Topology, combinatorics and arithmetics of line arrangements of odd regular polygons

Enrique ARTAL BARTOLO

Departamento de Matemáticas
Facultad de Ciencias
Instituto Universitario de Matemáticas y sus Aplicaciones
Universidad de Zaragoza

Aspects réels de la géométrie
Alex Degtyarev 60th birthday
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Work in progress with J.I. Cogolludo, J. Martín Morales and J. Viu Sos

Combinatorics and topology

Theorem (Rybnikov, 2011)

$\exists \mathscr{R}_i$, $i = 1, 2$, 13-line arrangements in $\mathbb{P}^2 = \mathbb{P}^2_{\mathbb{C}}$ same combinatorics and $\pi_1(\mathbb{P}^2 \setminus \bigcup \mathscr{R}_1) \ncong \pi_1(\mathbb{P}^2 \setminus \bigcup \mathscr{R}_2)$.
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Theorem (A_-Carmona-Cogolludo-Marco, 2005)

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Definition

Combinatorics $\mathscr{C} = (\mathscr{L}, \mathscr{P})$

- $\mathscr{L}$ finite set (of lines)
- $\mathscr{P} \subset \mathcal{P}(\mathscr{L})$, $\#P > 1$, $\forall P \in \mathscr{P}$ (points)
- $\forall L_1, L_2 \in \mathscr{L}$, $L_1 \neq L_2$, $\exists! P \in \mathscr{P}$ such that $L_1, L_2 \in P$ (denoted $P \in L_i$).

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Ordered combinatorics $\mathscr{C} = (\mathscr{L}, \mathscr{P})$

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Combinatorics associated to a line arrangement

$\mathscr{A}$ line arrangement in $\mathbb{P}^2_{\mathbb{K}}$ ($\mathbb{K}$ field) $\Rightarrow \mathscr{C}_{\mathscr{A}} := (\mathscr{A}, \{\text{multiple points}\})$.
$\mathscr{A}$ ordered $\Longrightarrow \mathscr{C}_{\mathscr{A}}$ ordered.

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Theorem (A_-Carmona-Cogolludo-Marco, 2005)

$\exists \mathscr{L}_\pm = \{L_{\pm,1}, \ldots, L_{\pm,10}\}$ ordered line arrangements in $\mathbb{P}^2_{\mathbb{R}}$ same ordered combinatorics and $\nexists : \Phi : \mathbb{P}^2 \xrightarrow{homeo} \mathbb{P}^2$ with $\Phi(L^{\mathbb{C}}_{+,i}) = L^{\mathbb{C}}_{-,i}$, $1 \leq i \leq 10$.

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Features of $\overline{\mathscr{Z}}_{\pm}$ and $\mathscr{Z}_{\pm}^{\mathbb{C}}$

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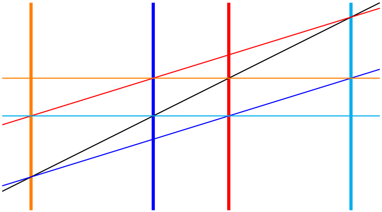

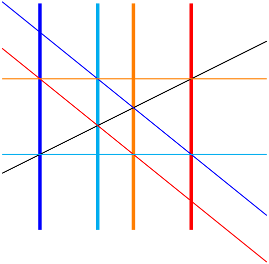

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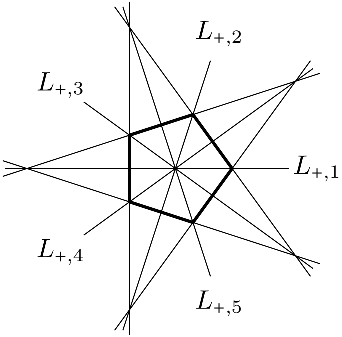

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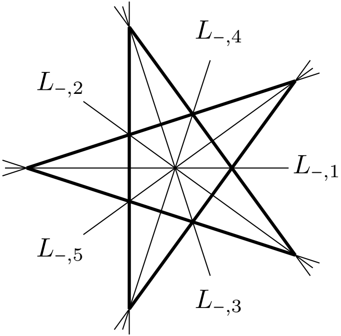

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- A more symmetric way to produce another $\overline{\mathscr{Z}}_{\pm}$.
- Study more general polygons (odd for simplicity).

Odd polygon arrangements

Definition

$\mathscr{A}_N$, $N := 2n + 1$, is the arrangement of sides and diagonals of a regular N-polygon $\Theta_N \subset \mathbb{C} \equiv \mathbb{R}^2 \subset \mathbb{P}^2_{\mathbb{R}}$.

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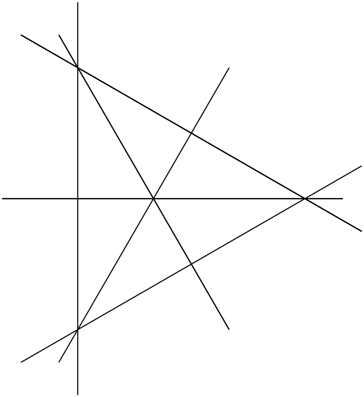
$N = 3$

Ceva arrangement



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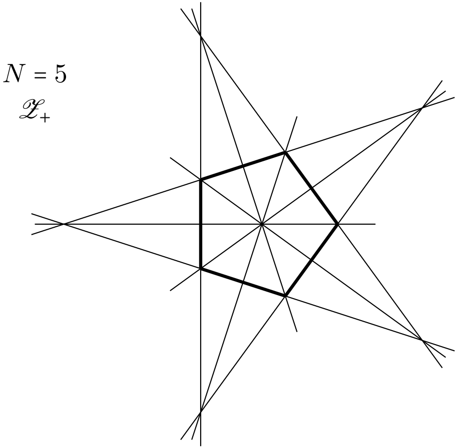

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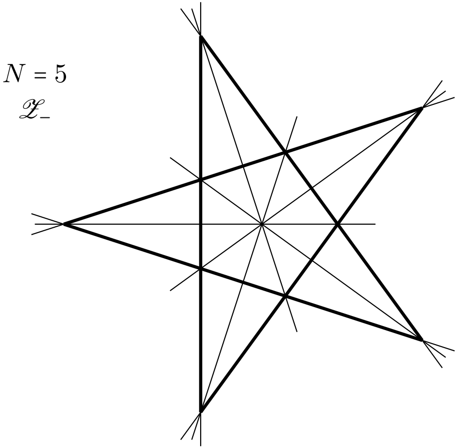

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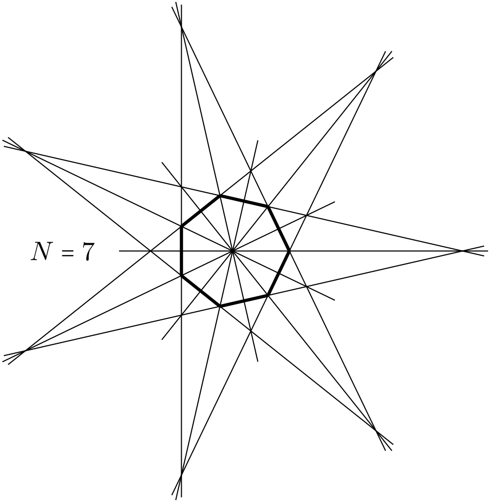

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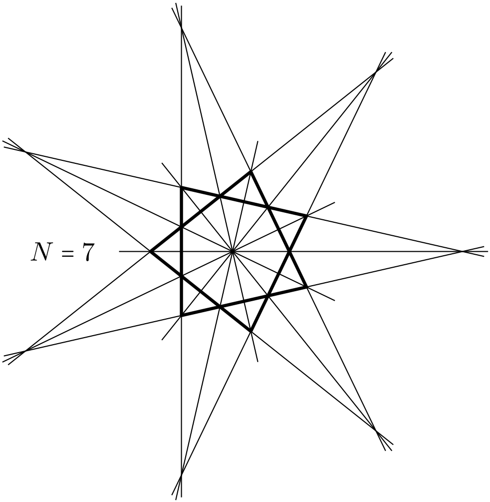

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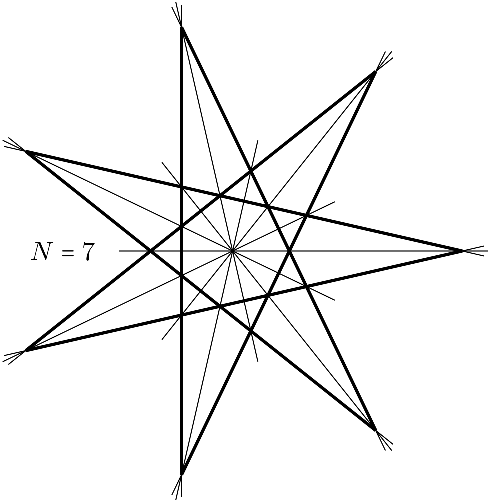

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Definition

Realization space of an ordered combinatorics $\mathscr{C}$:

$$\Sigma_{\mathbb{K}}(\mathscr{C}) := \{\mathscr{A} := (L_1, \ldots, L_m) \in (\check{\mathbb{P}}^2_{\mathbb{K}})^m \mid \mathscr{C}_{\mathscr{A}} = \mathscr{C}\}$$

Moduli space of an ordered combinatorics $\mathscr{C}$:

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Notation

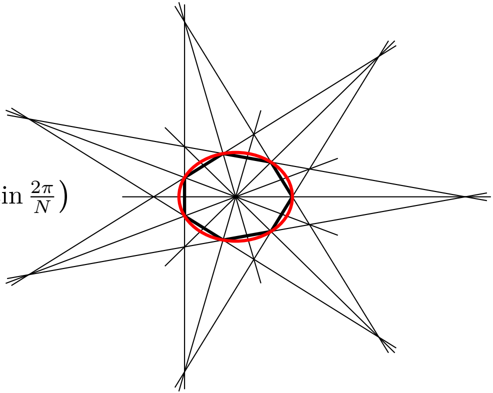
- $\mathscr{C}_N := \mathscr{C}_{\mathscr{A}_N}$, $\Sigma_N := \Sigma(\mathscr{C}_N)$, $\mathscr{M}_N := \mathscr{M}(\mathscr{C}_N)$
- $\mu_N := \{\eta \in \mathbb{C}^* \mid \eta^N = 1\}$, $D_\eta := \overline{\{t\eta \mid t \in \mathbb{R}\}} \subset \mathbb{P}^2_{\mathbb{R}}$
- L_η: line supporting edge of $\Theta_N \pitchfork D_\eta$.

Properties of $\mathscr{A}_N$ and $\mathscr{C}_N$

Field of definition

$\mathscr{A}_N$ has equations in $\mathbb{K}_N := \mathbb{Q}\left(\cos\frac{2\pi}{N}\right)$ of index $\frac{\Phi(N)}{2}$.

$$(x, y) \mapsto \left(x, y \sin\frac{2\pi}{N}\right)$$



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Sketch of the proof.

Each (convex or not) regular polygon yields one element of $\mathscr{M}_N$.

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- Place 5 points at fixed places

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$$1 \longrightarrow \mathbb{D}_{2N} \hookrightarrow \operatorname{Aut}\mathscr{C}_N \xrightarrow{\pi} \operatorname{Gal}\mathbb{K}_N \longrightarrow 1 \quad \textit{splits.}$$

Proposition

$\mathscr{M}_N \cong \operatorname{Gal}\mathbb{K}_N \cong \operatorname{Aut}\mu_N/\langle\zeta \mapsto \zeta^{-1}\rangle$

Sketch of the proof.

Each (convex or not) regular polygon yields one element of $\mathscr{M}_N$.

- Place 5 points at fixed places
- Place all the vertices of the *polygon* using variables.
- It can be done symmetric with respect to X-axis.

Properties of $\mathscr{A}_N$ and $\mathscr{C}_N$

Field of definition

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Each (convex or not) regular polygon yields one element of $\mathscr{M}_N$.

- Place 5 points at fixed places
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- It can be done symmetric with respect to X-axis.
- Apply dihedral action and use the regular polygon. □

Cyclic and dihedral actions

Cyclic action of μ_N

$$\begin{aligned} \mathbb{P}^2 &\longrightarrow \mathbb{P}^2 \\ [x:y:z] &\longmapsto \left[x\cos\tfrac{2\pi}{N} - y\sin\tfrac{2\pi}{N} : x\sin\tfrac{2\pi}{N} + y\cos\tfrac{2\pi}{N} : z\right] \end{aligned}$$

Fixed points: $[0:0:1], [1:i:0], [i:1:0]$.

Cyclic and dihedral actions

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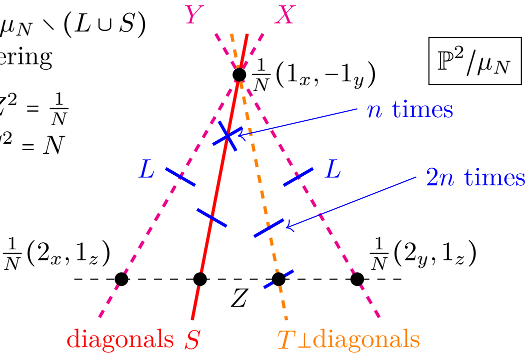
$\mathbb{P}^2 \smallsetminus \bigcup \mathscr{A}_N \longrightarrow \mathbb{P}^2/\mu_N \smallsetminus (L \cup S)$
cyclic orbifold covering

$$X^2 = Y^2 = Z^2 = \frac{1}{N}$$
$$L^2 = T^2 = S^2 = N$$



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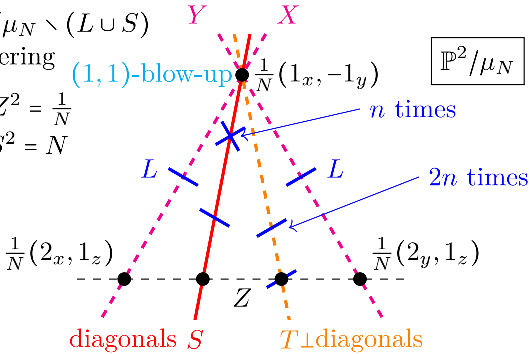


Cyclic and dihedral actions

$\boxed{\Sigma}$

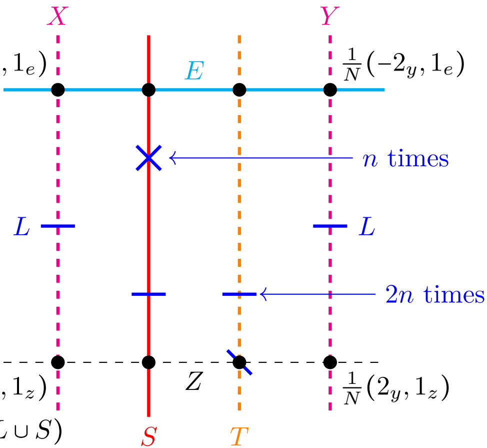
Normal ruled surface

$X^2 = Y^2 = T^2 = S^2 = 0$

$Z^2 = \frac{1}{N}$

$E^2 = -\frac{1}{N}$

$L^2 = N$



$\mathbb{P}^2 \smallsetminus \bigcup \mathscr{A}_N \longrightarrow \Sigma \smallsetminus (E \cup L \cup S)$

cyclic orbifold covering

μ_2 fixed points: S, $T \cap E$, $T \cap Z$

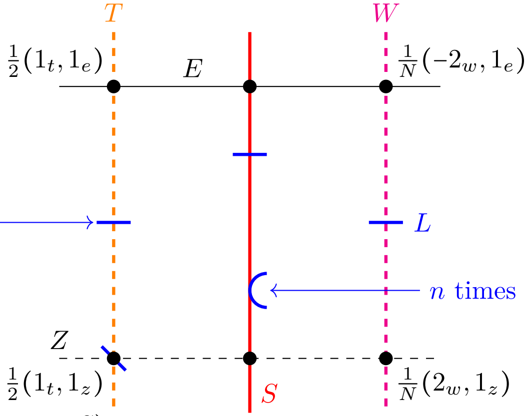
Cyclic and dihedral actions

$$W^2 = T^2 = S^2 = 0$$

$$-E^2 = Z^2 = \frac{1}{2N}$$

$$L^2 = \frac{N}{2}$$

Σ/μ_2 birational to $\mathbb{P}^2/\mathbb{D}_{2N}$



$\mathbb{P}^2 \smallsetminus \bigcup \mathscr{A}_N \longrightarrow \Sigma/\mu_2 \smallsetminus (E \cup L \cup S)$
dihedral orbifold covering

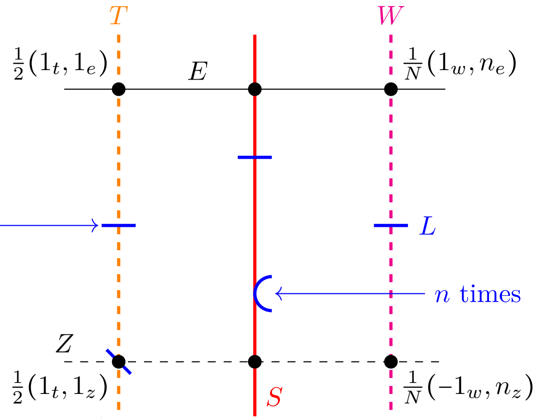
Cyclic and dihedral actions

$W^2 = T^2 = S^2 = 0$

$-E^2 = Z^2 = \frac{1}{2N}$

$L^2 = \frac{N}{2}$ $\quad$ n times

Σ/μ_2 birational to $\mathbb{P}^2/\mathbb{D}_{2N}$



$\mathbb{P}^2 \smallsetminus \bigcup \mathscr{A}_N \longrightarrow \Sigma/\mu_2 \smallsetminus (E \cup L \cup S)$

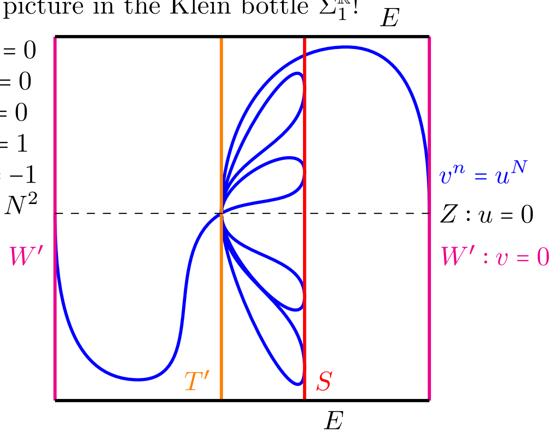
dihedral orbifold covering $\quad$ $(1, n)$-Nagata transformation at $E \cap W$

$(1, 1)$-Nagata transformation at $E \cap T$

Dessin d'enfants

Actual picture in the Klein bottle $\Sigma_1^{\mathbb{R}}$!

$W'^2 = 0$
$T'^2 = 0$
$S^2 = 0$
$Z^2 = 1$
$E^2 = -1$
$L^2 = N^2$



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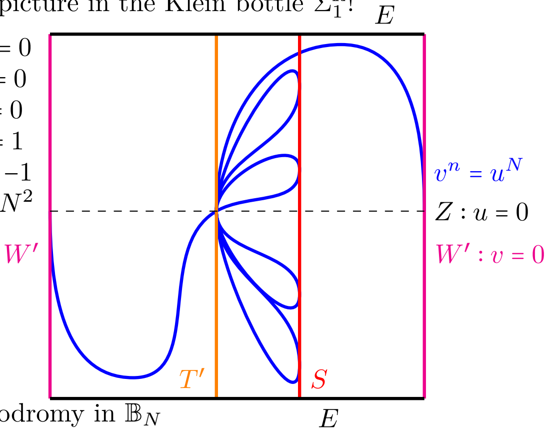

- Braid monodromy in $\mathbb{B}_N$
- $T' : \Delta_N,\ S : \prod_{j=1}^{n} \sigma_{2j},\ W' : \left(\prod_{j=1}^{n} \sigma_{2j}\sigma_{2j-1} \right)^n =: \tau$
- T, W are not in $E \cup L \cup S$!

Dessin d'enfants

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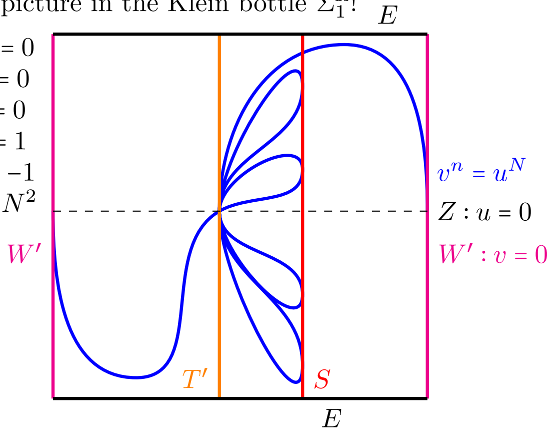

Theorem

$$\pi_1(\Sigma_1 \smallsetminus (L \cup S \cup E \cup T' \cup W')) = \langle \underset{1 \le i \le N}{x_i}, t, w \mid x_i^w = x_i^\tau, x_i^t = x_i^{\Delta_N} \rangle$$

Dessin d'enfants

Actual picture in the Klein bottle $\Sigma_1^{\mathbb{R}}$!

E

$W'^2 = 0$
$T'^2 = 0$
$S^2 = 0$
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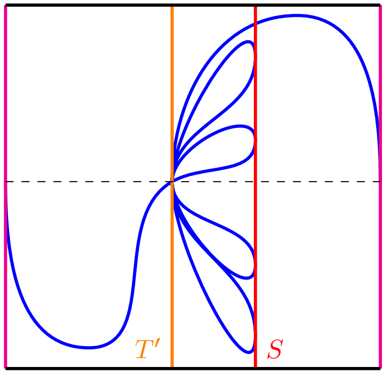
$L^2 = N^2$

W'



$v^n = u^N$
$Z : u = 0$
$W' : v = 0$

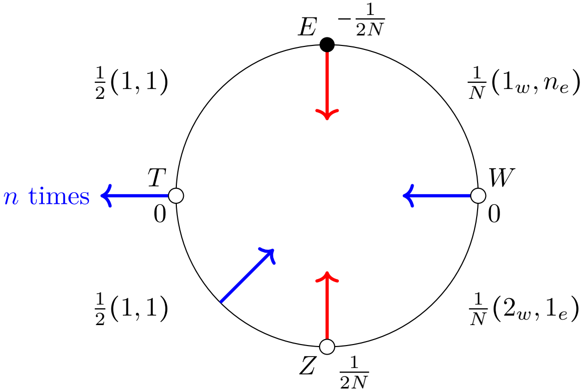
E

Theorem

$$\underbrace{\pi_1(\Sigma/\mu_2 \smallsetminus (L \cup S \cup E))}_{G} = \langle \underset{1 \le i \le N}{x_i}, t, w \mid x_i^w = x_i^\tau, x_i^t = x_i^{\Delta_N}, t^2 = C, w^N = C^n \rangle$$

$$C := \prod_{j=N}^{1} x_j, \qquad G \twoheadrightarrow \mu_N * \mu_2 \twoheadrightarrow \mathbb{D}_{2N}, \qquad \ker = \pi_1(\mathbb{P}^2 \smallsetminus \mathscr{A}_N)$$

Another birational model



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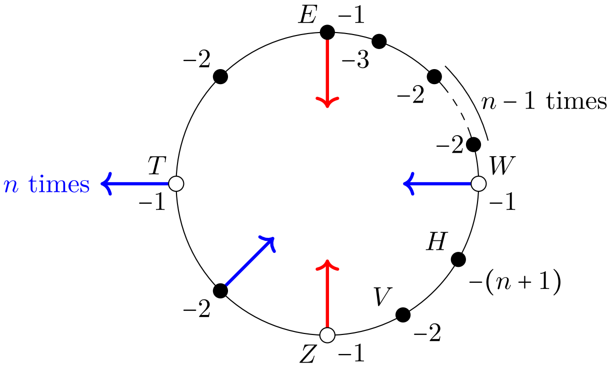

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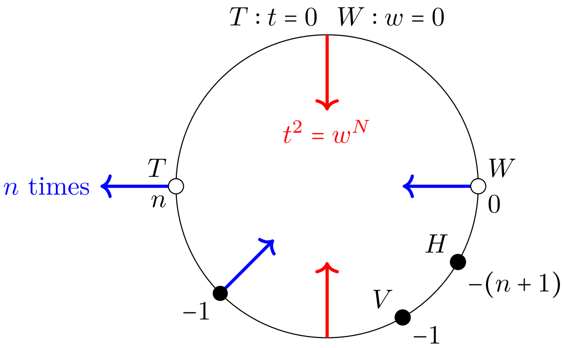

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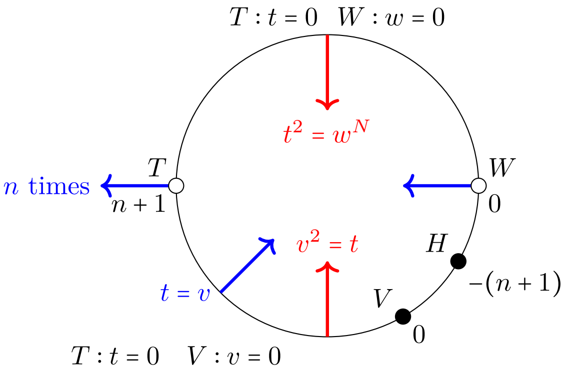
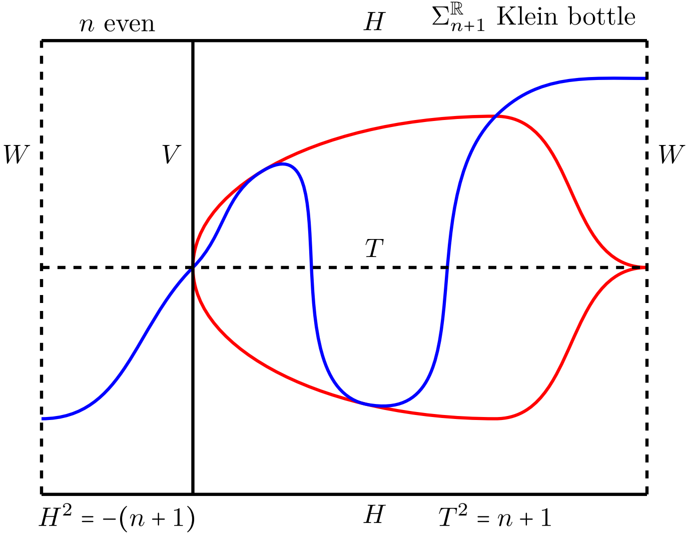


Real expression of the second birational model



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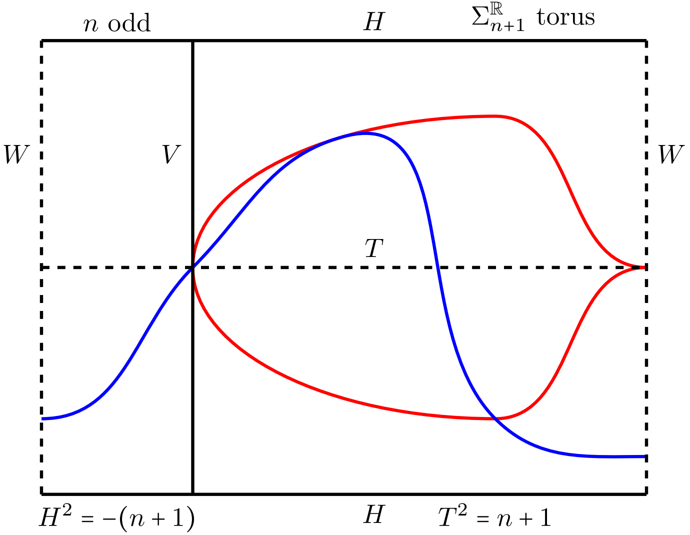

Short fundamental group presentation

Theorem

Let $n = 2m$. The orbifold fundamental group of $\mathbb{P}^2/\mathbb{D}_{2N} \setminus (E \cup L \cup S)$ has a presentation with two generators s (meridian of S) and v (meridian of V) with relations

- $R_1, \ldots, R_n$, *coming from the tacnodes (they do not depend on N) and S_{n+1} coming from the node.*
- $v^N = 1$

There is a meridian ℓ of L satisfying $l(sv)^2 = 1$.

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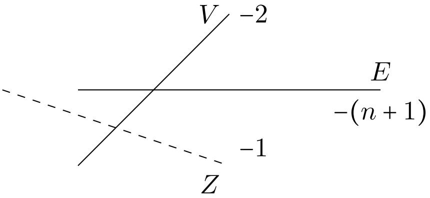

$e = v^2$

$v = e^{n+1}$

Short fundamental group presentation

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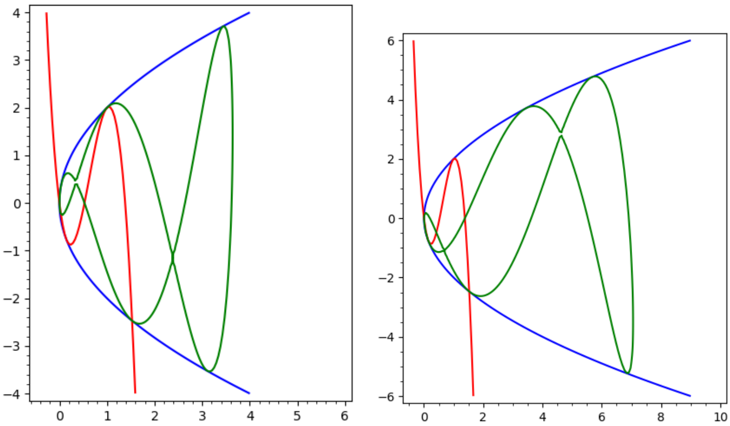
Epimorphism to $\mathbb{D}_{2N}$

- s is sent to a reflection
- v is sent to a rotation
- sv is a rotation and hence ℓ is in the kernel.
- The relations define commutation of ℓ with some element.

Quotient of an arrangement of 20 lines



- The real pictures determine the fundamental group
- Are they isomorphic? They have isomorphic profinite completions

Happy birthday, Alex!