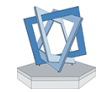
Topology, combinatorics and arithmetics of line arrangements of odd regular polygons

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Universidad de Zaragoza

Aspects réels de la géométrie
Alex Degtyarev 60th birthday
Marseille, November 3rd 2022

Work in progress with J.I. Cogolludo, J. Martín Morales and J. Viu Sos. Funded by MCIN/AEI/10.13039/501100011033 (PID2020-114750GB-C31) and Gobierno de Aragón (E22_20R: "Álgebra y Geometría").



Combinatorics and topology

Theorem (Rybnikov, 2011)

$\exists \mathscr{R}_i$, $i = 1, 2$, 13-*line arrangements in* $\mathbb{P}^2 = \mathbb{P}^2_{\mathbb{C}}$ *same combinatorics and* $\pi_1(\mathbb{P}^2 \smallsetminus \bigcup \mathscr{R}_1) \not\cong \pi_1(\mathbb{P}^2 \smallsetminus \bigcup \mathscr{R}_2)$.
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$\exists \overline{\mathscr{Z}}_{\pm}$ 11-*line arrangements in* $\mathbb{P}^2_{\mathbb{R}}$ *same combinatorics and* $\left(\mathbb{P}^2, \bigcup \overline{\mathscr{Z}}_{+}^{\mathbb{C}}\right)$ *and* $\left(\mathbb{P}^2, \bigcup \overline{\mathscr{Z}}_{-}^{\mathbb{C}}\right)$ *are not homeomorphic.*



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Definition

Combinatorics $\mathscr{C} = (\mathscr{L}, \mathscr{P})$

- $\mathscr{L}$ finite set (of lines)
- $\mathscr{P} \subset \mathcal{P}(\mathscr{L})$, $\#P > 1$, $\forall P \in \mathscr{P}$ (points)
- $\forall L_1, L_2 \in \mathscr{L}$, $L_1 \neq L_2$, $\exists! P \in \mathscr{P}$ such that $L_1, L_2 \in P$ (denoted $P \in L_i$).



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Combinatorics associated to a line arrangement

$\mathscr{A}$ line arrangement in $\mathbb{P}^2_{\mathbb{K}}$ ($\mathbb{K}$ field) $\Rightarrow \mathscr{C}_{\mathscr{A}} := (\mathscr{A}, \{\text{multiple points}\})$.
$\mathscr{A}$ ordered $\Longrightarrow \mathscr{C}_{\mathscr{A}}$ ordered.



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Theorem (A_-Carmona-Cogolludo-Marco, 2005)

$\exists \mathscr{L}_{\pm} = \{L_{\pm,1}, \ldots, L_{\pm,10}\}$ *ordered line arrangements in* $\mathbb{P}^2_{\mathbb{R}}$ *same ordered combinatorics and* $\nexists : \Phi : \mathbb{P}^2 \xrightarrow{homeo} \mathbb{P}^2$ *with* $\Phi(L^{\mathbb{C}}_{+,i}) = L^{\mathbb{C}}_{-,i}$, $1 \leq i \leq 10$.

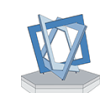
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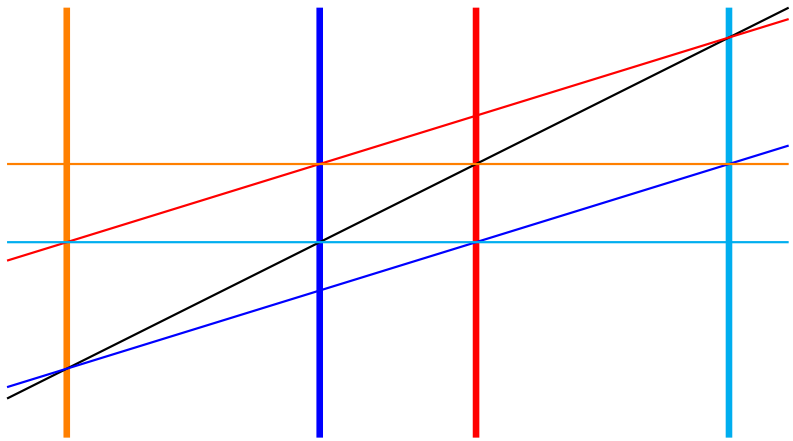
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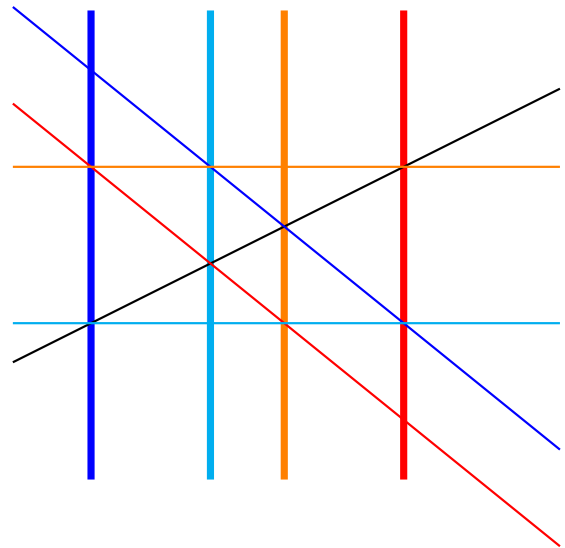
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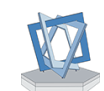


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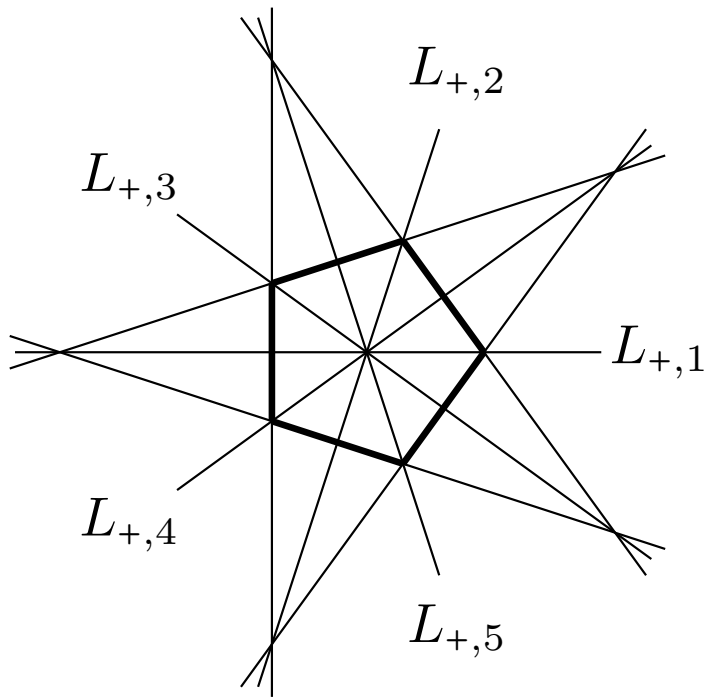






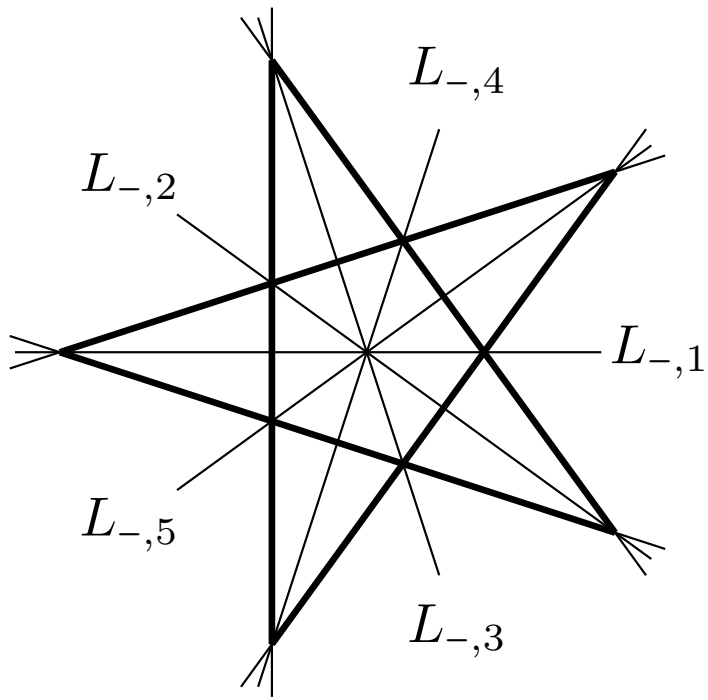
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- Study more general polygons (odd for simplicity).



Odd polygon arrangements

Definition

$\mathscr{A}_N$, $N := 2n + 1$, is the arrangement of sides and diagonals of a regular N-polygon $\Theta_N \subset \mathbb{C} \equiv \mathbb{R}^2 \subset \mathbb{P}^2_{\mathbb{R}}$.



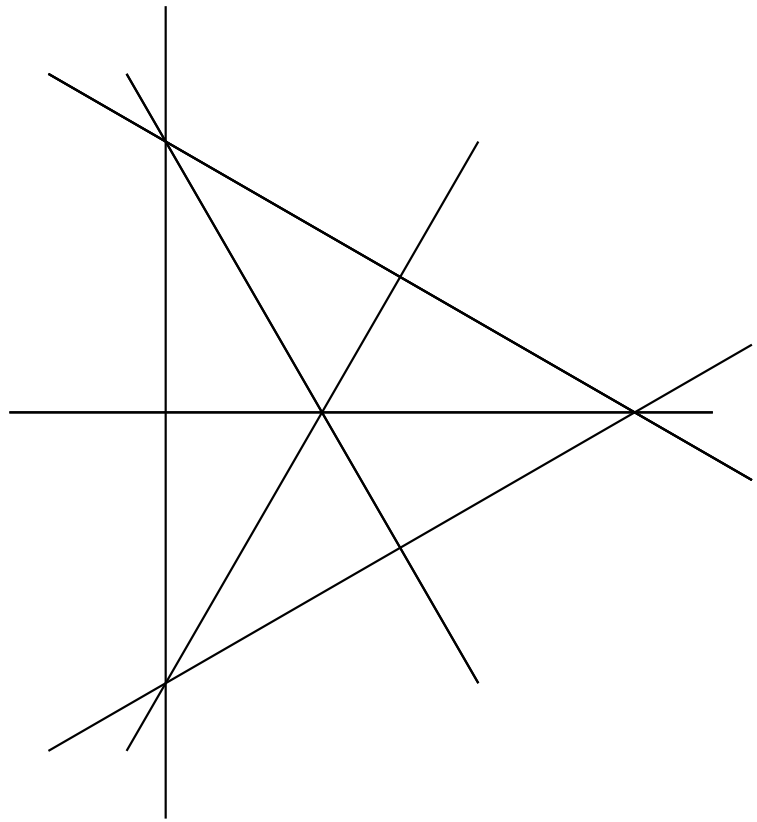
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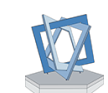
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Ceva arrangement





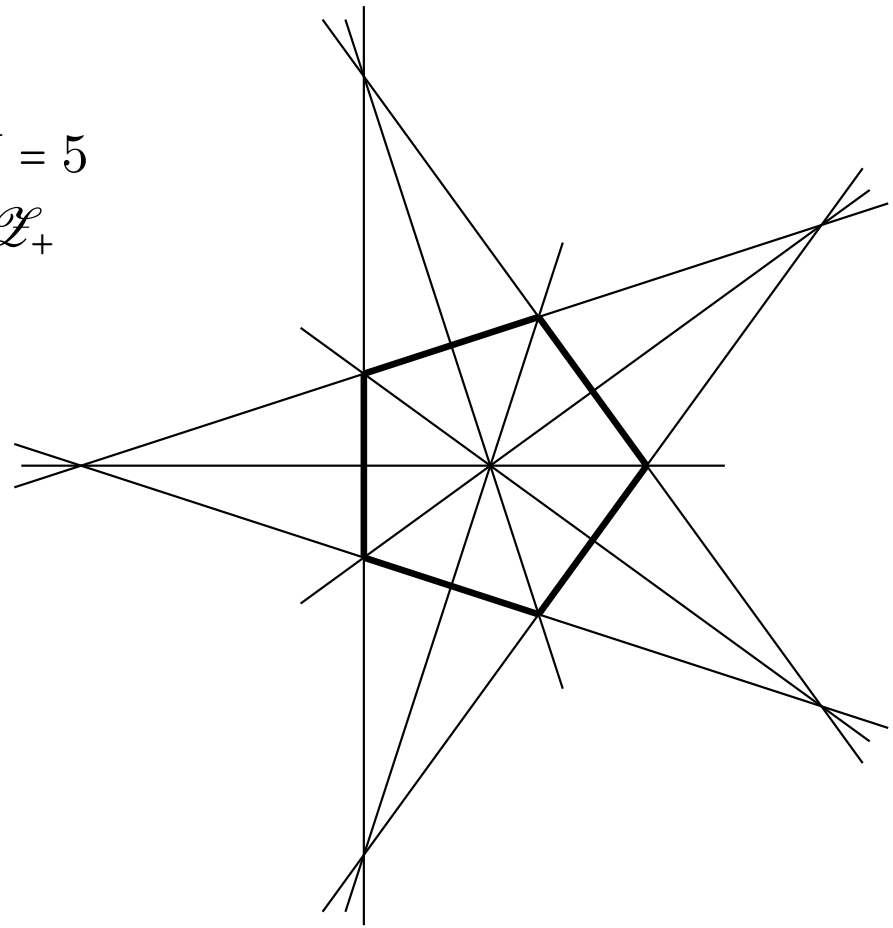
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$N = 5$

$\mathscr{Z}_+$





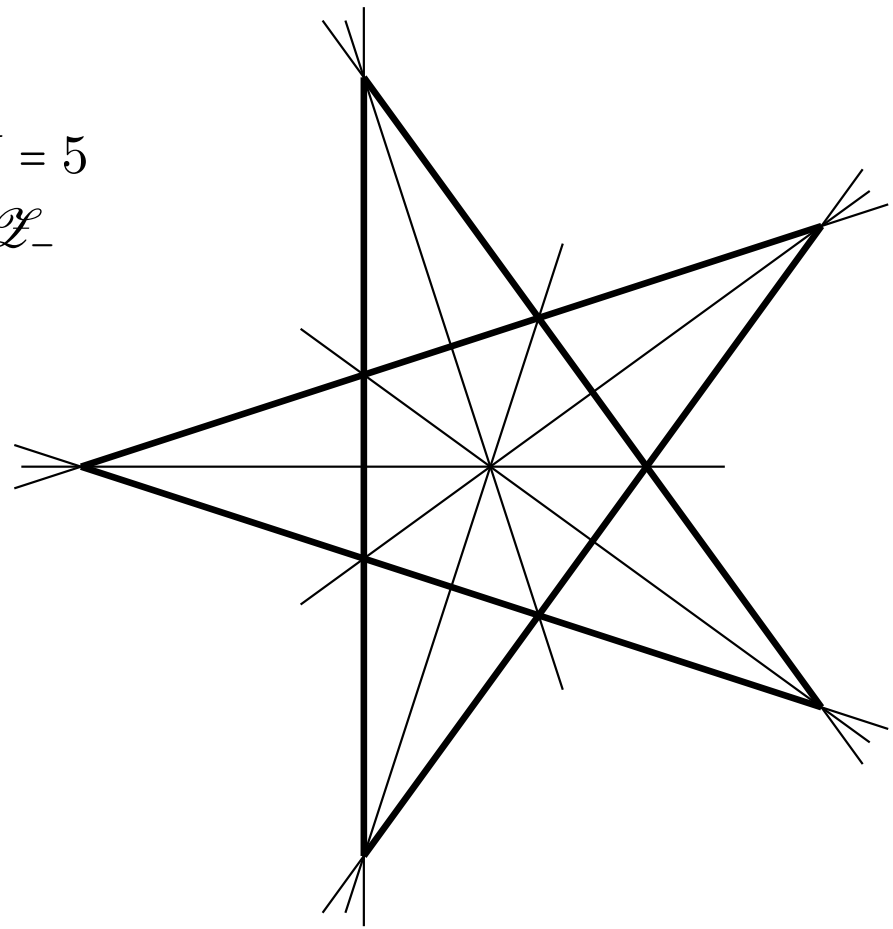
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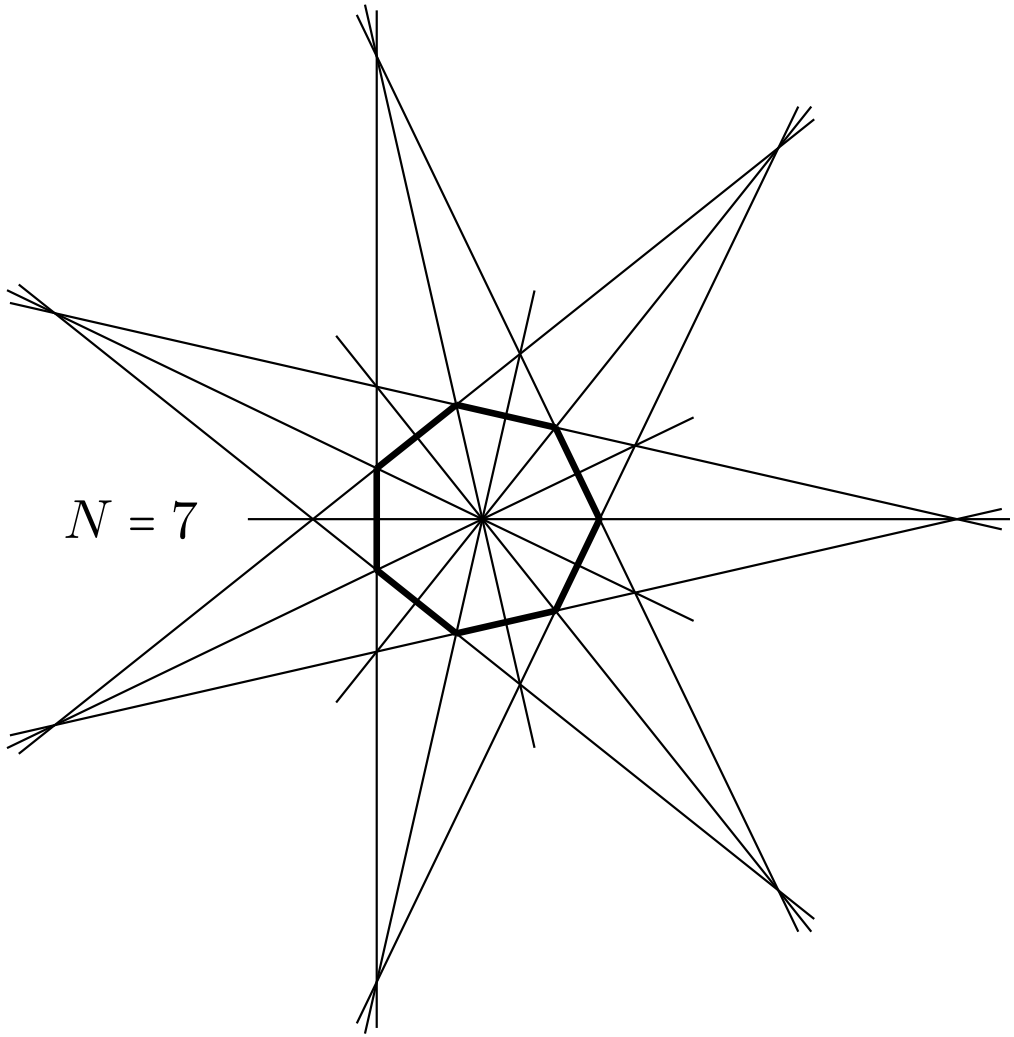
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Odd polygon arrangements

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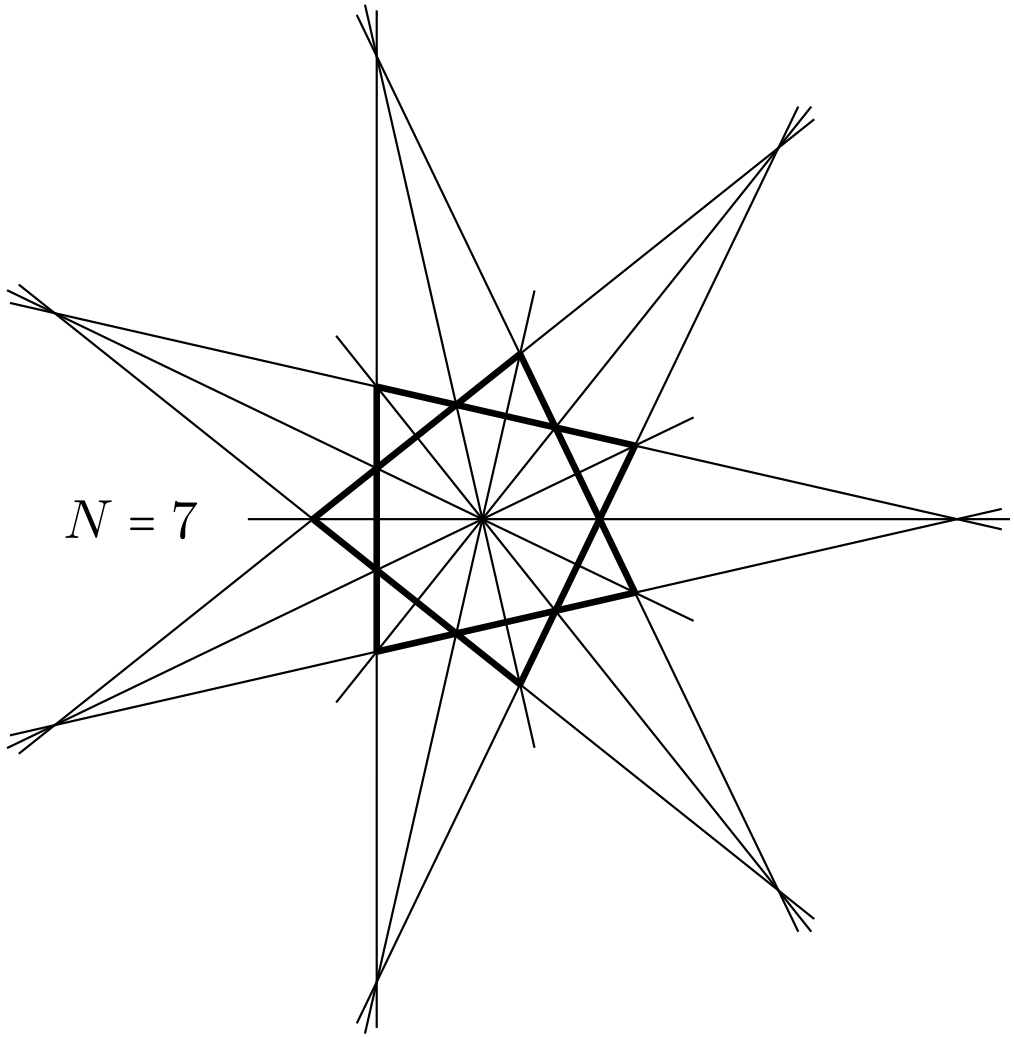
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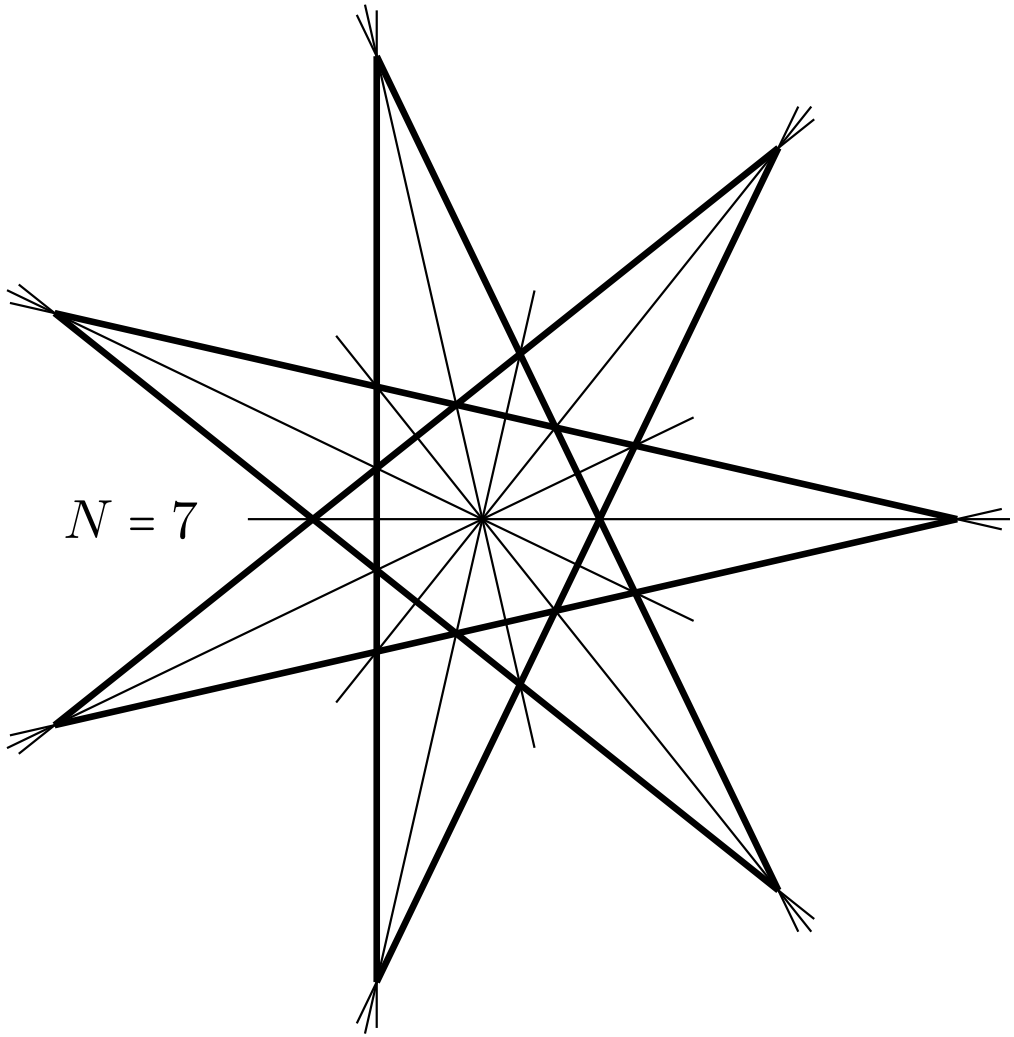
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Notation

- $\mathscr{C}_N := \mathscr{C}_{\mathscr{A}_N}$, $\Sigma_N := \Sigma(\mathscr{C}_N)$, $\mathscr{M}_N := \mathscr{M}(\mathscr{C}_N)$
- $\mu_N := \{\eta \in \mathbb{C}^* \mid \eta^N = 1\}$, $D_\eta := \overline{\{t\eta \mid t \in \mathbb{R}\}} \subset \mathbb{P}^2_{\mathbb{R}}$
- L_η: line supporting edge of $\Theta_N \pitchfork D_\eta$.

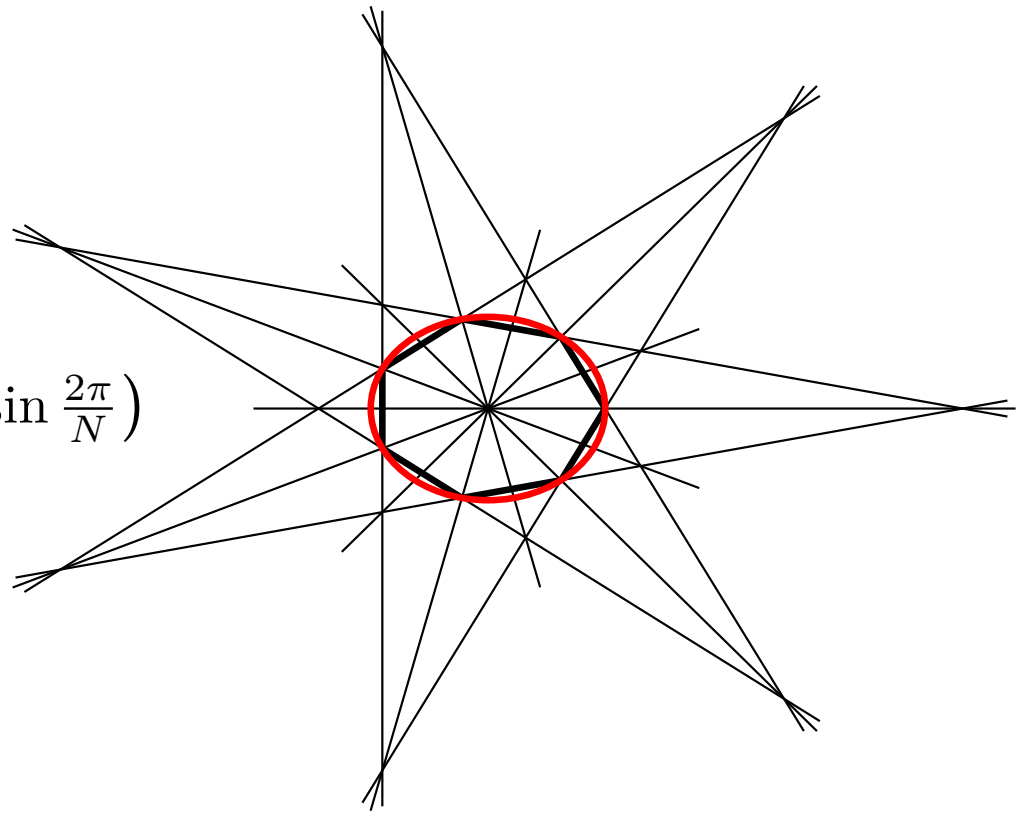


Properties of $\mathscr{A}_N$ and $\mathscr{C}_N$

Field of definition

$\mathscr{A}_N$ has equations in $\mathbb{K}_N := \mathbb{Q}\left(\cos\frac{2\pi}{N}\right)$ of index $\frac{\Phi(N)}{2}$.

$(x, y) \mapsto \left(x, y \sin\frac{2\pi}{N}\right)$



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Sketch of the proof.

Each (convex or not) regular polygon yields one element of $\mathscr{M}_N$.



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Each (convex or not) regular polygon yields one element of $\mathscr{M}_N$.

- Place 5 points at fixed places
- Place all the vertices of the *polygon* using variables.
- It can be done symmetric with respect to X-axis.



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- Place 5 points at fixed places
- Place all the vertices of the *polygon* using variables.
- It can be done symmetric with respect to X-axis.
- Apply dihedral action and use the regular polygon. □



Cyclic and dihedral actions

Cyclic action of μ_N

$$\begin{array}{ccc} \mathbb{P}^2 & \longrightarrow & \mathbb{P}^2 \\ [x:y:z] & \longmapsto & \left[x\cos\frac{2\pi}{N} - y\sin\frac{2\pi}{N} : x\sin\frac{2\pi}{N} + y\cos\frac{2\pi}{N} : z\right] \end{array}$$

Fixed points: $[0:0:1], [1:i:0], [i:1:0]$.



Cyclic and dihedral actions

Cyclic action of μ_N

$$\begin{array}{ccc} \mathbb{P}^2 & \longrightarrow & \mathbb{P}^2 \\ [x:y:z] & \longmapsto & \left[x \exp \frac{2i\pi}{N} : y \exp \frac{-2i\pi}{N} : z\right] \end{array}$$

Fixed points: $[0:0:1], [1:0:0], [0:1:0]$.



Cyclic and dihedral actions

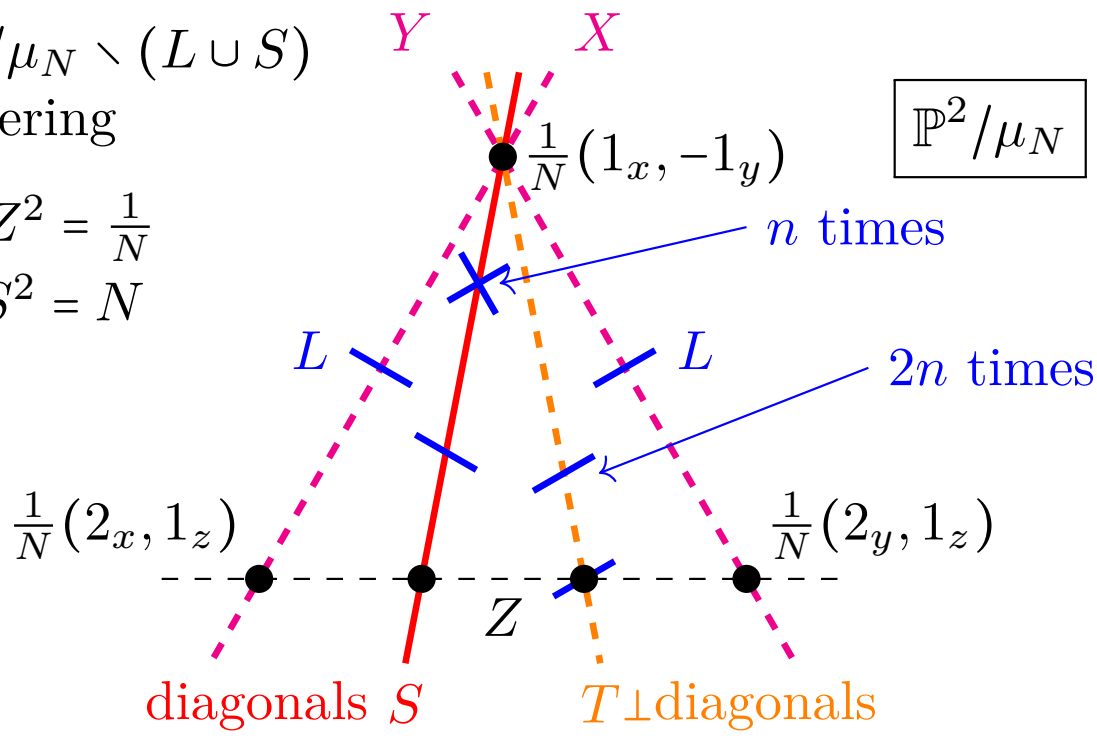
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$\mathbb{P}^2 \smallsetminus \bigcup \mathscr{A}_N \longrightarrow \mathbb{P}^2/\mu_N \smallsetminus (L \cup S)$
cyclic orbifold covering

$$X^2 = Y^2 = Z^2 = \tfrac{1}{N}$$
$$L^2 = T^2 = S^2 = N$$



Cyclic and dihedral actions

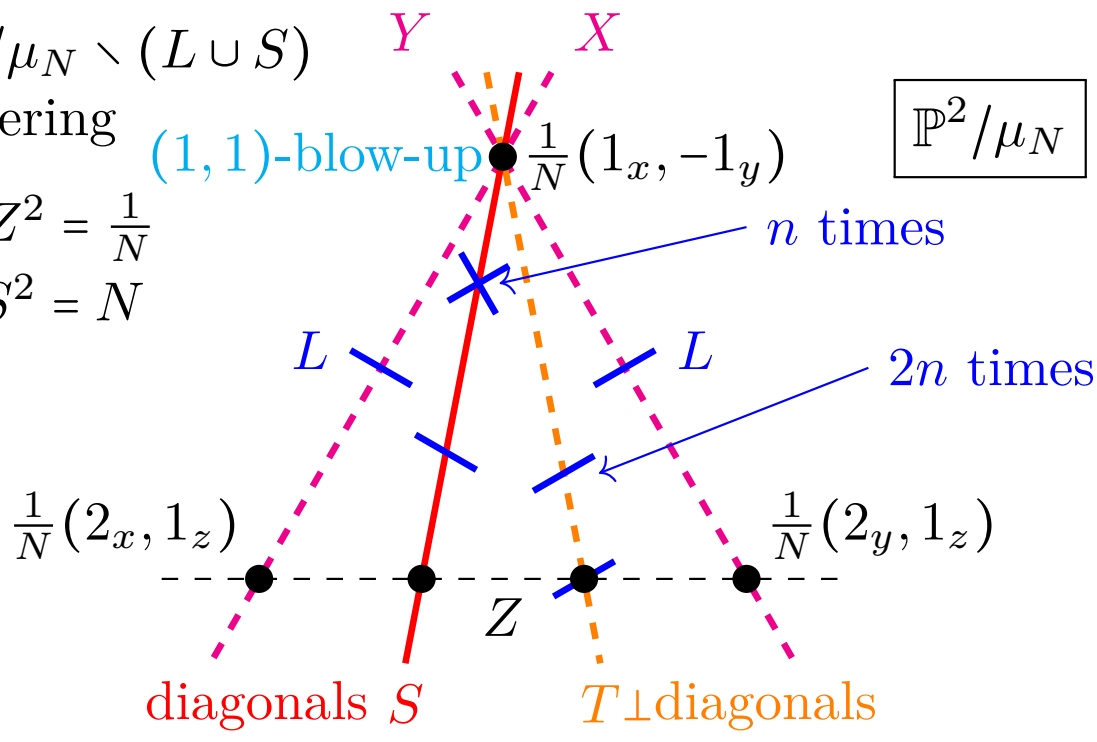
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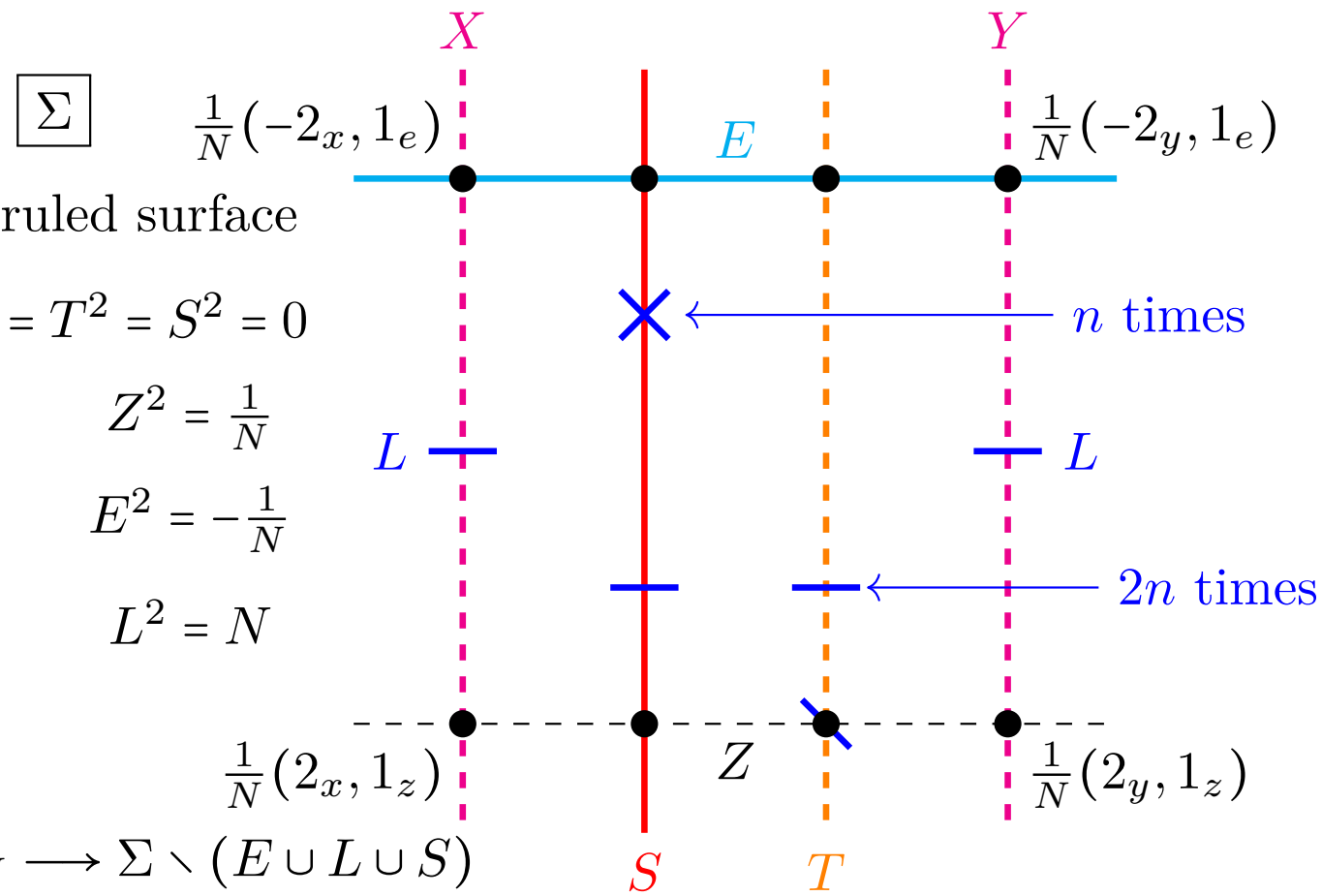
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Cyclic and dihedral actions



$\boxed{\Sigma}$

Normal ruled surface

$$X^2 = Y^2 = T^2 = S^2 = 0$$

$$Z^2 = \tfrac{1}{N}$$

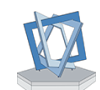
$$E^2 = -\tfrac{1}{N}$$

$$L^2 = N$$

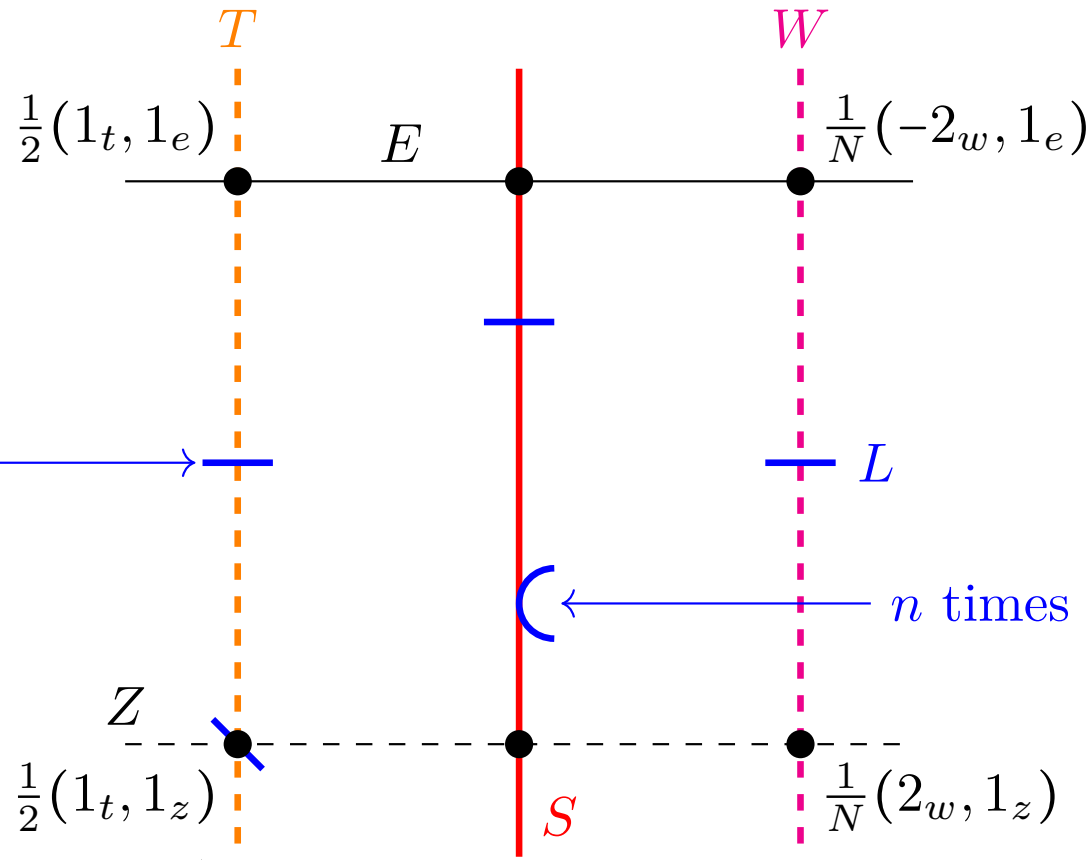
$$\mathbb{P}^2 \smallsetminus \bigcup \mathscr{A}_N \longrightarrow \Sigma \smallsetminus (E \cup L \cup S)$$

cyclic orbifold covering

μ_2 fixed points: S, $T \cap E$, $T \cap Z$



Cyclic and dihedral actions



$W^2 = T^2 = S^2 = 0$

$-E^2 = Z^2 = \frac{1}{2N}$

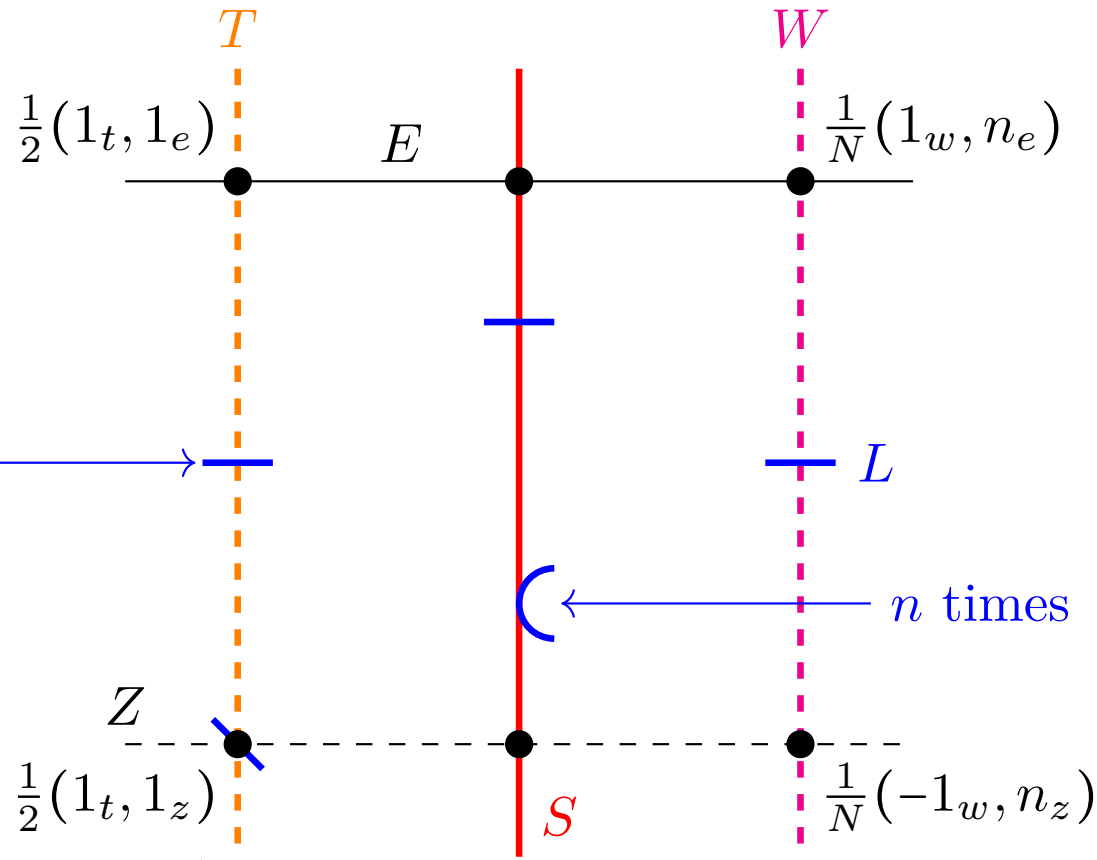
$L^2 = \frac{N}{2}$ n times

Σ/μ_2 birational to $\mathbb{P}^2/\mathbb{D}_{2N}$

$\mathbb{P}^2 \setminus \bigcup \mathscr{A}_N \longrightarrow \Sigma/\mu_2 \setminus (E \cup L \cup S)$
dihedral orbifold covering



Cyclic and dihedral actions



$W^2 = T^2 = S^2 = 0$

$-E^2 = Z^2 = \frac{1}{2N}$

$L^2 = \frac{N}{2}$

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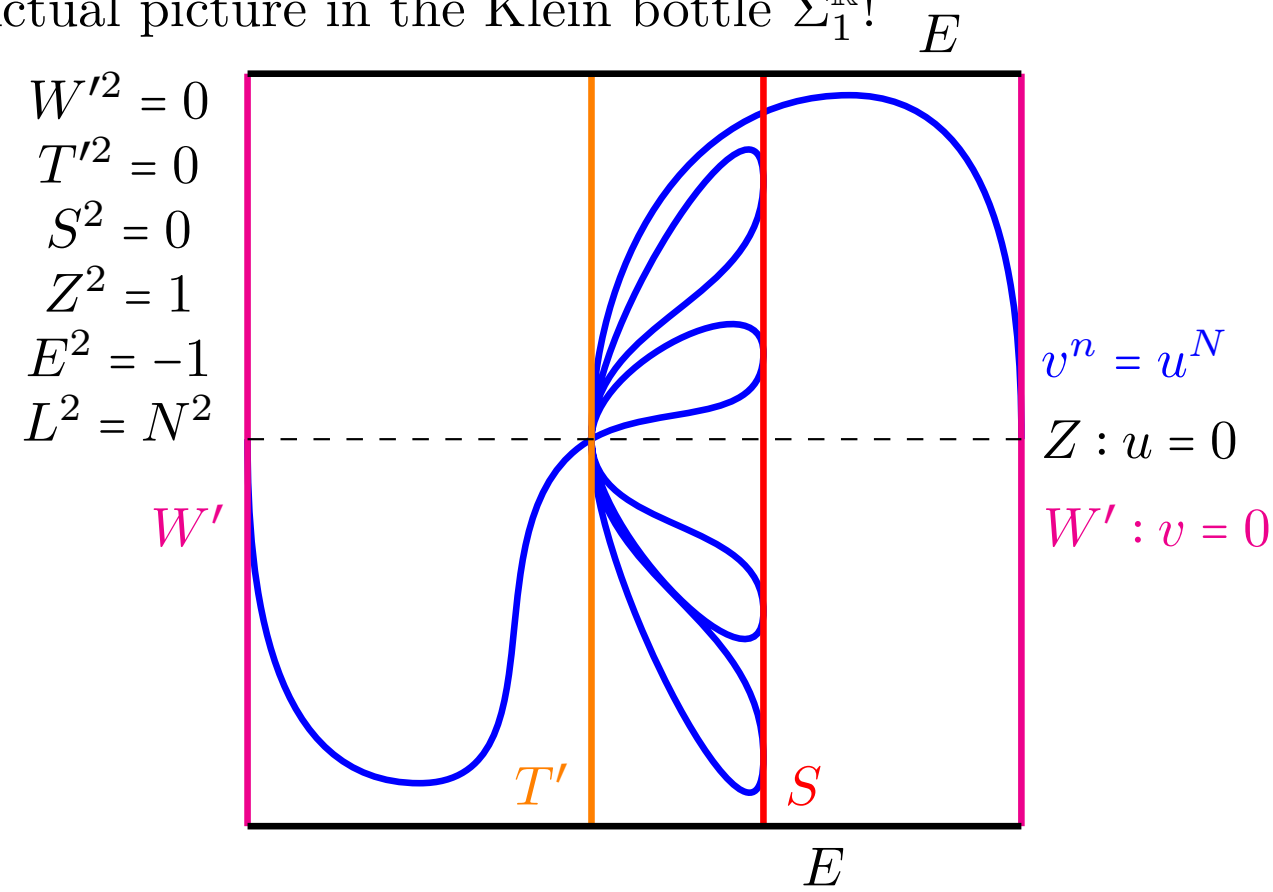
$(1,n)$-Nagata transformation at $E \cap W$

$(1,1)$-Nagata transformation at $E \cap T$



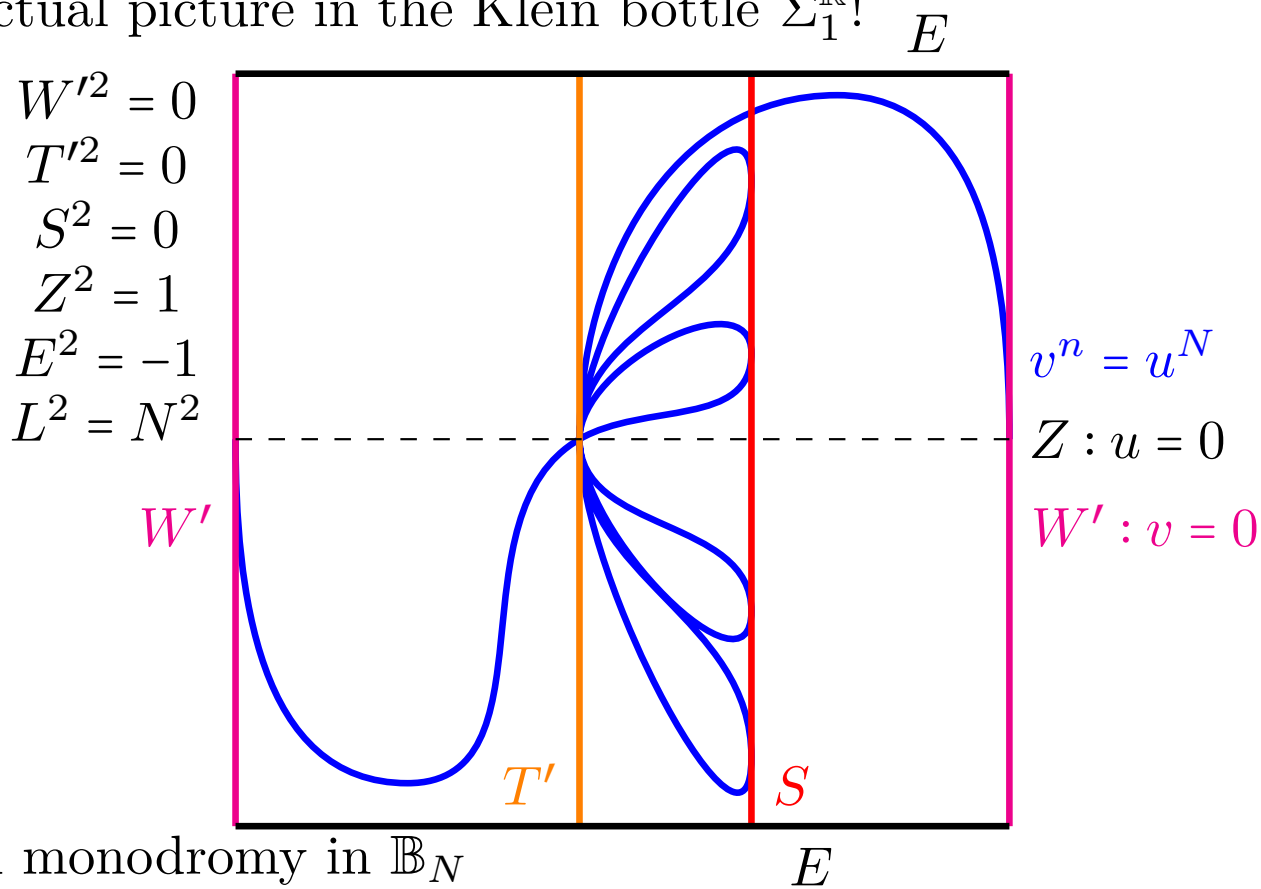
Dessin d'enfants

Actual picture in the Klein bottle $\Sigma_1^{\mathbb{R}}$!



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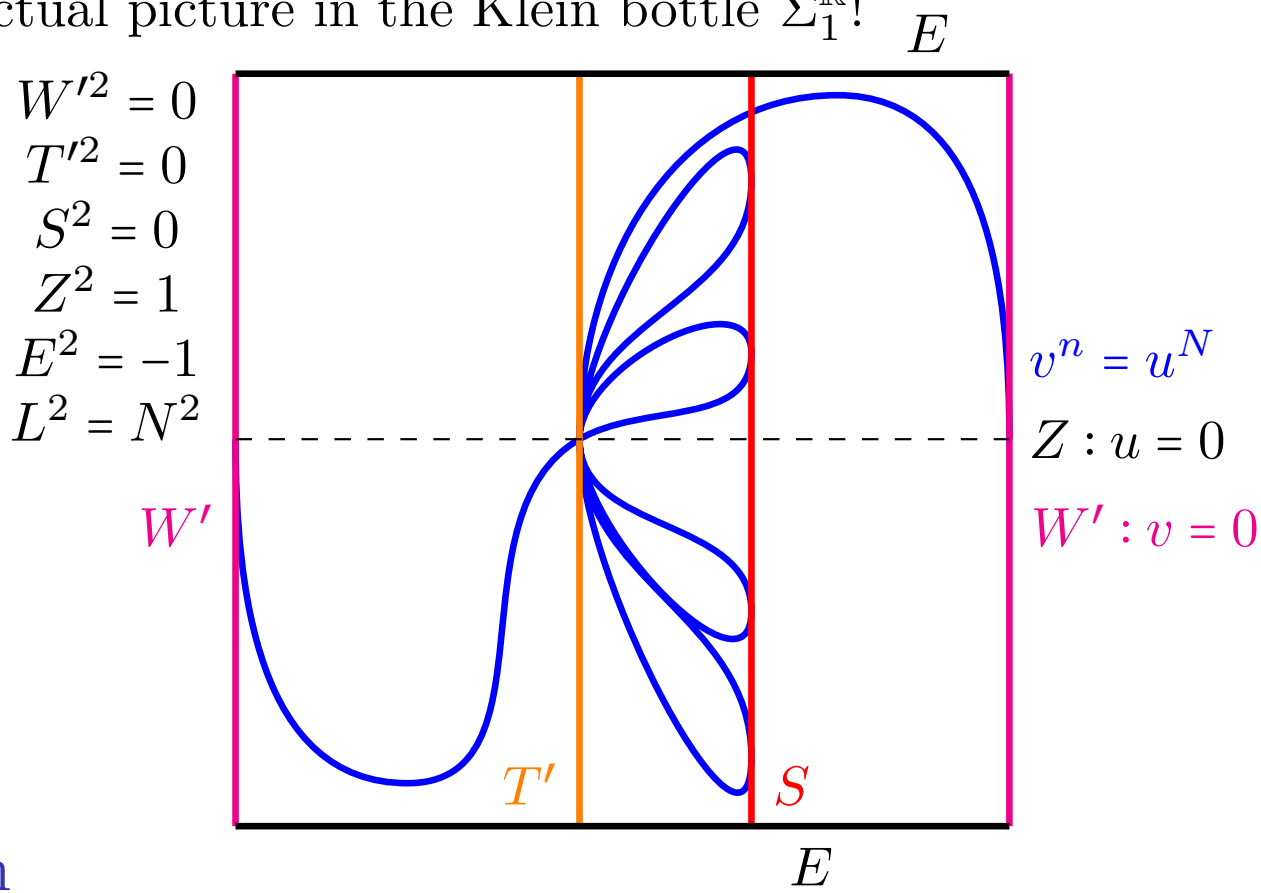


- Braid monodromy in $\mathbb{B}_N$
- $T' : \Delta_N$, $S : \prod_{j=1}^{n} \sigma_{2j}$, $W' : \left(\prod_{j=1}^{n} \sigma_{2j}\sigma_{2j-1} \right)^n =: \tau$
- T, W are not in $E \cup L \cup S$!



Dessin d'enfants

Actual picture in the Klein bottle $\Sigma_1^{\mathbb{R}}$!



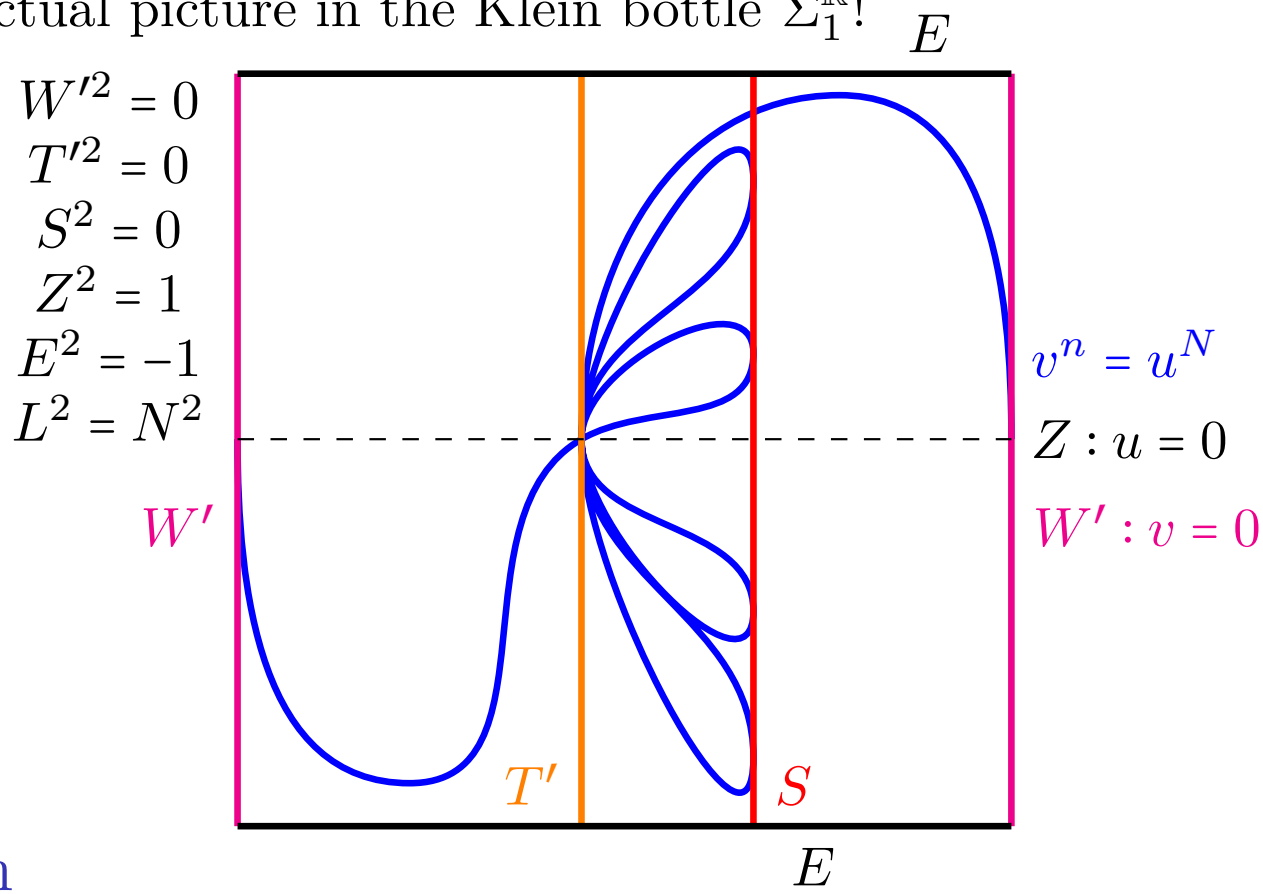
Theorem

$$\pi_1(\Sigma_1 \smallsetminus (L \cup S \cup E \cup T' \cup W')) = \left\langle \underset{1 \le i \le N}{x_i},\ t, w \;\middle|\; x_i^w = x_i^\tau, x_i^t = x_i^{\Delta_N} \right\rangle$$



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Actual picture in the Klein bottle $\Sigma_1^{\mathbb{R}}$!



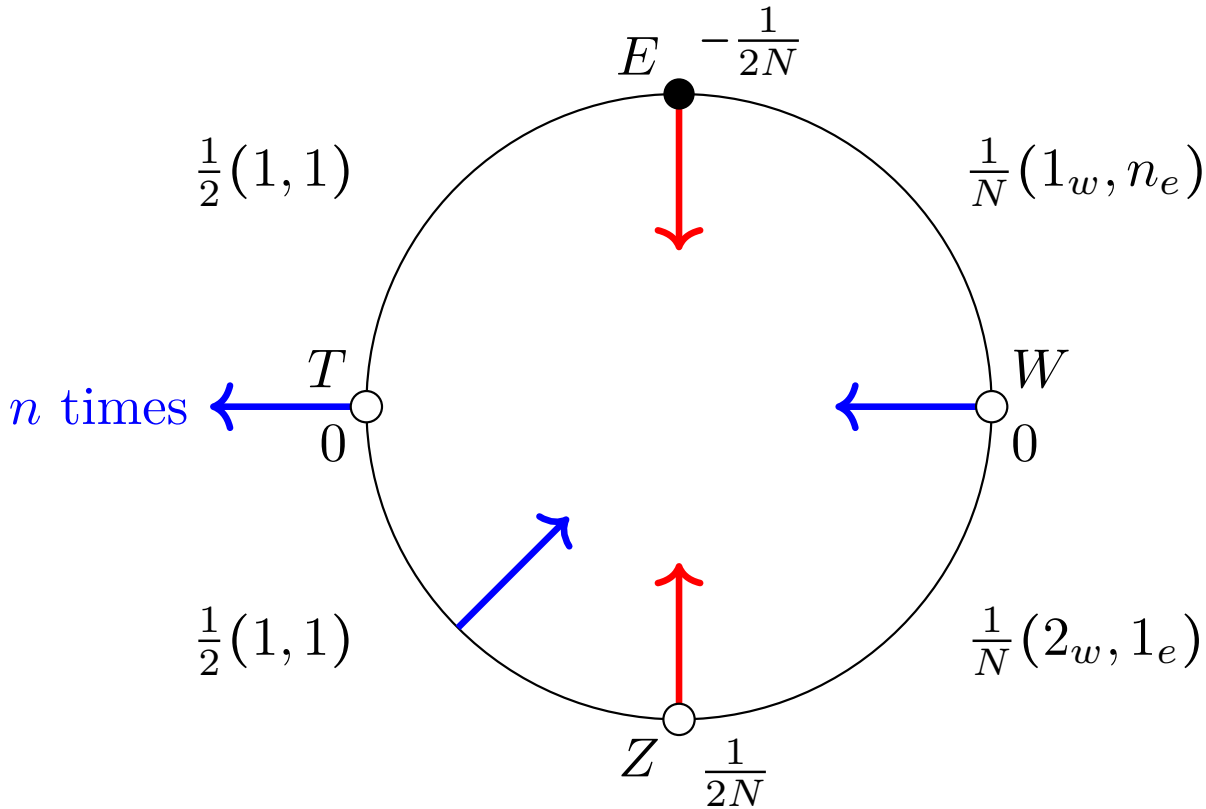
Theorem

$$\underbrace{\pi_1(\Sigma/\mu_2 \smallsetminus (L \cup S \cup E))}_{G} = \langle \underset{1\le i\le N}{x_i}, t, w \mid x_i^w = x_i^\tau, x_i^t = x_i^{\Delta_N}, t^2 = C, w^N = C^n \rangle$$

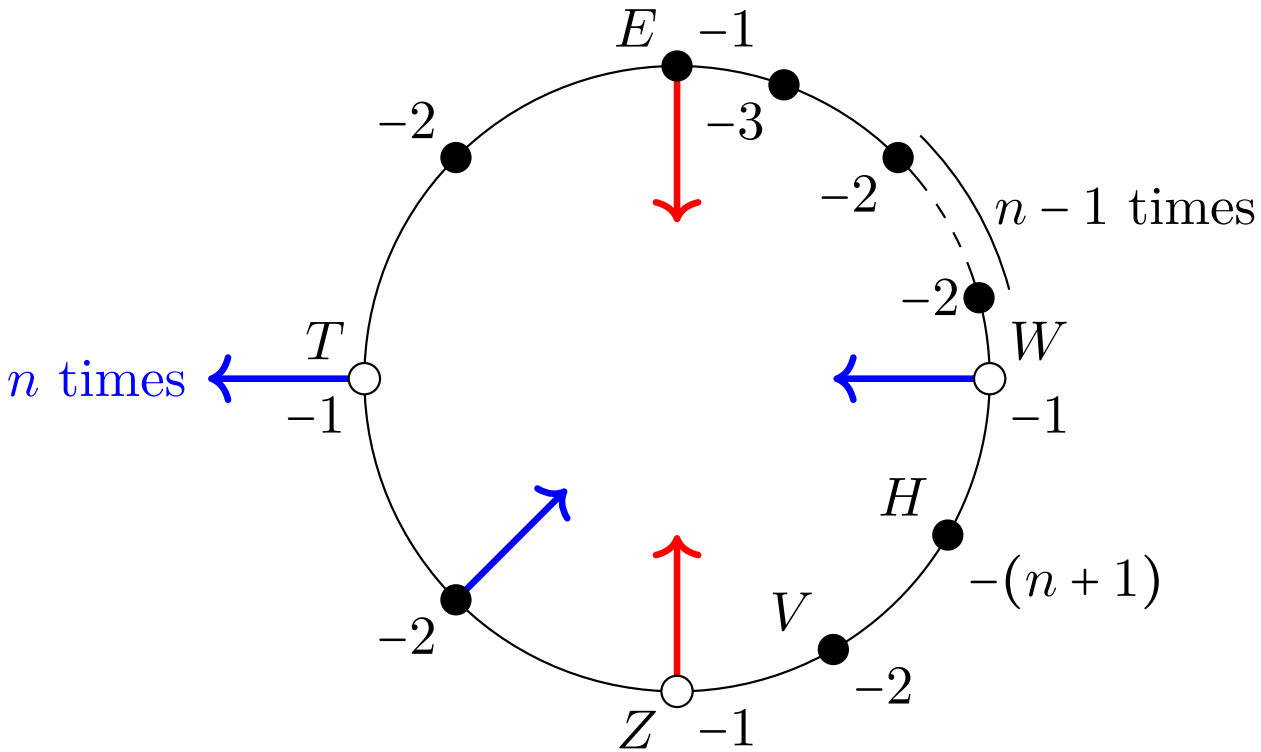
$$C := \prod_{j=N}^{1} x_j, \qquad G \twoheadrightarrow \mu_N * \mu_2 \twoheadrightarrow \mathbb{D}_{2N}, \qquad \ker = \pi_1(\mathbb{P}^2 \smallsetminus \mathscr{A}_N)$$



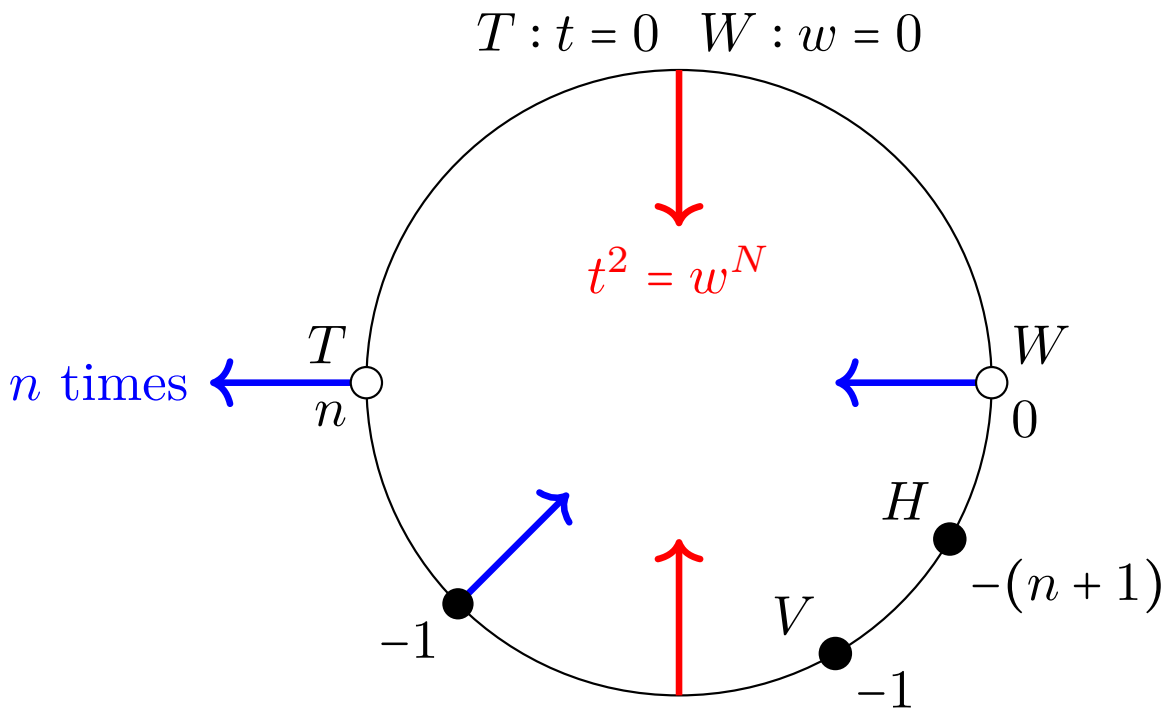
Another birational model



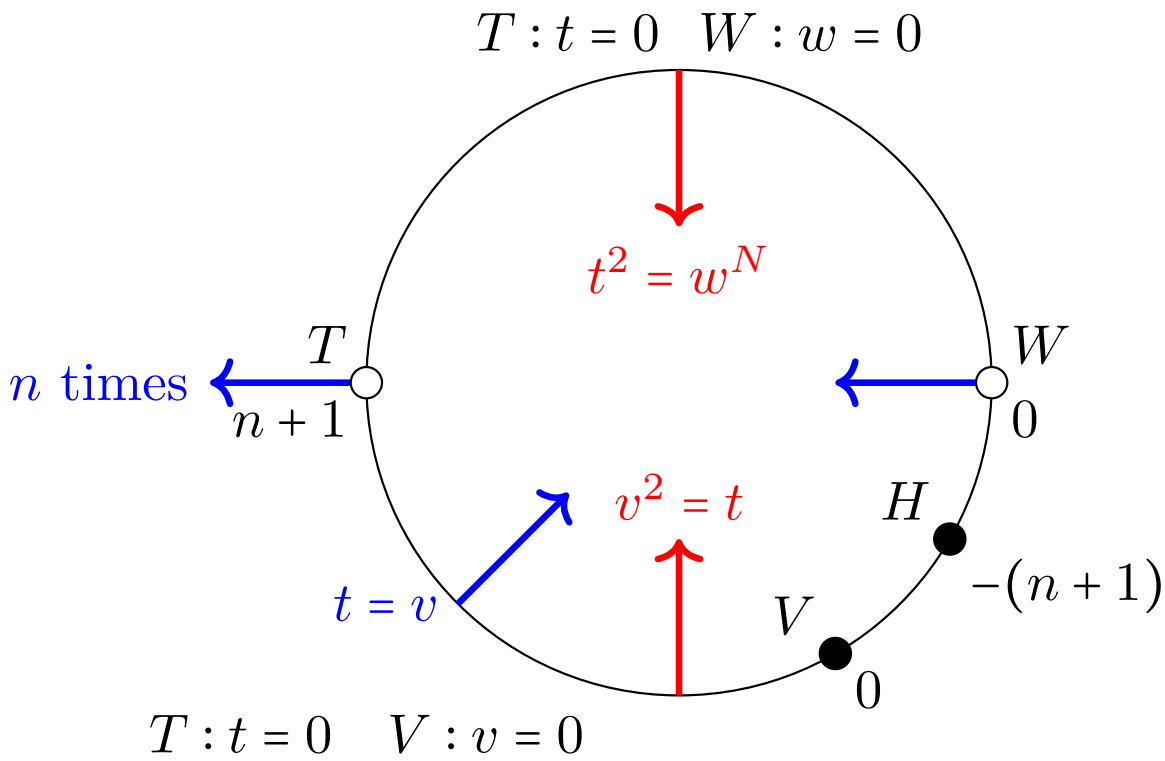
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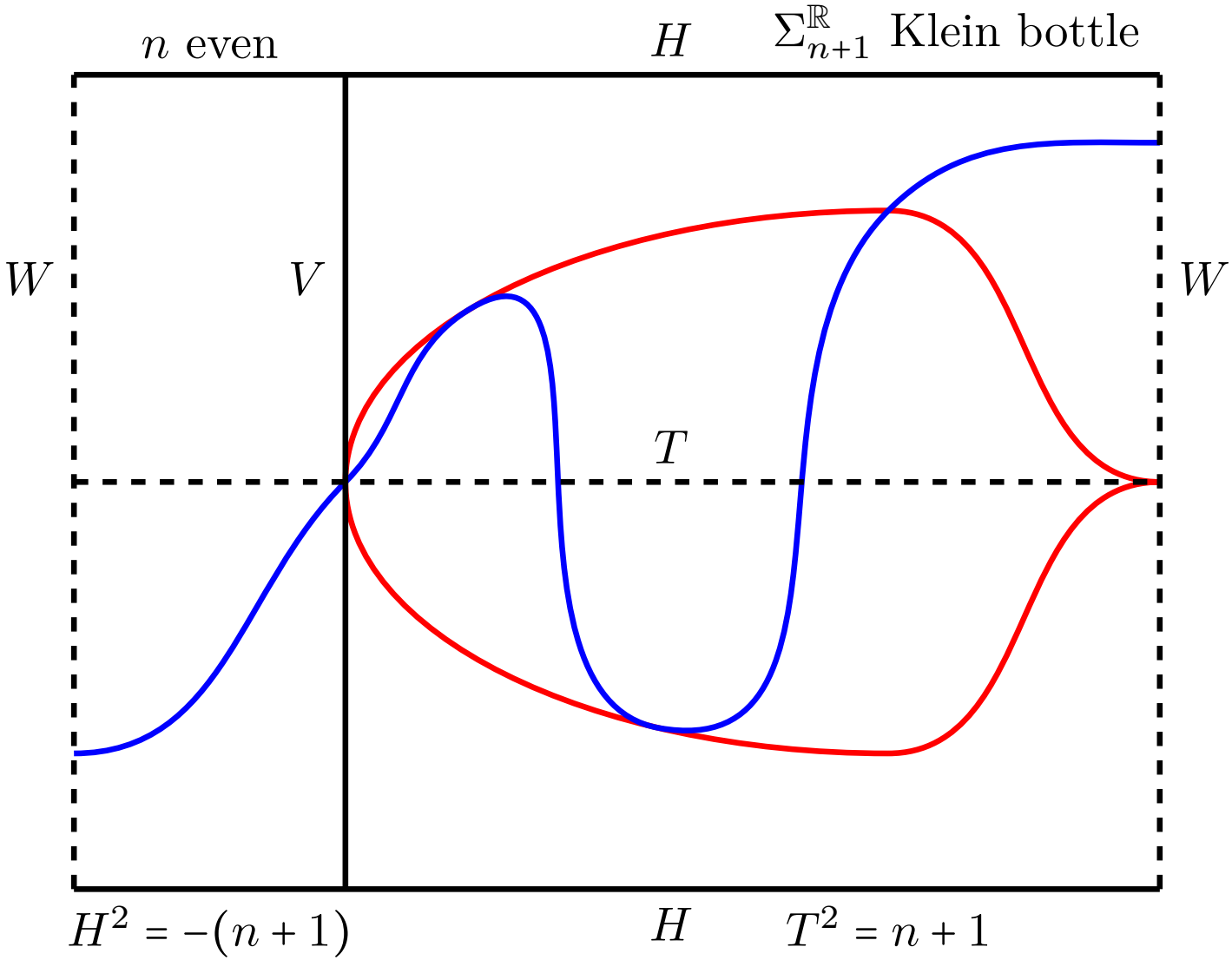
Another birational model



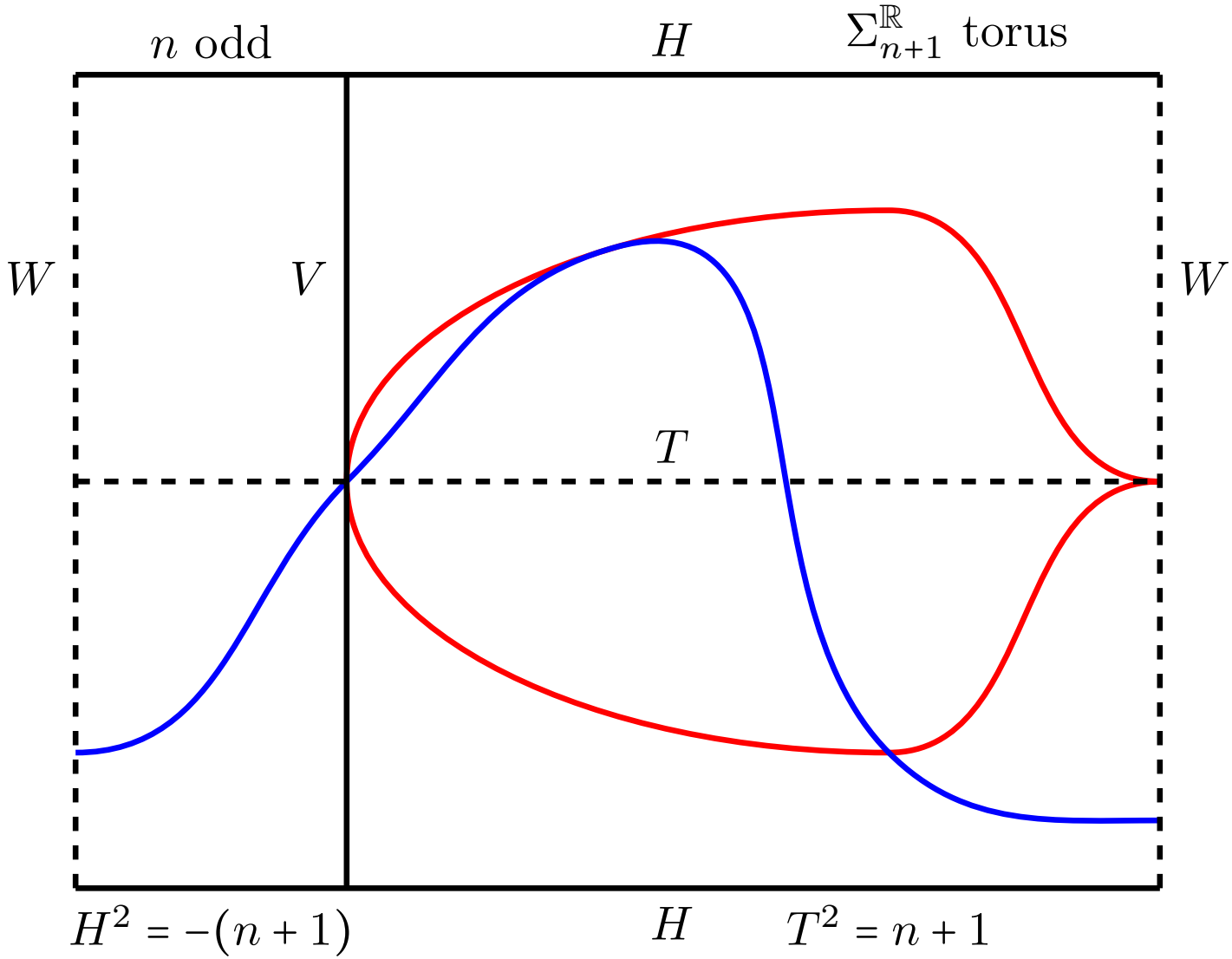
Another birational model



Real expression of the second birational model



Real expression of the second birational model



Short fundamental group presentation

Theorem

Let $n = 2m$. The orbifold fundamental group of $\mathbb{P}^2/\mathbb{D}_{2N} \setminus (E \cup L \cup S)$ has a presentation with two generators s (meridian of S) and v (meridian of V) with relations

- *$R_1, \ldots, R_n$, coming from the tacnodes (they do not depend on N) and S_{n+1} coming from the node.*
- $v^N = 1$

There is a meridian ℓ of L satisfying $l(sv)^2 = 1$.



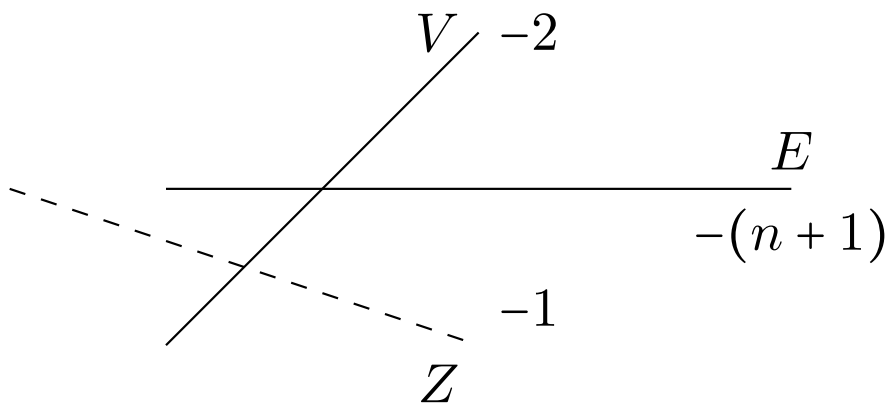
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$e = v^2$

$v = e^{n+1}$



Short fundamental group presentation

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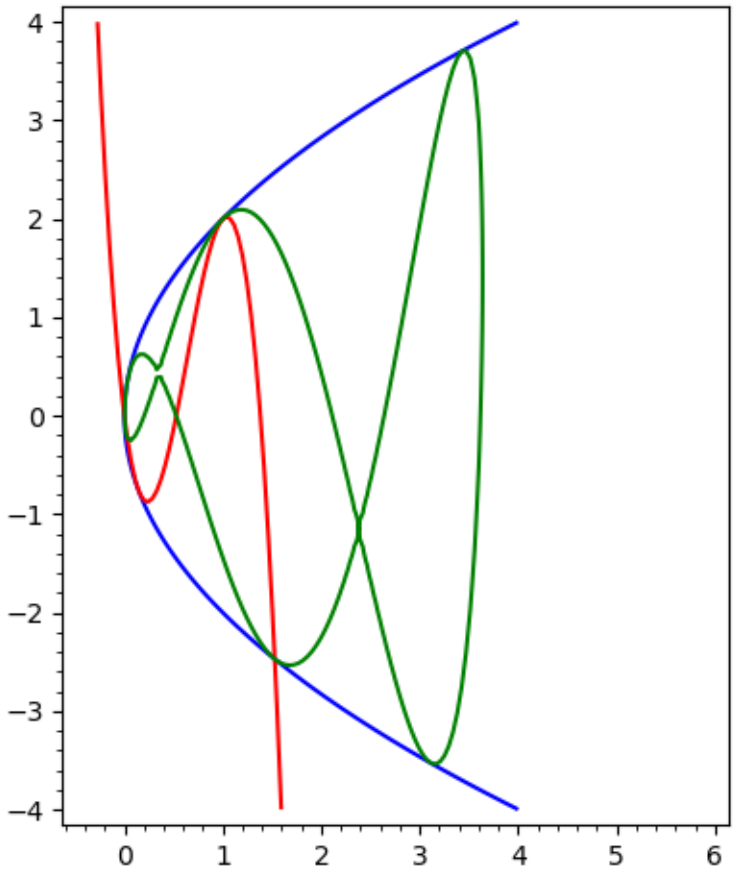
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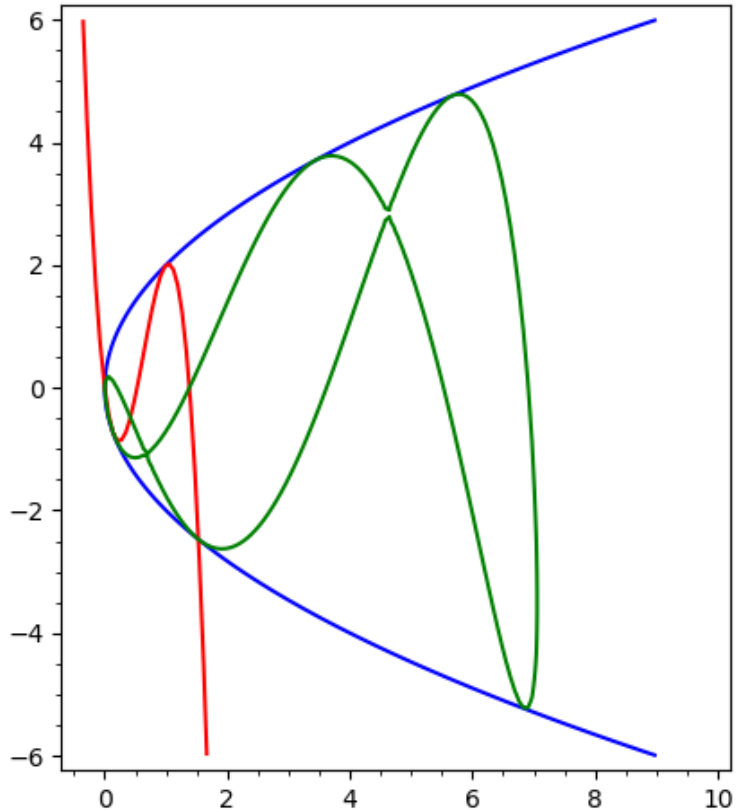
Epimorphism to $\mathbb{D}_{2N}$

- s is sent to a reflection
- v is sent to a rotation
- sv is a rotation and hence ℓ is in the kernel.
- The relations define commutation of ℓ with some element.

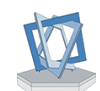


Quotient of an arrangement of 20 lines





- The real pictures determine the fundamental group
- Are they isomorphic? They have isomorphic profinite completions



Happy birthday, Alex!

