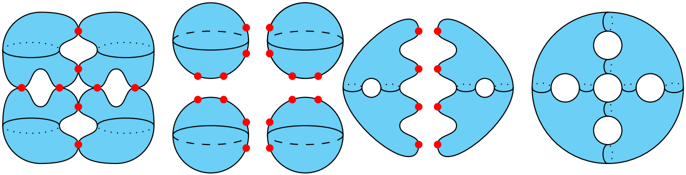

Generic singularities in dimension 3

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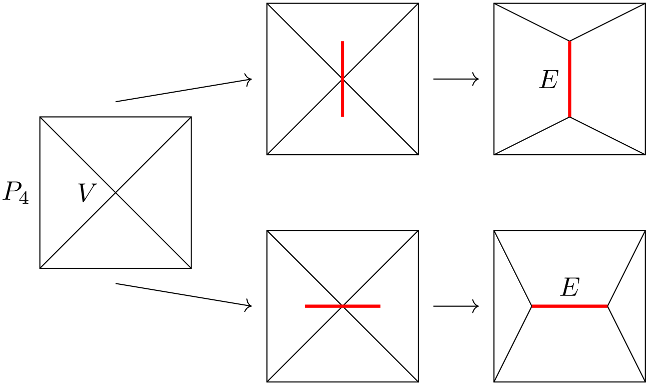
- $\check{P} := P \smallsetminus \{\text{4-vertices}\}$, $\check{Z} := \rho^{-1}(\check{P})$ smooth manifold with torus ends.

4-Pyramid



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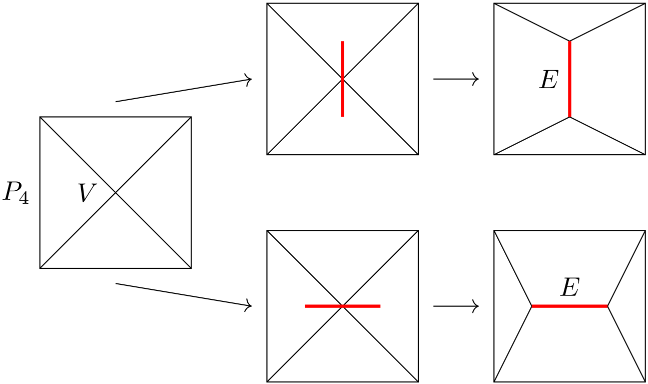

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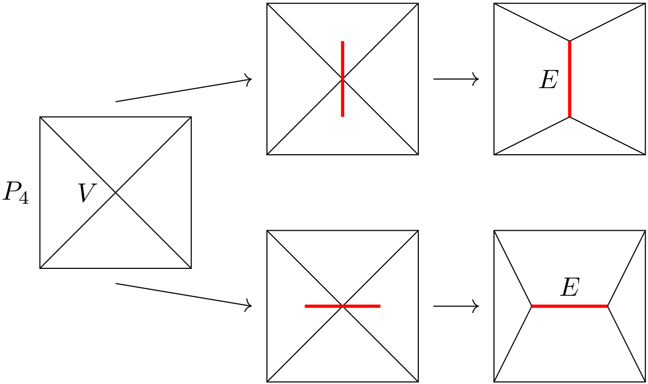

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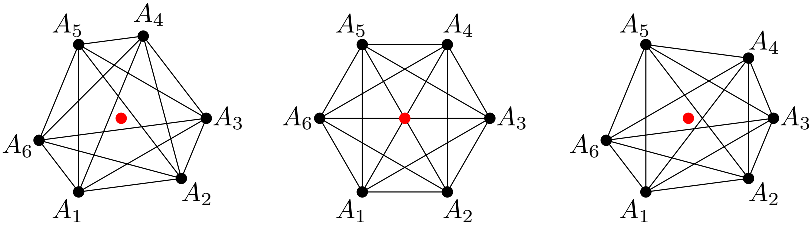
Triangular bipyramid B_3

Smoothings from $\mathcal{A}$



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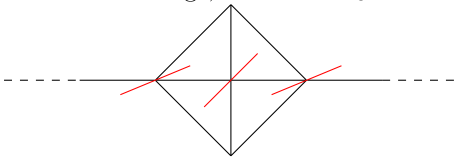

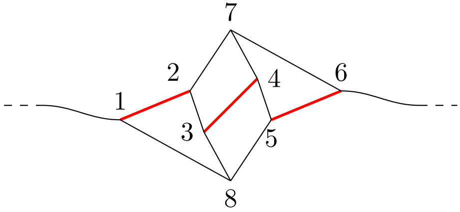

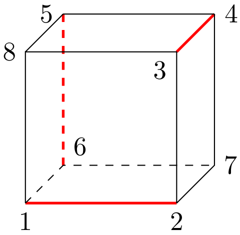

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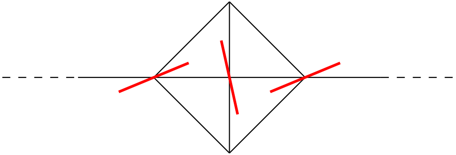

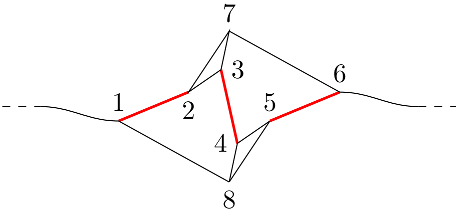

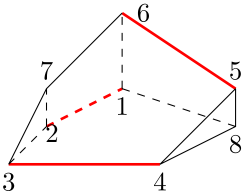

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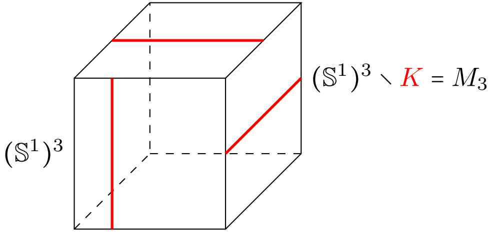
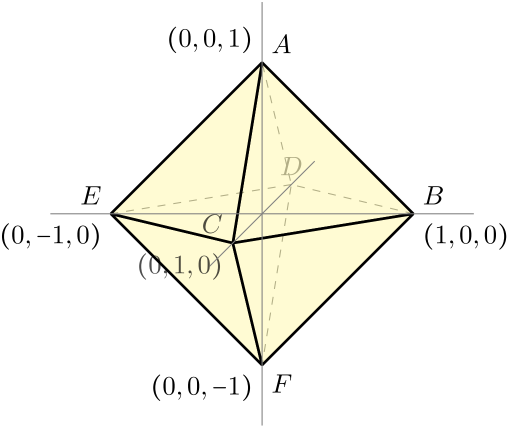


Octahedron

Face 1: ABC
Face 2: ABD
Face 3: ADE
Face 4: ACE
Face 5: BDF
Face 6: BCF
Face 7: CEF
Face 8: DEF



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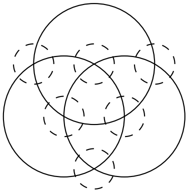

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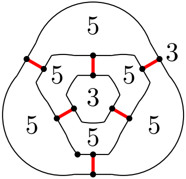

Octahedron

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Octahedron

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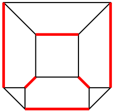

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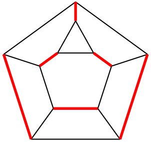

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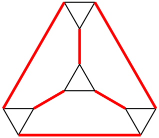

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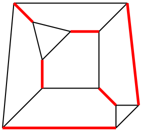

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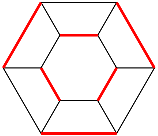

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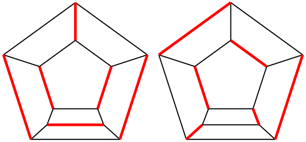

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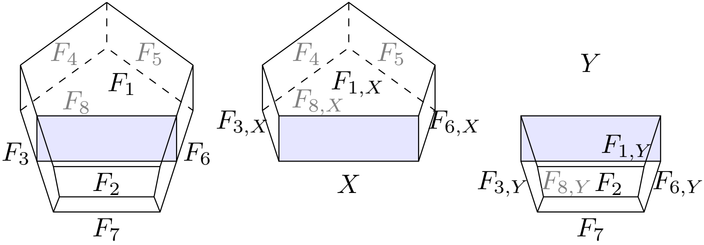

Octahedron

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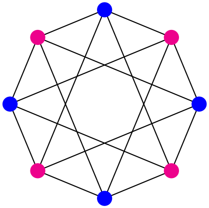

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- $\check{P}$: complete hyperbolic orbifold, $\check{Z}$ complete hyperbolic manifold with 96 cusps (torus ends).

¡¡¡¡Felicidades, Pepe!!!!