Orbifolds and line arrangements

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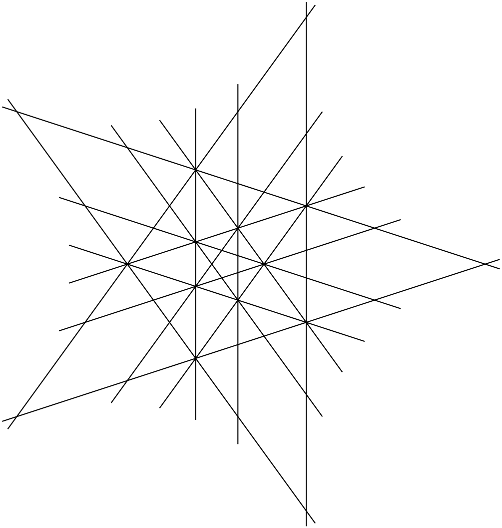
Introduction to the icosidodecahedron arrangement



- $\mathbb{P}^2(\mathbb{C})$
- Includes L_∞
- 16 lines
- Studied by Yoshinaga and his student Sugawara

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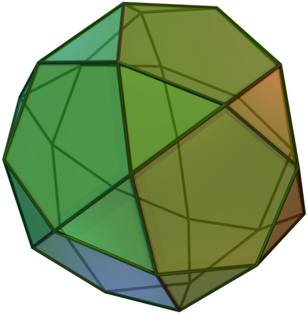

- 12 antipodal triangles
- 20 antipodal pentagones
- 16 lines joining the centers of opposite faces
- 16 points in $\mathbb{P}^2(\mathbb{R})$
- Consider dual lines

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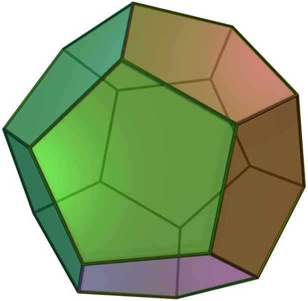

- 12 antipodal pentagones
- 20 antipodal vertices
- 16 lines joining the centers of opposite faces and vertices
- 16 points in $\mathbb{P}^2(\mathbb{R})$
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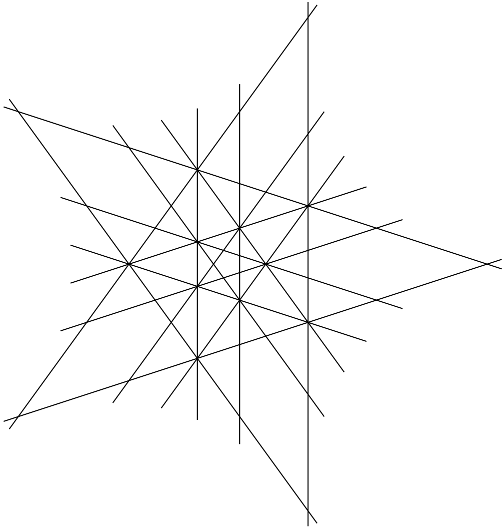

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Properties of the icosidodecahedron arrangement

- Symmetries: alternating group $\mathfrak{A}_5 \subset \mathrm{PGL}(3;\mathbb{R}) \subset \mathrm{PGL}(3;\mathbb{C})$
- 16 quadruple points
- 24 double points
- 6 lines with 5 quadruple points: $\mathfrak{A}_5$-orbit
- 10 lines with 3 quadruple points and 6 double points: $\mathfrak{A}_5$-orbit

How many $\mathfrak{A}_5$-invariant arrangements of lines?

Linear algebra problem. Eigenvectors of the matrices generated by

$$(1,\dots,5) \longmapsto \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\cos\frac{2\pi}{5} & \frac{1}{2}\cos\frac{2\pi}{5}+\frac{3}{4} \\ 0 & -1 & -\cos\frac{2\pi}{5} \end{bmatrix}$$

$$(1,2)(3,4) \longmapsto \begin{bmatrix} -\frac{4}{5}\cos\frac{2\pi}{5}-\frac{1}{5} & \frac{4}{5}\cos\frac{2\pi}{5}+\frac{1}{5} & \cos\frac{2\pi}{5} \\ \frac{6}{5}\cos\frac{2\pi}{5}+\frac{4}{5} & -\frac{1}{5}\cos\frac{2\pi}{5}+\frac{7}{10} & -\frac{1}{4} \\ \frac{8}{5}\cos\frac{2\pi}{5}+\frac{2}{5} & \frac{2}{5}\cos\frac{2\pi}{5}-\frac{2}{5} & \cos\frac{2\pi}{5}+\frac{1}{2} \end{bmatrix}$$

$$(5,3,1) \longmapsto \begin{bmatrix} -\frac{4}{5}\cos\frac{2\pi}{5}-\frac{1}{5} & -\frac{6}{5}\cos\frac{2\pi}{5}+\frac{1}{5} & \frac{1}{2} \\ \frac{6}{5}\cos\frac{2\pi}{5}+\frac{4}{5} & \frac{4}{5}\cos\frac{2\pi}{5}+\frac{1}{5} & \frac{1}{2} \\ \frac{8}{5}\cos\frac{2\pi}{5}+\frac{2}{5} & -\frac{8}{5}\cos\frac{2\pi}{5}-\frac{2}{5} & 0 \end{bmatrix}$$

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How many $\mathfrak{A}_5$-invariant arrangements of lines?

- Previous $\mathcal{A}_6$ and $\mathcal{A}_{10}$
- $\mathcal{A}_{12}$ and $\mathcal{A}_{20}$
- $\mathfrak{A}_5$ acts freely on the generic points of these arrangements.
- the isotropy group of generic points in $\mathcal{A}_{15}$: $\mathbb{Z}/2$.
- There are points in $\mathbb{P}^2$ with the following isotropy groups:
  - $\{1\}$
  - $\mathbb{Z}/n$, $n = 2, 3, 5$
  - $\mathbb{Z}/2 \times \mathbb{Z}/2$, $\mathbb{D}_3$, $\mathbb{D}_5$
- There is one invariant conic

Quotient

Questions

- What is the quotient $\mathbb{P}^2/\mathfrak{A}_5$?
- How does the image of the icosidodecahedron arrangement fit in?
- What about the isotropy groups? Think about orbifolds.

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Weighted projective planes

$$\omega := (p, q, r),\ \gcd \omega = 1, \quad \mathbb{P}^2_\omega :\ \mathbb{C}^3 \setminus \{0\}/\mathbb{C}^*$$
$$t \cdot (x, y, z) := (t^p x, t^q y, t^r z), \quad [(x, y, z)] = [x : y : z]_\omega .$$

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$$\begin{aligned} \mathbb{C}^2 &\longrightarrow \mathbb{P}^2_\omega \\ (x,y) &\longmapsto [x:y:1]_\omega \end{aligned}$$

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$$\begin{aligned} \mathbb{C}^2 &\longrightarrow \mathbb{P}^2_\omega \\ (x,y) &\longmapsto [x:y:1]_\omega \\ \left(\exp\tfrac{2i\pi p}{r}x, \exp\tfrac{2i\pi q}{r}y\right) &\longmapsto \left[\exp\tfrac{2i\pi p}{r}x : \exp\tfrac{2i\pi q}{r}y : 1\right]_\omega = [x:y:1]_\omega \end{aligned}$$

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Charts: algebraic (analytic) structure, orbifold structure

$$\begin{aligned} \mathbb{C}^2/\mu_r &\xrightarrow{\cong} \mathbb{P}^2_\omega \\ [(x,y)] &\longmapsto [x:y:1]_\omega \end{aligned}$$

$$\begin{aligned} \mu_r \times \mathbb{C}^2 &\longrightarrow \mathbb{C}^2 \\ (\zeta,(x,y)) &\longmapsto (\zeta^p x, \zeta^q y) \end{aligned} \qquad \mu_r := \{\zeta \in \mathbb{C}^* \mid \zeta^r = 1\}.$$

Complex orbifolds

Definition

X analytic variety. An orbifold structure $\mathscr{O}$ on X: *maximal atlas* $\{(U_i, \varphi_i, G_i)\}_{i \in I}$, φ_i analytic, G_i finite group acting holomorphically on U_i:

$$\begin{array}{ccc} & \mathbb{C}^n \supset U_i & \\ {}^{\varphi_i}\swarrow & & \searrow \\ X \supset \varphi_i(U_i) := V_i & \xrightarrow{\cong} & U_i/G_i \end{array}$$

such that the *charts* are *compatible*. An orbifold map $\Phi : (X; \mathscr{O}_X) \to (Y; \mathscr{O}_Y)$ is a holomorphic map $X \to Y$ together with *compatible* liftings to the *charts*.

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Remarks

- The key is not the quotient: local actions and isotropy groups.
- One can replace open subsets of $\mathbb{C}^n$ by analytic varieties
- Define also $\mathcal{C}^{(\infty)}$, topological orbifolds
- Notions of *orbifold fundamental group*, *orbifold covering*

Examples

- X holomorphic manifold, G group acting holomorphically with finite isotropy groups: orbifold structure on X/G
- Standard action of μ_n on $\mathbb{D}_r$, $0 < r \leq \infty$: an orbifold structure whose underlying variety is the disk $\mathbb{D}_{r^n}$. The orbifold will be denoted as the disk $\mathbb{D}_{r^n}$ with the origin marked with n.
- $\mathbb{P}^1$ with two points marked with n_1, n_2. If $n_1 \neq n_2$, it is not of the type X/G.
- Weighted projective planes
- $(-1) \cdot [x : y : z] := [x : y : -z] = [-x : y : z]$. Orbifold structure defined by $\mathbb{P}^2$ and μ_2.

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- $(-1) \cdot [x : y : z] := [x : y : -z] = [-x : y : z]$. Orbifold structure defined by $\mathbb{P}^2$ and μ_2.
  - Quotient structure defined by

$$\begin{aligned} \mathbb{P}^2 &\longrightarrow \mathbb{P}^2_\omega, \ \omega = (1,1,2) \\ [x : y : z] &\longmapsto [x : y : z^2]_\omega \end{aligned}$$

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  - Isotropy stratification:
    - $\{[x : y : z]_\omega \mid z \neq 0 \text{ and } (x, y) \neq (0, 0)\}$: trivial isotropy
    - $[0 : 0 : 1]_\omega$: μ_2 action on both variables
    - $\{[x : y : 0]_\omega \mid (x, y) \neq (0, 0)\}$: μ_2 action on one variable

Quotient equations

- $f_2 := \left(1 - 2\cos\frac{2\pi}{5}\right)x^2 + \frac{1}{2}\left(1 - \cos\frac{2\pi}{5}\right)y^2 + z^2$ invariant conic

$$f_6 := \left(20\cos\frac{2\pi}{5} - 6\right)x^5z + \left(20 - 70\cos\frac{2\pi}{5}\right)x^3y^2z+$$
$$\frac{25}{8}\left(4\cos\frac{2\pi}{5} - 1\right)xy^4z + \left(10 - 30\cos\frac{2\pi}{5}\right)x^4z^2+$$
$$\left(\frac{15}{2} - 20\cos\frac{2\pi}{5}\right)x^2y^2z^2 + \frac{25}{16}\left(1 - 2\cos\frac{2\pi}{5}\right)y^4z^2+$$
$$\left(10\cos\frac{2\pi}{5} - 5\right)x^2z^4 + \frac{5}{2}\left(\cos\frac{2\pi}{5} - 1\right)y^2z^4 + z^6$$

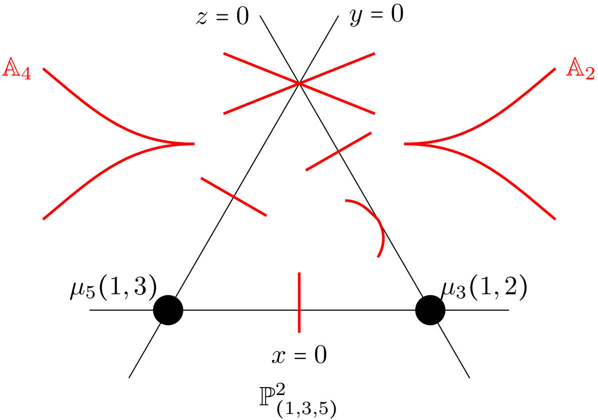
- f_{10} long equation of degree 10

$$\mathbb{P}^2 \xrightarrow{\Psi} \mathbb{P}^2_\omega, \;\; \omega = (1,3,5)$$
$$[x:y:z] \longmapsto [f_2 : f_6 : f_{10}]_\omega$$

- $\Psi(\mathcal{A}_{15})$ with equation (ω-degree 15)

$$z^3 + 65x^2yz^2 - 4x^5z^2 - 720xy^3z + 795x^4y^2z-$$
$$200x^7yz + 1728y^5 - 3440x^3y^4 + 2275x^6y^3 - 500x^9y^2 = 0$$

Orbifold stratification



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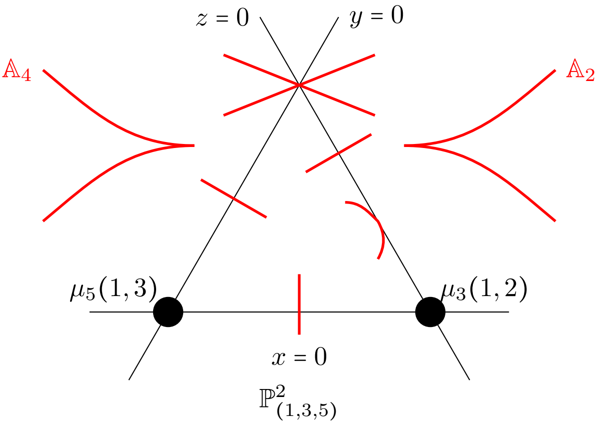
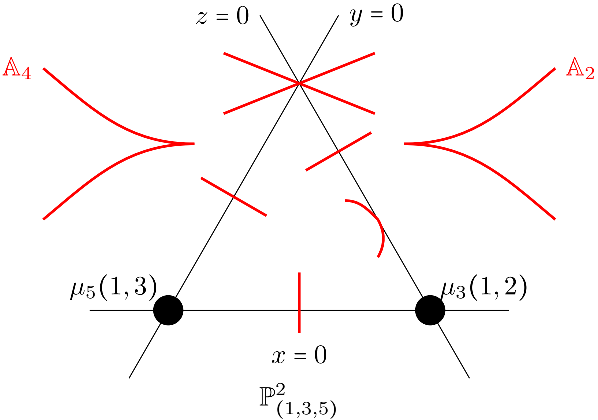


- $\mathbb{P}^2_{(1,3,5)} \setminus (\{\mu_3, \mu_5\} \cup \text{red})$: $\{1\}$
- $\mathbb{P}^2_{(1,3,5)} \setminus (\{\mu_3, \mu_5\} \cup \text{red})$: $\{1\}$
- $\mu_3 : \mathbb{Z}/3$
- $\mu_5 : \mathbb{Z}/5$
- $\text{red}^{\text{smooth}} : \mathbb{Z}/2$
- $\mathbb{A}_1 : \mathbb{Z}/2 \times \mathbb{Z}/2$
- $\mathbb{A}_3 : \mathbb{D}_3$
- $\mathbb{A}_4 : \mathbb{D}_5$

Orbifold stratification



- $\Psi(\mathcal{A}_6) = \{y = 0\}$
- $\Psi(\mathcal{A}_{10}) = \{z = 0\}$
- It is possible to compute π_1^{orb}