Topology of surface singularities: superisolated, Lê-Yomdin and weighted Lê-Yomdin

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GKŁW Seminar in Singularity Theory
March 24th 2023

Surfaces in $\mathbb{C}^3$

Superisolated and Lê-Yomdin singularities

Zariski tuples and Jordan form

Zariski tuples of non-Alexander type

Quasi-homogeneous and Weighted Lê-Yomdin singularities

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- In general, it does not determine the analytic type.

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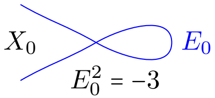
- $\sigma_0 : (X_0, E_0) \to (S, 0)$ minimal (non *good*) resolution:



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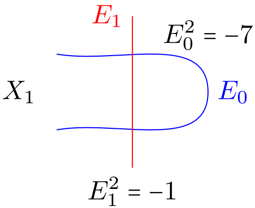

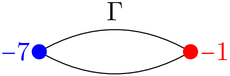

both genera vanish

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Plumbing (Neumann, Waldhausen)

- Vertex v weighted (g_v, e_v): M_v oriented $\mathbb{S}^1$-fiber bundle over a compact surface of genus g_v with Euler number e_v.
- Edge between v, w: empty two fiber solid tori in M_v, M_w and glue the boundaries interchanging meridians and fibers.
- *Normalized* graphs determine K topologically.
- Incidence matrix A_Γ; $\det S := \det(-A_\Gamma) \in \mathbb{Z}_{\geq 1}$.

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- Intersection and Seifert forms on $H_2(F; \mathbb{Z})$ (distinguished basis of vanishing cycles)

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- $\zeta(t) = \prod_{k=0}^{2} \Delta^{\rho}_{H^k(F;\mathbb{C})}(t)^{(-1)^k} = (t-1)\Delta^{\rho}_{H^2(F;\mathbb{C})}(t) = \prod_{i \in I} (t^{m_i} - 1)^{\chi(\check{E}_i)}$.

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- $\Delta^{\rho}_{H^2(F;\mathbb{C})}(t) = \dfrac{(t^d - 1)^{d^2 - 3d + 3}}{t - 1}$.
- Since it is homogeneous, the monodromy is semisimple (the geometric monodromy has finite order d).

Semistable resolution and Steenbrink spectral sequence

- $e = \operatorname{lcm}(m_i \mid i \in I)$, ν normalization, $D_0 \equiv \hat{S}$, D_i ramified cover over E_i.

$$\begin{array}{ccccc} (Z,D) & \xrightarrow{\quad\nu\quad} & (\tilde{X},E) & \xrightarrow{\tau_e} & (X,E) \\ & & \big\downarrow & & \big\downarrow{\scriptstyle F\circ\pi} \\ \text{Model for the} & & (\mathbb{C},0) & \longrightarrow & (\mathbb{C},0) \\ \text{Milnor fibration} & & t & \longmapsto & t^e \end{array}$$

Semistable resolution and Steenbrink spectral sequence

- Steenbrink spectral sequence degenerates at E_2, converges to $H^*(F;\mathbb{C})$ filtered by the weight filtration. E_1 step:

$$D^{[0]}:=\coprod_{i\in I_0} D_i, \quad D^{[1]}:=\coprod_{I_0} D_i\cap D_j, \quad D_0^{[1]}:=\coprod_{I} D_i\cap D_j$$

$$D^{[2]}:=\coprod_{I_0} D_i\cap D_j\cap D_k, \quad D_0^{[2]}:=\coprod_{I} D_i\cap D_j\cap D_k, \quad I_0:=I\smallsetminus\{0\}$$

$$\begin{array}{ccccc}
H^0(D_0^{[2]};\mathbb{C}) & & & & \\
\uparrow{\scriptstyle i^*} & & & & \\
H^0(D^{[1]};\mathbb{C}) & H^1(D^{[1]};\mathbb{C}) & H^2(D^{[1]};\mathbb{C}) & & \\
\uparrow{\scriptstyle i^*} & \uparrow{\scriptstyle i^*} & \uparrow{\scriptstyle i^*\oplus g} & & \\
H^0(D^{[0]};\mathbb{C}) & H^1(D^{[0]};\mathbb{C}) & H^2(D^{[0]};\mathbb{C})\oplus H^0(D_0^{[2]};\mathbb{C}) & H^3(D^{[0]};\mathbb{C}) & H^4(D^{[0]};\mathbb{C}) \\
 & & \uparrow{\scriptstyle g} & \uparrow{\scriptstyle g} & \uparrow{\scriptstyle g} \\
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- Δ_D, Δ_E, dual complexes of the normal crossing divisors D, E, $\pi_\Delta : \Delta_D \to \Delta_E$, $\rho_\Delta : \Delta_D \to \Delta_D$ the monodromy.
  - Δ_D is connected, $H^1(\Delta_D; \mathbb{C}) = 0$.
  - The characteristic polynomial of ρ_Δ on $H^2(\Delta_D)$ measures the Jordan blocks of size 3.
  - It corresponds to the first column.
- Second column: the characteristic polynomial of the monodromy acting on $H^1(D^{[1]}; \mathbb{C})/H^1(D^{[0]}; \mathbb{C})$ measures the Jordan blocks of size 2 (and eigenvalue $\neq 1$).

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  - The characteristic polynomial of ρ_Δ on $H^2(\Delta_D)$ measures the Jordan blocks of size 3.
  - It corresponds to the first column.
- Second column: the characteristic polynomial of the monodromy acting on $H^1(D^{[1]}; \mathbb{C})/H^1(D^{[0]}; \mathbb{C})$ measures the Jordan blocks of size 2 (and eigenvalue $\neq 1$).

Curves and homogeneous singularities

Homogeneous polynomials

$F(x, y, z) \in \mathbb{C}[x, y, z]$ homogeneous polynomial of degree $d > 0$:

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$(S,0) = \{f(x,y,z) = 0\}$, is *superisolated* (SIS) if $\operatorname{Sing}(C_m) \cap C_{m+1} = \emptyset$,

$$f(x,y,z) = \overbrace{f_m(x,y,z)}^{\neq 0} + f_{m+1}(x,y,z) + \cdots \in \mathbb{C}\{x,y,z\},$$

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- Generic deformation of $f_m(x,y,z) = 0$ to be isolated singularity.
- C_m must be reduced.

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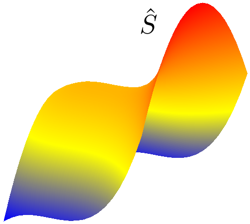

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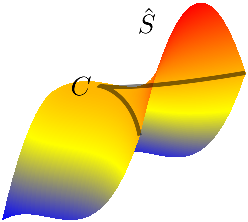

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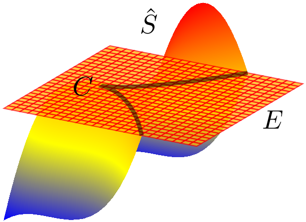

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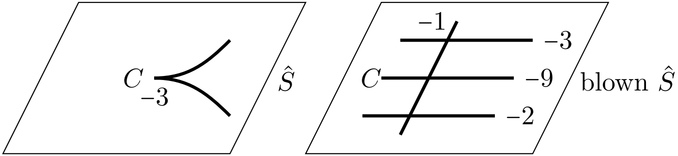

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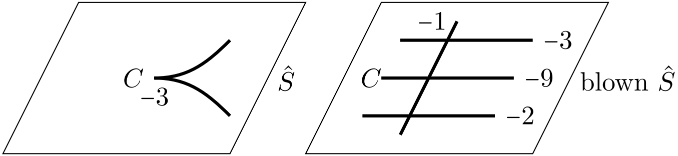

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S SIS with tangent cone $C_1 + \cdots + C_r$, $\deg C_i = d_i$. Then C in $\hat{S}$ is abstractly isomorphic to C in $\mathbb{P}^2$ and

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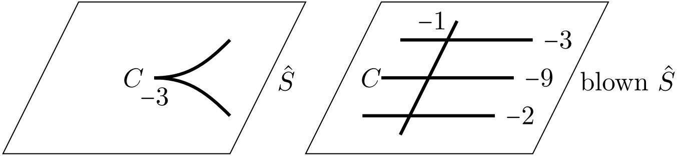

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$$\Delta(t) = \prod_{j=1}^{r} (t^{m_j} - 1)^{n_j} \Longrightarrow \Delta^{(k)}(t) = \prod_{j=1}^{r} \left(t^{\frac{m_j}{\gcd(m_j,k)}} - 1 \right)^{\gcd(m_j,k) n_j}$$

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Yau's conjecture

$(S,0) \subset (\mathbb{C}^3,0)$ isolated surface singularity. Then the embedded topological type of $(S,0)$ determines and is determined by $\Delta^{\rho}_{H^2(F;\mathbb{C})}$ and $\pi_1(K)$ (equivalent to the topological type of K).

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Corollary

$(S_i,0)$, $i=1,2$, SISs with tangent cones having the same combinatorics. They have the same characteristic polynomial of the monodromy and the same embedded topology.

Zariski tuples

Definition

$C_1, \ldots, C_r \subset \mathbb{P}^2$ curves with the same combinatorics. They form a *Zariski r-tuple* if $(\mathbb{P}^2, C_i)$ and $(\mathbb{P}^2, C_j)$ are not homeomorphic if $i \neq j$.

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A Zariski r-tuple is an *Alexander-Zariski r-tuple* if the Alexander polynomials are pairwise distinct.

Alexander-Zariski tuples polynomials and SIS

Alexander-Zariski polynomials exist (Zariski)

C sextic, $\#\operatorname{Sing}(C) = 6$, ordinary cusps.

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$D^{[0]} = D_o \coprod D_o^{[0]}$, where D_o is birationally equivalent to the d-cyclic cover of $\mathbb{P}^2$ ramified along the tangent cone. The number of 2-Jordan blocks (eigenvalues $\neq 1$):

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Corollary

Zariski's example provides a counterexample to Yau's conjecture.

Surfaces in $\mathbb{C}^3$

Superisolated and Lê-Yomdin singularities

Zariski tuples and Jordan form

Zariski tuples of non-Alexander type

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- What about the embedded topology of the associated SIS or k-Lê-Yomdin?
- One may construct *generic isolated singularities* with non-reduced tangent cone with distinct embedded topology.

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- Same question: What about the embedded topology of the associated SIS or k-Lê-Yomdin?

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- Possible counterexample: look for a SIS such that the topological type of the tangent cone cannot be realized over the reals.

Weighted Lê-Yomdin singularities

Definition

$\omega \in \mathbb{Z}^3_{>0}$ coprime, $(S,0) = \{F = 0\}$, $F \in \mathbb{C}\{x,y,z\}$, isolated surface singularity. Decompose F in ω-quasi-homogeneous forms:

$$F(x,y,z) = \overbrace{F_m(x,y,z)}^{\neq 0} + F_{m+k}(x,y,z) + \dots$$

Let $C_j := V_{\mathbb{P}}(F_m) \subset \mathbb{P}^2_\omega$ (as orbifold). Then S is a (k,ω)-*weighted Lê-Yomdin singularity* (WLYS) if at each $P \in C_m \cap C_{m+k}$ the *local equation* of C_m at P is of order 1.

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- Rephrase Lê-Yomdin singularity work replacing the first blow-up by an ω-weighted blow-up.

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- Notion of Zariski tuples can be extended.

Equisingular families in $\mathbb{P}^2_\omega$, $\omega = (2, 3, 1)$ and WLYS

Briançon-Speder WYLS

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Thanks for your attention!