

Combinatorics and essential coordinate components of characteristic varieties of line arrangements

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$$\begin{array}{ccc} \text{▶ } \mathcal{B} \subset \mathcal{A} \text{ sub-arrangement:} & \mathbb{T}_{G_{\mathcal{B}}} & \hookrightarrow & \mathbb{T}_G \\ & \downarrow & & \downarrow \\ & \{t_{\bar{L}} = 1, \bar{L} \in \mathcal{A} \setminus \mathcal{B}\} & = & (\mathbb{C}^*)^{\mathcal{B}} \hookrightarrow (\mathbb{C}^*)^{\mathcal{A}} \end{array}$$



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Characteristic varieties of \mathcal{A} :

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- ▶ **Determined by Betti numbers of finite cyclic covers.**



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Theorem (Libgober)

1. *There is an algorithm to compute non sub-essential coordinate components which depends on the position of multiple points of \mathcal{A} .*
2. *Sub-essential coordinate components are torsion points.*

Coverings and torsion characters

- ▶ $\xi \in \mathbb{T}_G$, $1 < \text{ord}(\xi) =: N < \infty$, $\xi(\mu_j) = \exp\left(2i\pi \frac{k_j}{N}\right)$,

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- ▶ $\tilde{\xi}: X_\xi \rightarrow X_\xi$ generator of the monodromy group of ρ .



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- ▶ $H^1(X_\xi; \mathbb{C})^\xi$ eigenspace for $\tilde{\xi}$ and eigenvalue $\exp\left(\frac{2i\pi}{N}\right)$.
- ▶ $H^1(X_\xi; \mathbb{C})^\xi \cong H^1(M(\mathcal{A}); \underline{\mathbb{C}}_\xi)$.



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- ▶ $\{\xi\}$ sub-essential coordinate \implies
 $H^1(\bar{X}_\xi; \mathbb{C}) \neq H^1(X_\xi; \mathbb{C}) \iff H^0(\bar{X}_\xi; \Omega_{\bar{X}_\xi}^1) \neq H^0(\bar{X}_\xi; \Omega_{\bar{X}_\xi}^1 \log(D_\xi))$

An explicit compactification

Notation

- ▶ $\mathcal{P} := \{\bar{L}_i \cap \bar{L}_j \mid 0 \leq i < j \leq n\}$.
- ▶ For $P \in \mathcal{P}$, $m(P) := \#\{\bar{L} \in \mathcal{A} \mid P \in \bar{L}\}$.
- ▶ $\mathcal{P}_{>2} := \{P \in \mathcal{P} \mid m(P) > 2\}$.

An explicit compactification

First Step

- ▶ Blow up the points of $\mathcal{P}_{>2}$, $\sigma : Y \rightarrow \mathbb{P}^2$.
- ▶ $\tilde{\mathcal{A}}$ irreducible components of $\sigma^{-1}(\bigcup \mathcal{A})$, normal crossing divisor:
 - ▶ \tilde{L} strict transform of \bar{L} , $\tilde{L}^2 = 1 - \#\mathcal{P}_{>2} \cap \bar{L}$.
 - ▶ $E_P := \sigma^{-1}(P)$, $P \in \mathcal{P}_{>2}$, exceptional component, $E_P^2 = -1$,
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Second Step

$$\tilde{Z}_\xi := \left\{ [x : y : z : T]_{(1,1,1,k)} \in \mathbb{P}^3_{(1,1,1,k)} \mid T^N = \prod_{j=0}^n \ell_j(x, y, z)^{k_j} \right\}$$

$$\begin{array}{ccccccc} \bar{X}_\xi & \xrightarrow{\pi} & \tilde{Y}_\xi^{\text{norm}} & \longrightarrow & \tilde{Y}_\xi & \longrightarrow & \tilde{Z}_\xi \\ & & & & \downarrow & & \downarrow \\ & & & & Y & \longrightarrow & \mathbb{P}^2 \end{array}$$



Exact sequences

$$\blacktriangleright 0 \rightarrow \Omega_{X_\xi}^1 \rightarrow \Omega_{X_\xi}^1 \log(D_\xi) \rightarrow \bigoplus_{D \in \mathcal{D}_\xi} i_* \mathcal{O}_D \rightarrow 0$$



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- ▶ $\mathcal{D}_\xi = \mathcal{D}_\xi^\pi \cup \bigcup_{B \in \tilde{\mathcal{A}}} \mathcal{B}_\xi(B):$
 - ▶ $\tilde{B} \in \mathcal{D}_\xi^\pi \iff \pi(\tilde{B})$ is a point
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- ▶ $\mathbb{C}\langle \mathcal{D}_\xi^\pi \rangle^\xi = 0$



Exact sequences

- ▶ $0 \rightarrow H^0(X_\xi; \Omega_{X_\xi}^1) \rightarrow H^0(X_\xi; \Omega_{X_\xi}^1 \log(D_\xi)) \rightarrow H(\mathcal{A}) \rightarrow 0$
- ▶ $0 \rightarrow H(\mathcal{A}) \rightarrow \bigoplus_{D \in \mathcal{D}_\xi} H^0(D; \mathcal{O}_D) \rightarrow H^1(X_\xi; \Omega_{X_\xi}^1) \subset H^2(X_\xi; \mathbb{C}).$

- ▶ $\mathcal{D}_\xi = \mathcal{D}_\xi^\pi \cup \bigcup_{B \in \tilde{\mathcal{A}}} \mathcal{B}_\xi(B):$

- ▶ $\tilde{B} \in \mathcal{D}_\xi^\pi \iff \pi(\tilde{B})$ is a point
- ▶ $\tilde{B} \in \mathcal{B}_\xi(B) \iff \tilde{B}$ projects onto B .

- ▶ $\bigoplus_{D \in \mathcal{D}_\xi} H^0(D; \mathcal{O}_D) = \mathbb{C}\langle \mathcal{D}_\xi^\pi \rangle \oplus \bigoplus_{B \in \tilde{\mathcal{A}}} \mathbb{C}\langle \mathcal{B}_\xi(B) \rangle$

- ▶ $\mathbb{C}\langle \mathcal{B}_\xi(B) \rangle^\xi = \begin{cases} 0 & \text{if } \#\mathcal{B}_\xi(B) < N \\ \mathbb{C}\left\langle \overbrace{\sum_{j=0}^{N-1} \exp\left(-2i\pi \frac{j}{N}\right) \xi^j(\tilde{B})}^{B(\xi)} \right\rangle & \text{if } \#\mathcal{B}_\xi(B) = N \end{cases}$

Inner components

Definition

B is *inner* if $\#\mathcal{B}_\xi(B) = N \iff t_B = 1$ and $t_C = 1$ for all C neighbor to B : \mathcal{U}_ξ set of inner components.

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Theorem

$$1. H^1(X_\xi; \mathbb{C})^\xi \cong H^1(\overline{X}_\xi; \mathbb{C})^\xi \oplus \ker \underbrace{\left(\bigoplus_{B \in \mathcal{U}_\xi} \mathbb{C} \langle B(\xi) \rangle \rightarrow H_2(\overline{X}_\xi; \mathbb{C}) \right)}_{K(\xi)}$$



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Theorem

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2. $\dim_{\mathbb{C}} K_\xi$ is the corank of the intersection form induced on $\bigoplus_{B \in \mathcal{U}_\xi} \mathbb{C} \langle B(\xi) \rangle$ by the usual intersection form of \bar{X}_ξ .



Twisted intersection form

$$\blacktriangleright \mathbb{C}\langle \mathcal{U}_\xi \rangle \equiv \bigoplus_{B \in \mathcal{U}_\xi} \mathbb{C}\langle B(\xi) \rangle, B \leftrightarrow \frac{1}{\sqrt{N}} B(\xi).$$



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- ▶ If $\Gamma_{\mathcal{U}_\xi}$ is a forest, then $\cdot_\xi = \cdot$.

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- ▶ $\Gamma_{\mathcal{U}_\xi}$ denotes the dual graph of \mathcal{U}_ξ .
- ▶ If $\Gamma_{\mathcal{U}_\xi}$ has cycles, it depends on the embedding of the boundary of a regular neighborhood of $\bigcup \mathcal{A}$ in $M(\mathcal{A})$ (Hironaka, Florens-Marco-Guerville)

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- ▶ If $\overrightarrow{BC} \notin \mathcal{T}_{\mathcal{U}_\xi}$ then it defines a cycle $\gamma_{B,C} \in H_1(\Gamma_{\mathcal{U}_\xi}; \mathbb{Z})$ which is well-defined mod $\ker \xi$ ($\gamma_{C,B} = \gamma_{B,C}^{-1}$).



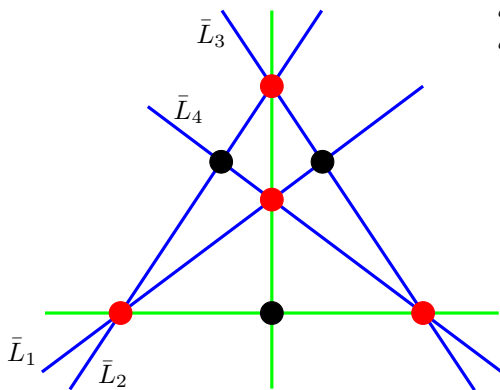
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$$\text{▶ } B \cdot_\xi C = \begin{cases} 0 & \text{if } B \cdot C = 0 \\ B \cdot C & \text{if either } B = C \text{ or } \overrightarrow{BC} \in \mathcal{T}_{\mathcal{U}_\xi} \\ \xi(\gamma_{B,C})(B \cdot C) & \text{if } \overrightarrow{BC} \notin \mathcal{T}_{\mathcal{U}_\xi} \end{cases}$$



Ceva arrangement

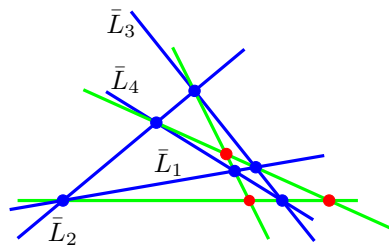


$$\begin{aligned}\xi(\mu_1) &= \zeta_d \\ \xi(\mu_2) &= \zeta_d^{-1} \\ \xi(\mu_3) &= \zeta_d \\ \xi(\mu_4) &= \zeta_d^{-1}\end{aligned}$$



$$\cdot \xi : \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

Ceva-extended arrangement



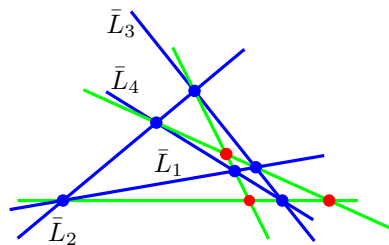
$$\xi(\mu_1) = -1$$

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$$\xi(\mu_3) = -1$$

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Ceva-extended arrangement

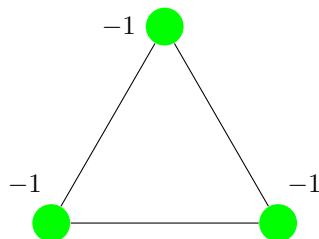


$$\xi(\mu_1) = -1$$

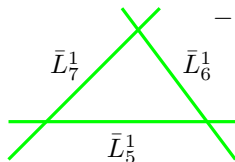
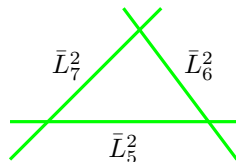
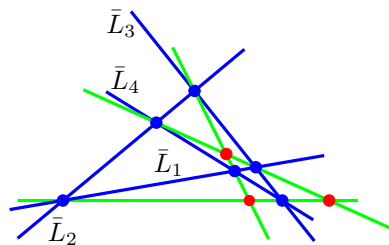
$$\xi(\mu_2) = -1$$

$$\xi(\mu_3) = -1$$

$$\xi(\mu_4) = -1$$



Ceva-extended arrangement

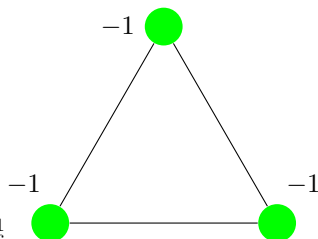


$$\xi(\mu_1) = -1$$

$$\xi(\mu_2) = -1$$

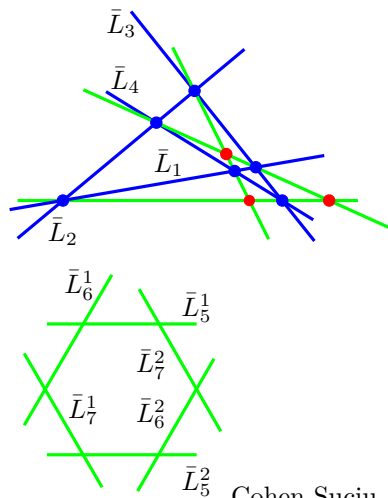
$$\xi(\mu_3) = -1$$

$$\xi(\mu_4) = -1$$



$$\cdot \xi : \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

Ceva-extended arrangement



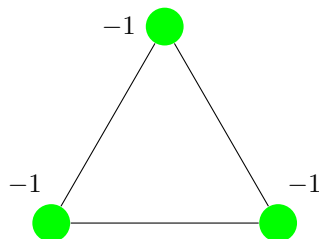
Cohen-Suciu

$$\xi(\mu_1) = -1$$

$$\xi(\mu_2) = -1$$

$$\xi(\mu_3) = -1$$

$$\xi(\mu_4) = -1$$



$$\cdot \xi : \begin{pmatrix} -1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & -1 \end{pmatrix}$$

Hesse arrangement

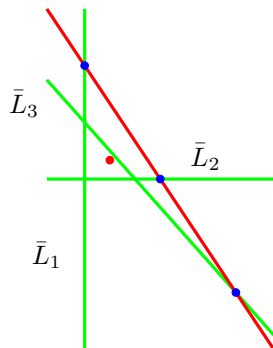
$$0 = xyz((x^3 + y^3 + z^3)^3 - 27xyz)$$



Hesse arrangement

$$0 = xyz((x^3 + y^3 + z^3)^3 - 27xyz)$$

$$0 = xyz(x^3 + y^3 + z^3 - 3xyz)(\dots)$$



$$\xi(\mu_1) = \xi(\mu_2) = \xi(\mu_3) = 1$$

$$\xi(\mu_4) = \xi(\mu_5) = \xi(\mu_6) = t \in \mathbb{C}^*$$

$$\xi(\mu_7) = \xi(\mu_8) = \xi(\mu_9) = s \in \mathbb{C}^*$$

$$\xi(\mu_{10}) = \xi(\mu_{11}) = \xi(\mu_{12}) = u \in \mathbb{C}^*$$

$$tsu = 1$$

$$\cdot \xi : \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$$

Multimesc!