# Topology of complex algebraic varieties

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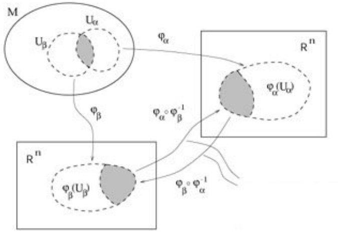


Algebraic Geometry: Equations

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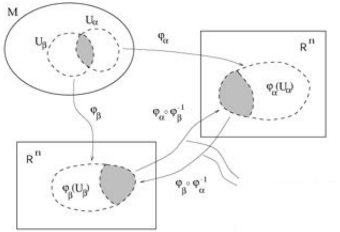


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Any compact differentiable manifold can be described by polynomial equations

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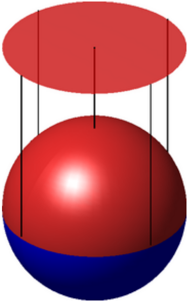

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Any compact differentiable manifold can be described by polynomial equations

- $x^2+y^2+z^2=1$
- $(x^2+y^2+z^2+R^2-r^2)^2=4R^2(x^2+y^2)$
- $x^2+y^2=1, z^2+t^2=1$

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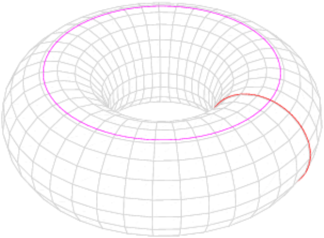

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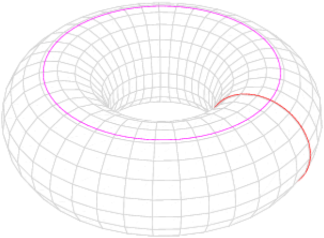

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- Complex manifolds are oriented even-dimensional manifolds
- Maximum principle $\Longrightarrow$ Compact submanifolds of $\mathbb{C}^N$ are finite sets

# Projective spaces

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- Smooth hypersurfaces: $r = 1$, $\frac{\partial f_1}{\partial x_j}(\mathbf{x}) = 0 \forall j \Leftrightarrow \mathbf{x} = 0$.
- $\mathbb{P}^N$ is the disjoint union of $\mathbb{C}^N$ ($x_0 \neq 0$) and $\mathbb{P}^{N-1}$ ($x_0 = 0$, *hyperplane at infinity*)

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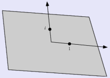

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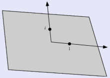

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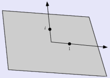

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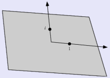

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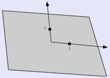

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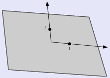

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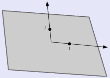

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- Kodaira's classification
- Important source of examples of compact oriented manifolds of dimension 4
- Simple invariants for the topological classification in the simply-connected case (Freedman)
- Donaldson invariants to distinguish homeomorphic but not diffeomorphic 4-manifolds are applied to complex surfaces

# Lefschetz hyperplane sections

**Theorem**

*Let $X^n$ be a (quasi)projective submanifold of $\mathbb{P}^N$ and let $H$ be a generic hyperplane. Then $X \cap H \hookrightarrow X$ induces isomorphisms in $\pi_j$, $0 \leq j < n-1$ and epimorphisms in $\pi_{n-1}$.*

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**Definition**

$\pi_0 \equiv \{\text{set of connected components}\}$

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**Definition**

$\pi_1 \equiv \{$homotopy classes of based maps $\mathbb{S}^1 \to X\}$. Any finitely presented group is the fundamental group of a compact space. Computation: Seifert-van Kampen theorem.

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**Definition**

$\pi_j \equiv \{$homotopy classes of based maps $\mathbb{S}^j \to X\}, j \leq 2$. Abelian Groups. Extremely difficult computation.

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## Example

$\pi_1(\mathbb{S}^1) \cong \mathbb{Z}$ given by $\tilde{\gamma}(1)$, $\pi_j(\mathbb{S}^1) = 0$ if $j > 1$.

$$\begin{array}{ccc} [0,1] & \xrightarrow{\tilde{\gamma}} & \mathbb{R} \\ e^{2i\pi\bullet} \downarrow & & \downarrow e^{2i\pi\bullet} \\ \mathbb{S}^1 & \xrightarrow{\gamma} & \mathbb{S}^1 \end{array}$$

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**Example**

$\pi_1(\mathbb{S}^2) \cong 0$, $\pi_2(\mathbb{S}^2) \cong \mathbb{Z}$.

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**Example**

$\pi_1(\mathbb{S}^1 \wedge \mathbb{S}^2) \cong \mathbb{Z}$, $\pi_2(\mathbb{S}^1 \wedge \mathbb{S}^2) \cong \mathbb{Z}^{\mathbb{Z}}$.

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## Consequence

Some geometric properties of $X$ can be studied in a hyperplane section. In order to understand $\pi_1$, it is enough to study complex surfaces.

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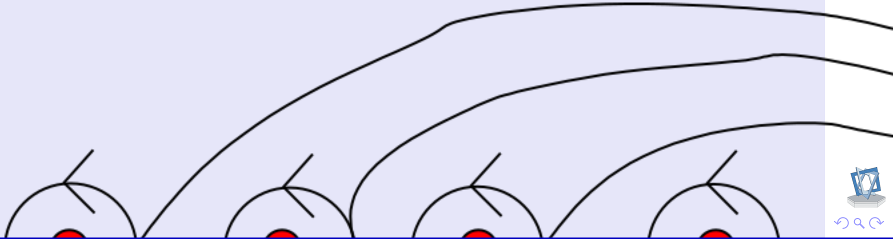
**Complements of curves**

$X := \mathbb{P}^2 \setminus C$, $C$ projective curve of degree $d$, $L$ *generic* line $\pi_1(L \setminus C) \twoheadrightarrow \pi(X)$



# Projection techniques

## Projection

Let $X^n \subset \mathbb{P}^N$ be a submanifold, let $H^{N-n-1} \subset \mathbb{P}^N$ be generic linear subspace, let $\mathbb{P}^n \subset \mathbb{P}^N$ be a subspace not containing $H$. The projection of center $H$ onto $\mathbb{P}^n$ defines a ramified covering $\pi : X \to \mathbb{P}^n$.

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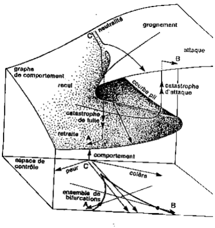

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## Example

$N = 3$, $n = 2$, $H := [0:0:1]$, $\mathbb{P}^2 = \{z = 0\}$, $X = \{f(x,y,z) = 0\}$, $f(x,y,z) = z^d + \dots$.

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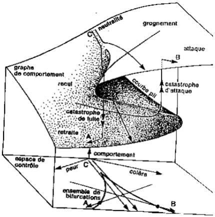


## Discriminant

There exists a hypersurface $\Delta \subset \mathbb{P}^n$ such that $\pi : X \setminus \pi^{-1}(\Delta) \to \mathbb{P}^n \setminus \Delta$ is an unramified covering. In the case of the example, $\Delta = \{\mathrm{disc}_z f = 0\}$.

# Unramified coverings

**Definition**

$\pi : X \to Y$ is an unramified covering if $\forall y \in Y$, $\exists U$ open neighbourhood of $y$ such that $\pi^{-1}(U) = \overset{\emptyset}{\bigcup}_{i\in I} U_i$ and $\pi_{|U_i} : U_i \to U$ is a homeomorphism.

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## Consequence

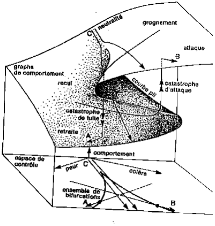
It is interesting to study $\pi_1(\mathbb{P}^2 \setminus C)$, $C$ plane projective curve.

# Singularities of plane curves

## Generic singularities of a discriminant

$g(x,y)=0$ local equation of the discriminant

- Two folding intersecting lines: $g(x,y)=g_2(x,y)+\dots=\ell_1(x,y)m_1(x,y)+\dots=xy+\dots$
- Cusp: $g(x,y)=\ell_1(x,y)^2+\dots=x^2+g_3(x,y)+\dots=x^2+y^3+\dots$, $g_3(0,y)\neq 0$



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## Local equations of simple singularities of plane curves

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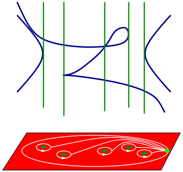

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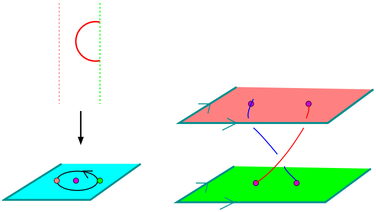

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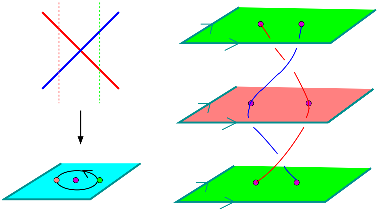

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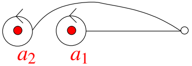
- Relations obtained by turning around on 

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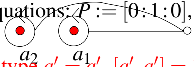


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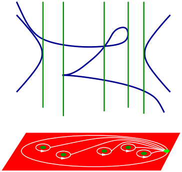

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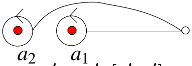
- Relations obtained by turning around on 

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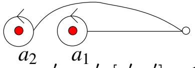


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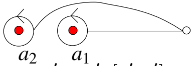
- Relations obtained by turning around on 

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It two curves are in a connected family of equisingular curves, then they are isotopic

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*On the irregularity of cyclic multiple planes*, Ann. Math. **32** (1931), 445–489:
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**Theorem (Degtyarev, 2006)**

*The family of sextics with six cusps has two connected components*