Characteristic Varieties of Quasiprojective Varieties

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- Zariski-Lefschetz: it is enough to consider curves and surfaces.

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- Any finitely presented group is the fundamental group of compact orientable 4-manifold.
- Kähler groups are fundamental groups of compact Kähler 4-manifolds: there are obstructions for a group to be Kähler.

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- $1 \to \mathbb{T}_G^{\mathbf{1}} \to \mathbb{T}_G \to \operatorname{Hom}(\operatorname{Tors} H, \mathbb{C}^*) \to 1.$

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$$\begin{array}{l}\tilde{X} \text{ maximal abelian covering of } X\\ \downarrow \pi \\ X\end{array}$$

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$$\begin{array}{ccc} \tilde{X} & \circlearrowleft & H \\ \Big\downarrow \pi & & \\ X & & \end{array}$$

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- $C_*(\tilde{X};\mathbb{C})$ is complex of free Λ-modules.

Twisted Cohomology

Local system of coefficients

- $\xi \in \mathbb{T}_G$.
- Define a *locally constant* sheaf $\underline{\mathbb{C}}_\xi$.
- $\tilde{X} \times \mathbb{C}$ sheaf over X ($\mathbb{C}$ with the discrete topology).
- Action of H given by $(\tilde{x}, t)^h := (\tilde{x}^h, \xi(h^{-1})t)$.
- $\underline{\mathbb{C}}_\xi$ is the quotient of $\tilde{X} \times \mathbb{C}$ by this action.

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Definition of twisted cohomology

$H^1(X; \underline{\mathbb{C}}_\xi)$ means *sheaf cohomology*. It depends only on G!

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- Consider the complexes $C_*(X;\mathbb{C}_\xi) := C_*(\tilde{X};\mathbb{C}) \otimes_\Lambda \mathbb{C}_\xi$ and $C^*(X;\mathbb{C}_\xi) := C^*(\tilde{X};\mathbb{C}) \otimes_\Lambda \mathbb{C}_\xi$.

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- For computational purposes we will consider $H_1(X;\underline{\mathbb{C}}_\xi)$.

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- $k - 1 \leq \dim H_1(X; \mathbb{C}_\xi) \leq k$ (it equals $k \Leftrightarrow \xi = \mathbf{1}$).

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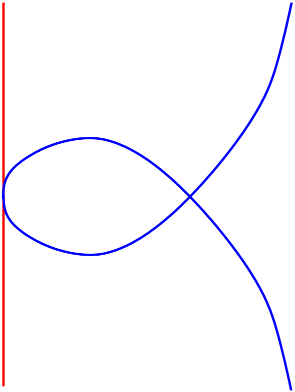
- $\dim H_1(X; \underline{\mathbb{C}}_\xi) = 1$ if $\xi(t)$ is a primitive 6^{th}-root of unity.

A curve group



$C : (x+z)z(y^2z - x^2z - x^3) = 0$

$G = \langle x, y, z | [x, y] = 1, y = z^{-1}xz, [yx, z] = 1 \rangle.$

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- $\dim H_1(X; \underline{\mathbb{C}}_\xi) = 1$ if $\xi(t) = 1$, $\xi(s) = -1$.

Deligne's theory I

Deligne extensions

- $X = \bar{X} \setminus \mathcal{D}$, $\bar{X}$ smooth projective, $\mathcal{D}$ normal crossing divisor.
- Flat line bundle (L_ξ, ∇) on X where $L_\xi = \underline{\mathbb{C}}_\xi \otimes \mathcal{O}_X$.
- $(\bar{L}_\xi, \bar{\nabla})$ extension to $\bar{X}$
- A lot of extensions are possible: parametrized by $\mathbb{Z}^n$, n number of irreducible components of $\mathcal{D}$.

Deligne's theory II

Example

- Let $X = \mathbb{C}^*$, x a generator of $\pi_1(X)$ realized by $t \mapsto \exp(2i\pi t)$.
- Let ξ a character such that $\xi(x) = \exp(-2i\pi\alpha)$, for some $\alpha \in \mathbb{C}$
- Take a trivial bundle on $\mathbb{C}$ with section σ such that $\nabla(\tau) = 0$ for $\tau : z \mapsto z^{-\alpha}\sigma$ multivalued section.
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Definition

The Deligne extension is defined as follows: the real part of the residues are in $[0, 1)$.

Deligne's theory III

Theorem

1. $H^1(X;\underline{\mathbb{C}}_\xi)=H^{\mathcal{O}}_\xi\oplus H^{\bar{\mathcal{O}}}_\xi$.
2. *If the residues are not positive integers then,* $H^{\mathcal{O}}_\xi$ *is the homology of*

$$H^0(\bar{X};\bar{L}_\xi)\xrightarrow{\bar{\nabla}}H^0(\bar{X};\Omega^1_{\bar{X}}(\log\mathcal{D})\otimes\bar{L}_\xi)\xrightarrow{\bar{\nabla}}H^0(\bar{X};\Omega^2_{\bar{X}}(\log\mathcal{D})\otimes\bar{L}_\xi)$$

and $H^{\bar{\mathcal{O}}}_\xi$ *is* $\ker\left(\bar{\nabla}:H^1(\bar{X};\bar{L}_\xi)\to H^1(\bar{X};\Omega^1_{\bar{X}}(\log\mathcal{D})\otimes\bar{L}_\xi\right)$.

3. ξ *is unitary,* $\tilde{L}_\xi$ *is the Deligne extension:*

$$H^{\mathcal{O}}_\xi=H^0(\bar{X};\Omega^1_{\bar{X}}(\log\mathcal{D})\otimes\tilde{L}_\xi),\quad H^{\bar{\mathcal{O}}}_\xi=H^1(\bar{X};\tilde{L}_\xi).$$

Definition of Characteristic Varieties

Definition

The k-th *characteristic variety* of X (or G) is the subvariety of $\mathbb{T}_G$, defined by $\mathcal{V}_k(G) := \{\xi \in \mathbb{T}_G \mid \dim H^1(X, \underline{\mathbb{C}}_\xi) \geq k\}$.

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- $\mathcal{V}_k$ is the set of zeros of the k^{th}-Fitting ideal of the Alexander invariant (except possibly at $\mathbf{1}$).
- If ξ is torsion then its depth (the maximal k such that $\xi \in \mathcal{V}_k$) is related with jumpings of Betti numbers of finite abelian coverings.

Computations

Hypersurface case: Libgober approach

- Libgober gives a method to compute *most* irreducible components of $\mathcal{V}_k(\mathbb{P}^2 \setminus C)$ without computing the fundamental group.
- Using Sakuma's formula the problem is reduced to compute equivariant Betti numbers of finite abelian coverings of $X := \mathbb{P}^2 \setminus C$:

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  - Let σ a torsion element of T_G of order $\ell > 1$
  - ρ is associated to a cyclic ℓ-fold covering $\rho_\sigma : Y_\sigma \to X$
  - There is a natural decomposition of $H^1(Y_\sigma;\mathbb{C}) = \oplus_{j=0}^{\ell-1} H_{\sigma^j}$
  - $\sigma \in \mathcal{V}_k(G) \Leftrightarrow \dim H_\sigma \geq k$.

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- Using quasiadjunction polytopes and ideals for the singular points, one obtains a finite combinatorial partition; it is enough to compute the twisted cohomology for one character in each partition.
- The *position* of the quasiadjunction ideals on each point is the key point in the computation.

Comments

Epimorphisms

If $G_1 \to G_2$ is an epimorphism, then $\mathbb{T}_{G_2}$ is a subtorus of $\mathbb{T}_{G_1}$ and $\mathcal{V}_k(G_2) \subset \mathcal{V}_k(G_1)$.

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Theorem (Arapura)

Let Σ be an irreducible component of $\mathcal{V}_1(G)$. Then,

1. *If $\dim \Sigma > 0$ then there exists a surjective morphism $\rho : X \to C$, C algebraic curve, and a torsion element σ such that $\Sigma = \sigma\rho^*(H^1(C;\mathbb{C}^*))$.*
2. *If $\dim \Sigma = 0$ then Σ is unitary.*

In particular, positive dimensional irreducible components are subtori translated by torsion elements.

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Uncovered Components

- Isolated unitary non torsion points are not visible with Libgober's method.
- Following Deligne, if ξ is torsion then, for the Deligne extension $H_\xi^{\mathcal{O}} = H^0(\bar{X}; \Omega^1_{\bar{X}}(\log \mathcal{D}) \otimes \tilde{L}_\xi), \quad H_\xi^{\bar{\mathcal{O}}} = H^1(\bar{X}; \tilde{L}_\xi)$.
- Libgober proves that if ξ ramifies along any irreducible component of C then $H^0(\bar{X}; \Omega^1_{\bar{X}}(\log \mathcal{D}) \otimes \tilde{L}_\xi) = H^0(\bar{X}; \Omega^1_{\bar{X}} \otimes \tilde{L}_\xi)$.

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Consequence

Behaviour of torsion characters determine characteristic varieties.

Orbifold groups

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For an orbifold X_φ, let $p_1, \ldots, p_n$ the points such that $\varphi(p_j) := m_j > 1$. Then

$$\pi_1^{\text{orb}} := \pi_1(X \setminus \{p_1, \ldots, p_n\}) / \langle \mu_j^{m_j} = 1 \rangle$$

where μ_j is a meridian of p_j. We denote X_φ by $X_{m_1,\ldots,m_n}$.

Orbifold maps

Definition

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Remark

If $\bar{X}$ is rational only morphisms on rational orbifolds are allowed.

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- For projective orbifolds of genus g, then the component through $\mathbf{1}$ has depth $2g-2$. The other components are in $\mathcal{V}_{2g-1}$ (at least).

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- If Σ has depth k and $\mathbf{1} \notin \Sigma$ then, $\dim \Sigma \leq k+1$ ($\leq k$ if $\dim \Sigma$ odd). Moreover $\operatorname{Sh}\Sigma$ has depth $\dim \Sigma - 2$ or $\dim \Sigma - 1$ (depth $\dim \Sigma - 1$ if $\dim \Sigma$ odd).

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- If $\mathbf{1} \notin \Sigma$ and $\dim \Sigma = 1$, then its shadow is not an irreducible component of $\mathcal{V}_1(G)$.
- Let Σ_1 be an irreducible component of $\mathcal{V}_k(G)$ and let Σ_2 be an irreducible component of $\mathcal{V}_\ell(G)$. If $\xi \in \Sigma_1 \cap \Sigma_2$ then it is a torsion point and $\xi \in \mathcal{V}_{k+\ell}(G)$.

An Artin group

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Let $G := \langle x, y, z | [x, y] = 1, (yz)^2 = (zy)^2, (xz)^3 = (zx)^3 \rangle$; $\mathcal{V}_2(G) = \emptyset$ and $\mathcal{V}_1(G)$ has 5 irreducible components of dimension 1 Σ_i such that $\Sigma_i \cap \Sigma_{i+1}$ consists of one point (of torsion type). Then G is not quasiprojective.

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Theorem

Let $G_{p,q,r}$ the Artin group associated to a triangle with sides p, q, r

- *If $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \geq 1$ then there exists an affine curve $C_{p,q,r}$ such that $G_{p,q,r} = \pi_1(\mathbb{C}^2 \setminus C_{p,q,r})$*
- *If p, q, r are even, not all of them equal and $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$ then the groups $G_{p,q,r}$ are not quasiprojective.*

Thanks for your attention