Cyclotomic Zariski tuples with abelian fundamental group. From the pair to the complement

Enrique ARTAL BARTOLO

Departamento de Matemáticas
Facultad de Ciencias
Instituto Universitario de Matemáticas y sus Aplicaciones
Universidad de Zaragoza

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Joint work with J.I. Cogolludo and J. Martín

Realization space I

$\mathscr{M}^{2d}_{3(d,d+1)}$

Set of projective plane curves in $\mathbb{P}^2 := \mathbb{CP}^2$ of degree d with three singular types of topological type as $v^d - u^{d+1} = 0$.

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- Replace $\mathcal{C}$ by $\Phi(\mathcal{C})$: $\operatorname{Sing}(\mathcal{C}) = \{[1:0:0], [0:1:0], [0:0:1]\}$.

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Diagonal automorphisms

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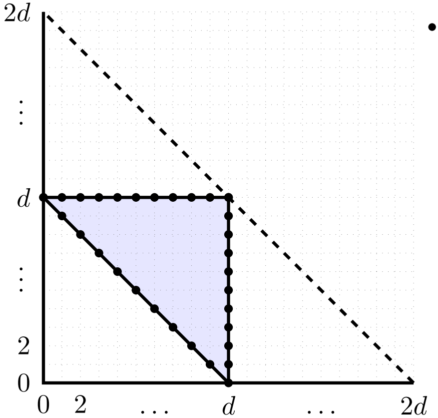

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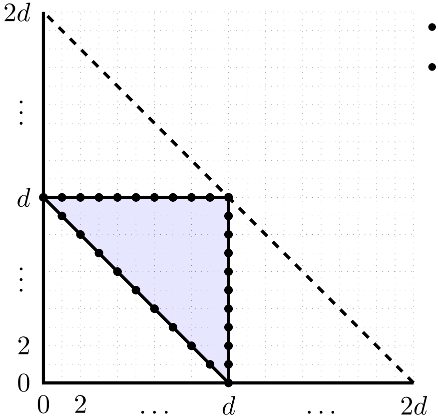

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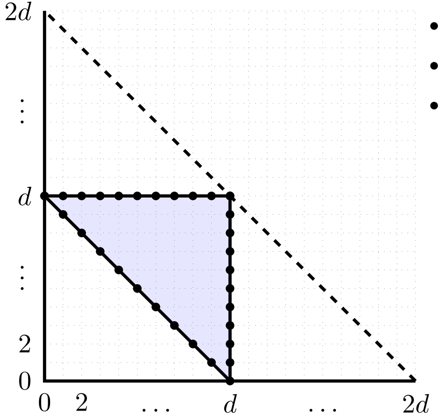

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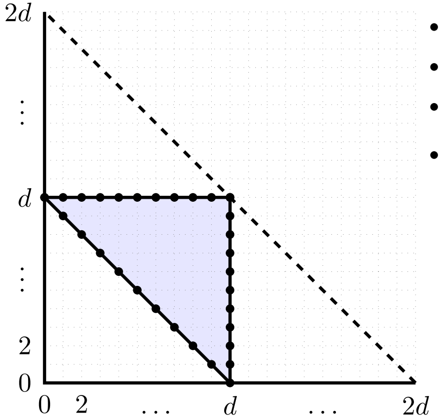

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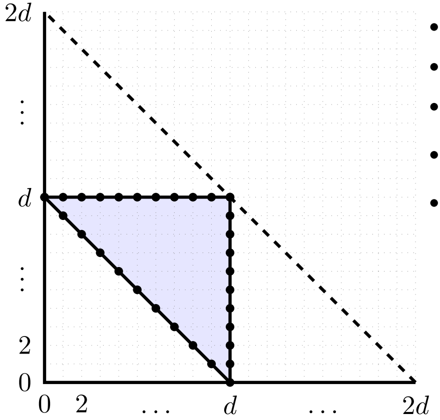

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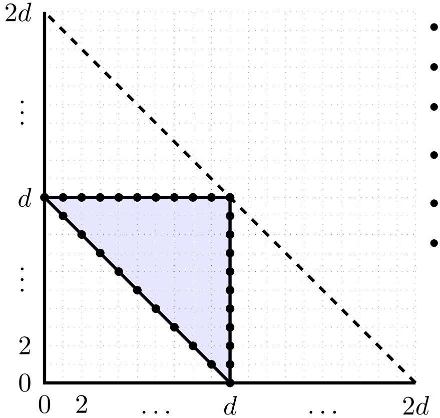

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- $Z^d(Y+\omega Z)^d, \omega^d = 1$.

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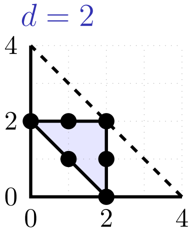
Decomposition via roots of unity

$\mathscr{M}^{2d}_{3(d,d+1)}$ decomposes in $\left\lfloor \frac{d}{2} \right\rfloor + 1$ subsets $\mathscr{M}^{2d}_{3(d,d+1)}(\omega)$ parametrized by the sets $\{\omega, \omega^{-1}\}$, when $\omega^d = 1$.

Small degrees



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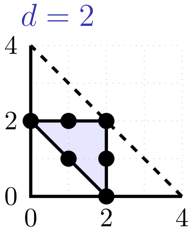
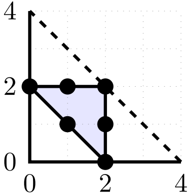


$\boxed{\omega = 1}$ $(YZ + XZ + XY)^2$, $\nexists\mathcal{C}$

Small degrees

$d = 2$



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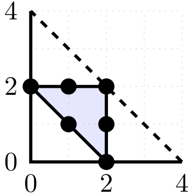

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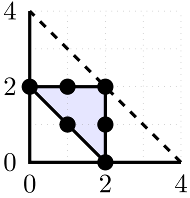

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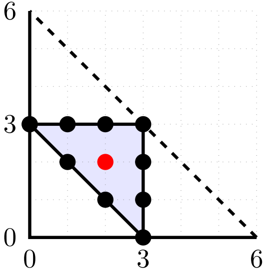

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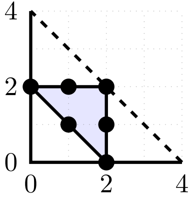

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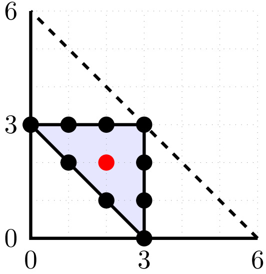

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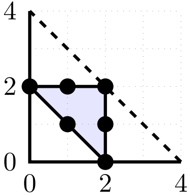

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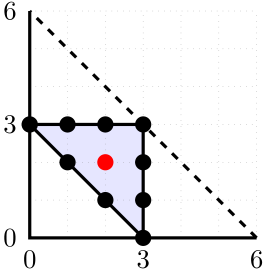

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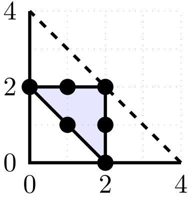

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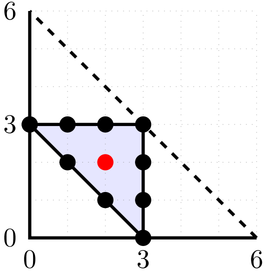

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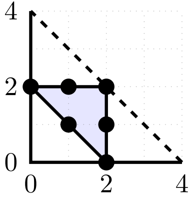

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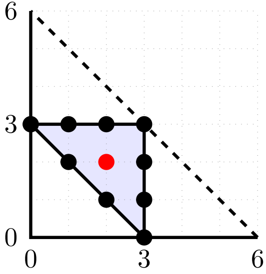

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Theorem

If $d \geq 3$, $\mathscr{M}^{2d}_{3(d,d+1)}$ *has* $\left\lfloor \frac{d}{2} \right\rfloor + 1$ *connected components parametrized by the sets* $\{\omega, \omega^{-1}\}$, *when* $\omega^d = 1$.

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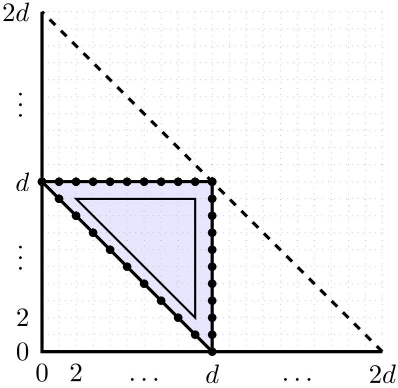
Proof.



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- $\mathcal{L}$ line passing through two singular points

Realization space

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If $d \geq 3$, $\mathscr{M}^{2d}_{3(d,d+1)}$ *has* $\left\lfloor \frac{d}{2} \right\rfloor + 1$ *connected components parametrized by the sets* $\{\omega, \omega^{-1}\}$, *when* $\omega^d = 1$.

$\mathscr{M}^{2d+3}_{3\langle uv((u+v)^d+v^{d+1})\rangle}$

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Theorem

If $d \geq 3$, $\mathscr{M}^{2d+3}_{3\langle uv((u+v)^d+v^{d+1})\rangle}$ *has* $\left\lfloor \frac{d-1}{2} \right\rfloor + 1$ *connected components parametrized by* $\{\{\omega, \omega^{-1}\} \mid \omega^d = 1\}$

Shirane curves

Cremona transformation

$\mathscr{M}_{(d),(d),(d)}$ is the space of of curves $\mathcal{E}$ of degree $d+3$ with four irreducible components: a smooth curve $\mathcal{S}$ of degree d and three non-concurrent lines $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ such that $\mathcal{S} \cap \mathcal{L}_i$ has one point.

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Definition

A *Shirane curve* $\mathcal{T}$ of type $((a_1, \ldots, a_r), (b_1, \ldots, b_s), (c_1, \ldots, c_t))$, $\sum a_i = \sum b_i = \sum c_i = d$ is formed by a smooth curve $\mathcal{S}$ of degree d and three lines $\mathcal{L}_a, \mathcal{L}_b, \mathcal{L}_c$ such that:

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- $\mathcal{S} \cap \mathcal{L}_b = \{Q_1, \dots, Q_s\}$, $(\mathcal{S} \cdot \mathcal{L}_b)_{Q_i} = b_i$

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- $\mathcal{S} \cap \mathcal{L}_c = \{R_1, \ldots, R_t\}$, $(\mathcal{S} \cdot \mathcal{L}_c)_{R_i} = c_i$

Shirane curves and coverings

Theorem (Shirane)

$SH_{((a_1,\dots,a_r),(b_1,\dots,b_s),(c_1,\dots,c_t))}$ has $\left\lfloor \frac{m}{2} \right\rfloor + 1$ or m components having pairwise distinct topological embeddings in $\mathbb{P}^2$, $m = \gcd(a_i, b_j, c_k)$.

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- $\mathcal{S} = \{G_\omega = 0\}$, $\mathcal{L}_a = \{X = 0\}$, $\mathcal{L}_b = \{Y = 0\}$ and $\mathcal{L}_c = \{Z = 0\}$.

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- $\rho_\omega^{-1}(\mathcal{L}_b) = \bigcup_{\zeta^d=1} \mathcal{L}_b^\zeta$, $\mathcal{L}_b^\zeta = \{Y = 0, T = \zeta(X+Z)\}$

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  $\omega^{\pm 1} \rightsquigarrow$ an invariant of ρ_ω restricted to $\{XYZG_\omega(X,Y,Z) \neq 0\}$

Shirane curves and coverings

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Proof for type $((d),(d),(d))$.

$$\begin{tikzcd}
(X_{\omega_1}, \mathcal{U}_{\omega_1}) \arrow[d, "\rho_{\omega_1}"] & (X_{\omega_2}, \mathcal{U}_{\omega_2}) \arrow[d, "\rho_{\omega_2}"] \\
(\mathbb{P}^2, \mathcal{T}_{\omega_1}) \arrow[r, "\cong", "\Phi"'] & (\mathbb{P}^2, \mathcal{T}_{\omega_2})
\end{tikzcd}$$

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$$\begin{tikzcd}
H_1(\mathbb{P}^2 \setminus \mathcal{T}_{\omega_1}; \mathbb{Z}) \arrow[d, "\sigma_{\omega_1}"] & (X_{\omega_1}, \mathcal{U}_{\omega_1}) \arrow[d, "\rho_{\omega_1}"] & (X_{\omega_2}, \mathcal{U}_{\omega_2}) \arrow[d, "\rho_{\omega_2}"] & H_1(\mathbb{P}^2 \setminus \mathcal{T}_{\omega_2}; \mathbb{Z}) \arrow[d, "\sigma_{\omega_2}"] \\
\mathbb{Z}/d\mathbb{Z} & (\mathbb{P}^2, \mathcal{T}_{\omega_1}) \arrow[r, "\cong", "\Phi"'] & (\mathbb{P}^2, \mathcal{T}_{\omega_2}) & \mathbb{Z}/d\mathbb{Z}
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\end{tikzcd}$$

$$\mu_{\mathcal{S}_{\omega_1}} \xmapsto{\sigma_{\omega_1}} 1 \bmod d \qquad \mu_{\mathcal{L}^{\omega_1}_{\bullet}} \xmapsto{\sigma_{\omega_1}} 0 \bmod d$$

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$$\mu_{\mathcal{S}_{\omega_2}} \xmapsto{\sigma_{\omega_2}} 1 \bmod d \qquad \mu_{\mathcal{L}_\bullet^{\omega_2}} \xmapsto{\sigma_{\omega_2}} 0 \bmod d$$

Shirane curves and coverings

Theorem (Shirane)

$SH_{((a_1,\dots,a_r),(b_1,\dots,b_s),(c_1,\dots,c_t))}$ has $\left\lfloor \frac{m}{2} \right\rfloor + 1$ or m components having pairwise distinct topological embeddings in $\mathbb{P}^2$, $m = \gcd(a_i, b_j, c_k)$.

Proof for type $((d), (d), (d))$.

$$\begin{tikzcd}
H_1(\mathbb{P}^2 \setminus \mathcal{T}_{\omega_1}; \mathbb{Z}) \arrow[d, "\sigma_{\omega_1}"] & (X_{\omega_1}, \mathcal{U}_{\omega_1}) \arrow[d, "\rho_{\omega_1}"] & (X_{\omega_2}, \mathcal{U}_{\omega_2}) \arrow[d, "\rho_{\omega_2}"] & H_1(\mathbb{P}^2 \setminus \mathcal{T}_{\omega_2}; \mathbb{Z}) \arrow[d, "\sigma_{\omega_2}"] \\
\mathbb{Z}/d\mathbb{Z} & (\mathbb{P}^2, \mathcal{T}_{\omega_1}) \arrow[r, "\cong", "\Phi"'] & (\mathbb{P}^2, \mathcal{T}_{\omega_2}) & \mathbb{Z}/d\mathbb{Z}
\end{tikzcd}$$

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\mu_{\mathcal{S}_{\omega_1}} \xmapsto{\sigma_{\omega_1}} 1 \bmod d & \qquad & \mu_{\mathcal{L}_\bullet^{\omega_1}} \xmapsto{\sigma_{\omega_1}} 0 \bmod d \\
\mu_{\mathcal{S}_{\omega_2}} \xmapsto{\sigma_{\omega_2}} 1 \bmod d & \qquad & \mu_{\mathcal{L}_\bullet^{\omega_2}} \xmapsto{\sigma_{\omega_2}} 0 \bmod d \\
\mu_{\mathcal{S}_{\omega_1}} \xmapsto{\Phi_*} \mu_{\mathcal{S}_{\omega_2}}^{\pm 1} & \qquad & \mu_{\mathcal{L}_\bullet^{\omega_1}} \xmapsto{\Phi_*} \mu_{\mathcal{L}_\bullet^{\omega_2}}^{\pm 1}
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H_1(\mathbb{P}^2 \setminus \mathcal{T}_{\omega_1}; \mathbb{Z}) \arrow[d, "\sigma_{\omega_1}"] & (X_{\omega_1}, \mathcal{U}_{\omega_1}) \arrow[r, dashed, "\tilde{\Phi}?"] \arrow[d, "\rho_{\omega_1}"] & (X_{\omega_2}, \mathcal{U}_{\omega_2}) \arrow[d, "\rho_{\omega_2}"] & H_1(\mathbb{P}^2 \setminus \mathcal{T}_{\omega_2}; \mathbb{Z}) \arrow[d, "\sigma_{\omega_2}"] \\
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\end{array}$$

Relative position of $\mathcal{L}_\bullet^\zeta \Longrightarrow \omega_2 = \omega_1^{\pm 1}$ □

Fundamental group

Examples

- $\mathcal{C}_1 \in \mathscr{M}^4_{3\mathbb{A}_2}$, the triscuspidal quartic:
  $\pi_1(\mathbb{P}^2 \setminus \mathcal{C})$ non-abelian of order 12 (Zariski)

Fundamental group

Examples

- $\mathcal{T}_1 \in \mathscr{M}_{((2),(2),(2))}$, smooth conic with three tangents:
  $\pi_1(\mathbb{P}^2 \setminus \mathcal{C})$ Artin group of the triangle $(2, 4, 4)$ (non-abelian)

Fundamental group

Examples

- $\mathcal{C}_1 \in \mathscr{M}^4_{3\mathbb{A}_2}$, the triscuspidal quartic: $\pi_1(\mathbb{P}^2 \setminus \mathcal{C})$ non-abelian of order 12 (Zariski)
- $\mathcal{C}_1 \in \mathscr{M}^4_{3\mathbb{A}_2}$, sextic with three $\mathbb{E}_6$ points tangent to a conic: $\pi_1(\mathbb{P}^2 \setminus \mathcal{C}) \cong \mathbb{Z}/2 * \mathbb{Z}/3$ (A-Carmona)

Fundamental group

Examples

- $\mathcal{C}_1 \in \mathscr{M}^4_{3\mathbb{A}_2}$, the triscuspidal quartic:
  $\pi_1(\mathbb{P}^2 \setminus \mathcal{C})$ non-abelian of order 12 (Zariski)
- $\mathcal{T}_1 \in \mathscr{M}_{((3),(3),(3))}$, smooth cubic with three tangents at aligned inflection points:
  $\pi_1(\mathbb{P}^2 \setminus \mathcal{C})$ non-abelian

Fundamental group

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- $\mathcal{C}_\zeta \in \mathscr{M}^4_{3\mathbb{A}_2}$, sextic with three $\mathbb{E}_6$ points non-tangent to a conic: $\pi_1(\mathbb{P}^2 \setminus \mathcal{C}) \cong \mathbb{Z}/2 \times \mathbb{Z}/3$ (A-Carmona)

Fundamental group

Examples

- $\mathcal{C}_1 \in \mathscr{M}^4_{3\mathbb{A}_2}$, the triscuspidal quartic: $\pi_1(\mathbb{P}^2 \setminus \mathcal{C})$ non-abelian of order 12 (Zariski)
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  $\pi_1(\mathbb{P}^2 \setminus \mathcal{C})$ abelian

Fundamental group

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Question

How many such groups are abelian?

Fundamental group

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Question

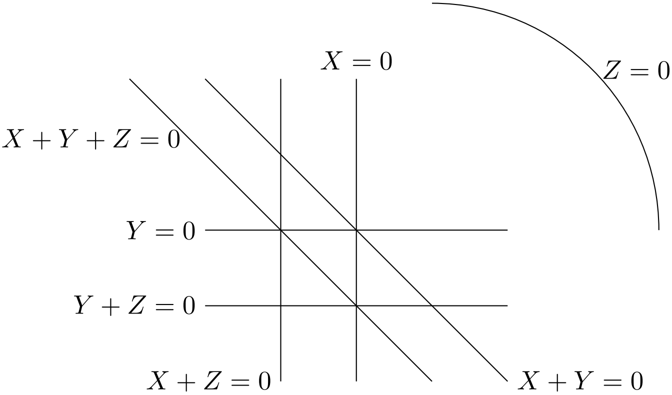
How many such groups are abelian?

Strategy

Study a curve with plenty of extremal flexes: Fermat curves $X^d + Y^d + Z^d = 0$.

$$\text{Tangent lines: } (X^d + Y^d)(Y^d + Z^d)(Z^d + X^d) = 0$$

An arrangement of lines



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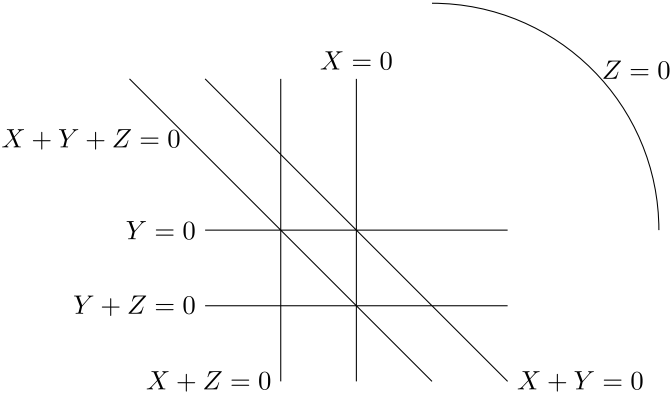

$$1 = [\mu_{xz}, \mu_{xy}] = [\mu_{xz}, \mu_{xyz}, \mu_y] = [\mu_{xz}, \mu_{yz}]$$

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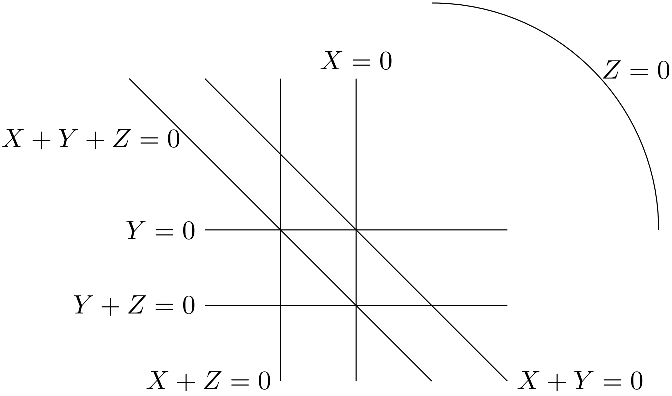

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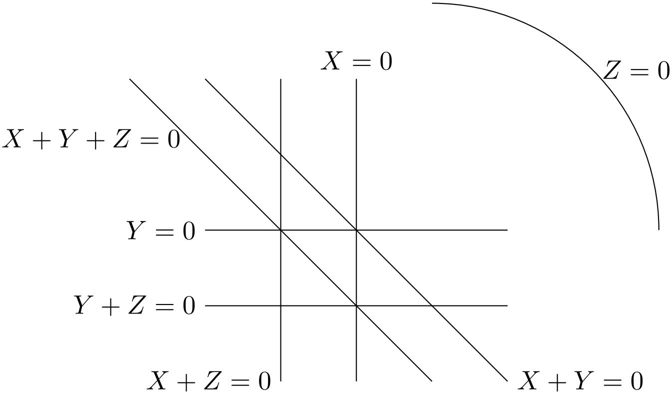

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$$1 = \mu_{yz}\mu_{xyz}\mu_y\mu_{xy}\mu_{xz}\mu_x\mu_z = \mu_x^d = \mu_y^d = \mu_z^d$$

Fundamental group of triangular curves

Theorem

Let $\mathcal{T}_\omega \in \mathscr{M}_{((d),(d),(d))}$. *If either* $d > 3$ *or* $(d, \omega) = (3, \zeta)$, *then* $\pi_1(\mathbb{P}^2 \setminus \mathcal{T}_\omega)$ *is abelian.*

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Proof.

$\square$

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Proof.

- Consider the orbifold fundamental group

$$\frac{\pi_1(\mathbb{P}^2 \setminus \{(X+Y+Z)(X+Y)(Y+Z)(Z+X)XYZ = 0\})}{\langle \mu_x^d, \mu_y^d, \mu_z^d \rangle}$$

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- Kummer cover $\pi_d : \mathbb{P}^2 \to \mathbb{P}^2$, $[X:Y:Z] \mapsto [X^d : Y^d : Z^d]$

☐

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☐

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☐

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Kill meridians to obtain the result ☐

Fundamental group of Shirane curves

Theorem (Classic)

X projective surface, $A, B \subset X$ with no common irreducible components, $B = \bigcup_j B_j$. Then, $\pi_1(X \setminus A) \cong \pi_1(X \setminus (A \cup B))/\langle \mu_{B_j} \rangle$, μ_{B_j} meridians.

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Corollary

Let $\mathcal{D}_\omega \in \mathscr{M}^{2d+3}_{3\langle uv((u+v)^d+v^{d+1})\rangle}$. If either $d > 3$ or $(d, \omega) = (3, \zeta)$, then $\pi_1(\mathbb{P}^2 \setminus \mathcal{D}_\omega)$ is abelian.

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Theorem (Zariski,Dimca)

$\{\mathcal{C}_t\}_{t\in[0,1]}$ family of projective plane curves, equisingular for $t \in (0, 1]$) with $\mathcal{C}_1$ reduced. Then $\exists \pi_1(\mathbb{P}^2 \setminus \mathcal{C}_0) \twoheadrightarrow \pi_1(\mathbb{P}^2 \setminus \mathcal{C}_1)$.

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Corollary (Degeneration of $((d),(d),(d))$ curves)

All Shirane curves have abelian fundamental group except from $((2),(2),(2))$ and $((3),(3),(3))$ (with aligned intersection points).

Homeomorphisms of complements

Theorem

$SH_{((a_1,\ldots,a_r),(b_1,\ldots,b_s),(c_1,\ldots,c_t))}$ has m or $\left\lfloor \frac{m}{2} \right\rfloor + 1$ components having pairwise distinct topological complements in $\mathbb{P}^2$, $m = \gcd(a_i, b_j, c_k)$.

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Proof of Case $((d),(d),(d))$.

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- $W_\omega := \mathbb{P}^2 \setminus \mathcal{T}_\omega$

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- $W_\omega := \mathbb{P}^2 \setminus \mathcal{T}_\omega$
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- $M_\omega := \partial U_\omega = \partial E_\omega$.

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- Covering monodromy:

$$\begin{tikzcd}
\pi_1(M_\omega) \arrow[r, two heads] \arrow[dd, "i_*"] \arrow[rrd, bend left, "\tilde{\sigma}_\omega"] & H_1(M_\omega;\mathbb{Z}) \arrow[dd, "i_*"] \arrow[rd] & \\
 & & \mu_d \\
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\end{tikzcd}$$

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- $\mu_{\mathcal{S}}$ and μ_{cycle} determined by homemorphism type of M_ω (combinatorics!) and σ_ω. Waldhausen's classification of graph manifolds

Homeomorphisms of complements

Theorem

$SH_{((a_1,\ldots,a_r),(b_1,\ldots,b_s),(c_1,\ldots,c_t))}$ has m or $\left\lfloor \frac{m}{2} \right\rfloor + 1$ components having pairwise distinct topological complements in $\mathbb{P}^2$, $m = \gcd(a_i, b_j, c_k)$.

Proof of Case $((d),(d),(d))$.

- $W_\omega := \mathbb{P}^2 \setminus \mathcal{T}_\omega$
- U_ω closed regular neighbourhood of $\mathcal{T}_\omega$, $E_\omega := \mathbb{P}^2 \setminus \mathring{U}_\omega$, $\mathring{E}_\omega \cong W_\omega$
- $M_\omega := \partial U_\omega = \partial E_\omega$.
- Covering monodromy:

$$\begin{tikzcd}
\pi_1(M_\omega) \arrow[r, two heads] \arrow[d, "i_*"] \arrow[rr, bend left, "\tilde{\sigma}_\omega"] & H_1(M_\omega;\mathbb{Z}) \arrow[d, "i_*"] \arrow[r] & \mu_d \\
\pi_1(E_\omega) \arrow[r, two heads] \arrow[rru, bend right, "\sigma_\omega"'] & H_1(E_\omega;\mathbb{Z}) \arrow[ru] &
\end{tikzcd}$$

- $\mu_{\mathcal{S}}$ and μ_{cycle} determined by homemorphism type of M_ω (combinatorics!) and σ_ω. Waldhausen's classification of graph manifolds
- $\tilde{\sigma}_d(\mu_{\mathcal{S}}) = \exp\left(\frac{2\pi i}{d}\right)$, $\tilde{\sigma}_d(\mu_{\text{cycle}}) = \omega$ $\square$

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- Wrong! Bad behavior of regular neighborhoods under homeomorphisms

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- Lines essential to distinguish components in $\mathscr{M}^{2d+3}_{3\langle uv((u+v)^d + v^{d+1})\rangle}$

Thank you