Character Varieties and Peripheral Polynomials

Enrique ARTAL BARTOLO

Departamento de Matemáticas
Facultad de Ciencias
Instituto Universitario de Matemáticas y sus Aplicaciones
Universidad de Zaragoza

Congreso Bienal RSME
Special Session in Honour of Maite LOZANO
Zaragoza, January 30th 2017

Character Varieties

Base reference

Hugh M. Hilden, María Teresa Lozano, José María Montesinos,
Peripheral polynomials of hyperbolic knots.
Topology Appl. **150** (2005), no. 1-3, 267–288

Character Varieties

Base reference

Hugh M. Hilden, María Teresa Lozano, José María Montesinos,
Peripheral polynomials of hyperbolic knots.
Topology Appl. **150** (2005), no. 1-3, 267–288

- G f.p. group $R(G) := \mathrm{Hom}(G, \mathrm{SL}(2; \mathbb{C}))$: $\mathbb{C}$-algebraic variety defined over $\mathbb{Q}$.

Character Varieties

Base reference

Hugh M. Hilden, María Teresa Lozano, José María Montesinos,
Peripheral polynomials of hyperbolic knots.
Topology Appl. **150** (2005), no. 1-3, 267–288

- G f.p. group $R(G) := \mathrm{Hom}(G, \mathrm{SL}(2;\mathbb{C}))$: $\mathbb{C}$-algebraic variety defined over $\mathbb{Q}$.
- $X(G) = R(G)/G$ (conjugation action) character variety (traces of characters).

Character Varieties

Base reference

Hugh M. Hilden, María Teresa Lozano, José María Montesinos,
Peripheral polynomials of hyperbolic knots.
Topology Appl. **150** (2005), no. 1-3, 267–288

- G f.p. group $R(G) := \mathrm{Hom}(G, \mathrm{SL}(2;\mathbb{C}))$: $\mathbb{C}$-algebraic variety defined over $\mathbb{Q}$.
- $X(G) = R(G)/G$ (conjugation action) character variety (traces of characters).
- $g \in G$: $t_g : R(G) \to \mathbb{C}$, $\rho \mapsto \mathrm{tr}(\rho(g))$.

Character Varieties

Base reference

Hugh M. Hilden, María Teresa Lozano, José María Montesinos,
Peripheral polynomials of hyperbolic knots.
Topology Appl. **150** (2005), no. 1-3, 267–288

- G f.p. group $R(G) := \mathrm{Hom}(G, \mathrm{SL}(2;\mathbb{C}))$: $\mathbb{C}$-algebraic variety defined over $\mathbb{Q}$.
- $X(G) = R(G)/G$ (conjugation action) character variety (traces of characters).
- $g \in G$: $t_g : R(G) \to \mathbb{C}$, $\rho \mapsto \mathrm{tr}(\rho(g))$.
- The ring generated by $\{t_g \mid g \in G\}$ is f.g.: $X(G)$ is the associated affine algebraic variety.

Character Varieties

Base reference

Hugh M. Hilden, María Teresa Lozano, José María Montesinos,
Peripheral polynomials of hyperbolic knots.
Topology Appl. **150** (2005), no. 1-3, 267–288

- G f.p. group $R(G) := \mathrm{Hom}(G, \mathrm{SL}(2;\mathbb{C}))$: $\mathbb{C}$-algebraic variety defined over $\mathbb{Q}$.
- $X(G) = R(G)/G$ (conjugation action) character variety (traces of characters).
- $g \in G$: $t_g : R(G) \to \mathbb{C}$, $\rho \mapsto \mathrm{tr}(\rho(g))$.
- The ring generated by $\{t_g \mid g \in G\}$ is f.g.: $X(G)$ is the associated affine algebraic variety.
- $g_1, \ldots, g_n$ generating system, $t_{i_1,\ldots,i_r} := t_{g_{i_1}\cdot\ldots\cdot g_{i_r}}$.

Character Varieties

Base reference

Hugh M. Hilden, María Teresa Lozano, José María Montesinos,
Peripheral polynomials of hyperbolic knots.
Topology Appl. **150** (2005), no. 1-3, 267–288

- G f.p. group $R(G) := \mathrm{Hom}(G, \mathrm{SL}(2;\mathbb{C}))$: $\mathbb{C}$-algebraic variety defined over $\mathbb{Q}$.
- $X(G) = R(G)/G$ (conjugation action) character variety (traces of characters).
- $g \in G$: $t_g : R(G) \to \mathbb{C}$, $\rho \mapsto \mathrm{tr}(\rho(g))$.
- The ring generated by $\{t_g \mid g \in G\}$ is f.g.: $X(G)$ is the associated affine algebraic variety.
- $g_1, \dots, g_n$ generating system, $t_{i_1,\dots,i_r} := t_{g_{i_1}\cdot\dots\cdot g_{i_r}}$.
- t_i, $1 \le i \le n$, $t_{i,j}$, $1 \le i < j \le n$, $t_{i,j,k}$, $1 \le i < j < k \le n$ generate this ring.

Examples and knots

Example (González-Acuña, Montesinos)

- $G = \langle m, \ell \mid [m, \ell] = 1 \rangle = \mathbb{Z}^2$

Examples and knots

Example (González-Acuña, Montesinos)

- $G = \langle m, \ell \mid [m, \ell] = 1 \rangle = \mathbb{Z}^2$
- $x = t_m$, $y = t_\ell$, $z = t_{m\ell}$, $X(G) = \{\underbrace{x^2 + y^2 + z^2 - xyz - 4}_{D(x,y,z)} = 0\}$

Examples and knots

Example (González-Acuña, Montesinos)

- $G = \langle m, \ell \mid [m, \ell] = 1 \rangle = \mathbb{Z}^2$
- $x = t_m,\ y = t_\ell,\ z = t_{m\ell},\ X(G) = \{\underbrace{x^2 + y^2 + z^2 - xyz - 4}_{D(x,y,z)} = 0\}$

Knot groups

- $K \subset \mathbb{S}^3$ knot, $G := \pi_1(\mathbb{S}^3 \setminus K) = \left\langle \underset{1 \leq i \leq n}{a_i} \,\middle|\, \underset{1 \leq h < n}{r_h} \right\rangle$,

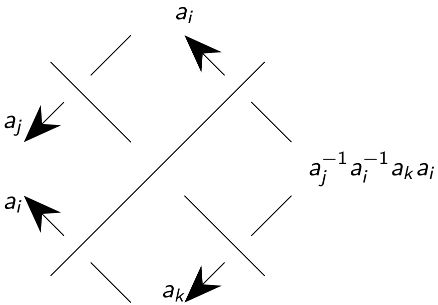
Examples and knots

Example (González-Acuña, Montesinos)

- $G = \langle m, \ell \mid [m,\ell] = 1 \rangle = \mathbb{Z}^2$
- $x = t_m,\ y = t_\ell,\ z = t_{m\ell},\ X(G) = \{\underbrace{x^2 + y^2 + z^2 - xyz - 4}_{D(x,y,z)} = 0\}$

Knot groups

- $K \subset \mathbb{S}^3$ knot, $G := \pi_1(\mathbb{S}^3 \setminus K) = \left\langle \begin{array}{c|c} a_i & r_h \\ 1\le i\le n & 1\le h<n \end{array} \right\rangle$,



Examples and knots

Example (González-Acuña, Montesinos)

- $G = \langle m, \ell \mid [m, \ell] = 1 \rangle = \mathbb{Z}^2$
- $x = t_m,\ y = t_\ell,\ z = t_{m\ell},\ X(G) = \{\underbrace{x^2 + y^2 + z^2 - xyz - 4}_{D(x,y,z)} = 0\}$

Knot groups

- $K \subset \mathbb{S}^3$ knot, $G := \pi_1(\mathbb{S}^3 \setminus K) = \left\langle \underset{1 \le i \le n}{a_i} \middle| \underset{1 \le h < n}{r_h} \right\rangle,\ r_h : a_j^{-1} a_i^{-1} a_k a_i$

Examples and knots

Example (González-Acuña, Montesinos)

- $G = \langle m, \ell \mid [m, \ell] = 1 \rangle = \mathbb{Z}^2$
- $x = t_m,\ y = t_\ell,\ z = t_{m\ell},\ X(G) = \{\underbrace{x^2 + y^2 + z^2 - xyz - 4}_{D(x,y,z)} = 0\}$

Knot groups

- $K \subset \mathbb{S}^3$ knot, $G := \pi_1(\mathbb{S}^3 \setminus K) = \left\langle \underset{1 \leq i \leq n}{a_i} \,\middle|\, \underset{1 \leq h < n}{r_h} \right\rangle$, $r_h : a_j^{-1} a_i^{-1} a_k a_i$
- $p := 1 + \frac{n(n-1)}{2} + \frac{n(n-1)(n-2)}{6}$ coordinates

Examples and knots

Example (González-Acuña, Montesinos)

- $G = \langle m, \ell \mid [m, \ell] = 1 \rangle = \mathbb{Z}^2$
- $x = t_m,\ y = t_\ell,\ z = t_{m\ell},\ X(G) = \{\underbrace{x^2 + y^2 + z^2 - xyz - 4}_{D(x,y,z)} = 0\}$

Knot groups

- $K \subset \mathbb{S}^3$ knot, $G := \pi_1(\mathbb{S}^3 \setminus K) = \left\langle \underset{1\le i\le n}{a_i} \,\middle|\, \underset{1\le h<n}{r_h} \right\rangle$, $r_h : a_j^{-1} a_i^{-1} a_k a_i$
- $p := 1 + \frac{n(n-1)}{2} + \frac{n(n-1)(n-2)}{6}$ coordinates
- Two commuting involutions: J (complex conjugation), I $(\rho(a_i) \mapsto -\rho(a_i))$

Examples and knots

Example (González-Acuña, Montesinos)

- $G = \langle m, \ell \mid [m,\ell] = 1 \rangle = \mathbb{Z}^2$
- $x = t_m,\ y = t_\ell,\ z = t_{m\ell},\ X(G) = \{\underbrace{x^2 + y^2 + z^2 - xyz - 4}_{D(x,y,z)} = 0\}$

Knot groups

- $K \subset \mathbb{S}^3$ knot, $G := \pi_1(\mathbb{S}^3 \setminus K) = \left\langle \underset{1\leq i\leq n}{a_i} \,\middle|\, \underset{1\leq h<n}{r_h} \right\rangle$, $r_h : a_j^{-1} a_i^{-1} a_k a_i$
- $p := 1 + \frac{n(n-1)}{2} + \frac{n(n-1)(n-2)}{6}$ coordinates
- Two commuting involutions: J (complex conjugation), I $(\rho(a_i) \mapsto -\rho(a_i))$
- $\tau = \partial E(K)$ (exterior of the knot), $V \subset \tau$ simple closed curve defines $t_V : X(G) \subset \mathbb{C}^p \to \mathbb{C}$.

Figure Eight Knot I

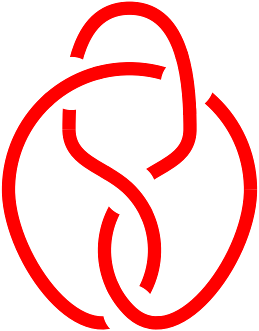
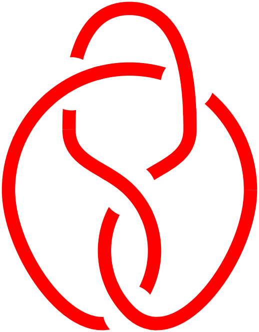
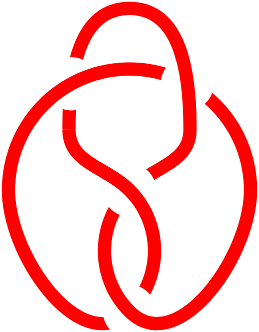


Figure Eight Knot I



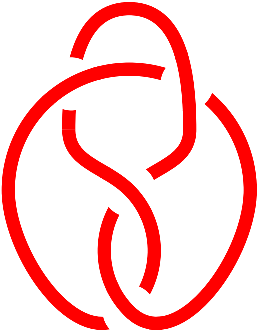
- $G = \langle g_1, g_2 \mid g_2^{g_1 \cdot g_2^{-1}} = g_1^{g_2^{-1} \cdot g_1} \rangle$

Figure Eight Knot I



- $G = \langle g_1, g_2 \mid g_2^{g_1 \cdot g_2^{-1}} = g_1^{g_2^{-1} \cdot g_1} \rangle$
- $y = t_{g_1} = t_{g_2}$, $z = t_{g_1 g_2}$,
  $X(G) = \{(1 - 2y^2 + z + y^2 z - z^2)(y^2 - z - 2) = 0\}$ (Whittemore)

Figure Eight Knot I



- $G = \langle g_1, g_2 \mid g_2^{g_1 \cdot g_2^{-1}} = g_1^{g_2^{-1} \cdot g_1} \rangle$
- $y = t_{g_1} = t_{g_2}$, $z = t_{g_1 g_2}$,
  $X(G) = \{(1 - 2y^2 + z + y^2 z - z^2)(y^2 - z - 2) = 0\}$ (Whittemore)
- Meridian $m = g_2$, canonical longitude $\ell = g_1 g_2^{-1} g_1^{-1} g_2^2 g_1^{-1} g_2^{-1} g_1$

Hyperbolic knots

- K is hyperbolic if $\mathbb{S}^3 \setminus K$ admits a finite volume complete hyperbolic metric.

Hyperbolic knots

- K is hyperbolic if $\mathbb{S}^3 \setminus K$ admits a finite volume complete hyperbolic metric.
- Holonomy: $\tilde{\rho} : G = \pi_1(\mathbb{S}^3 \setminus K) \to \mathrm{Iso}^+(\mathbb{H}^3) \equiv \mathrm{PSL}(2; \mathbb{C})$.

Hyperbolic knots

- K is hyperbolic if $\mathbb{S}^3 \setminus K$ admits a finite volume complete hyperbolic metric.
- Holonomy: $\tilde{\rho} : G = \pi_1(\mathbb{S}^3 \setminus K) \to \mathrm{Iso}^+(\mathbb{H}^3) \equiv \mathrm{PSL}(2;\mathbb{C})$.
- (González-Acuña) It lifts to $\rho : G \to \mathrm{SL}(2;\mathbb{C})$,
  $\rho, I(\rho), \bar{\rho}, I(\bar{\rho}) \in R(G)$, $[\rho], [I(\rho)], [\bar{\rho}], [I(\bar{\rho})] \in X(G)$.

Hyperbolic knots

- K is hyperbolic if $\mathbb{S}^3 \setminus K$ admits a finite volume complete hyperbolic metric.
- Holonomy: $\tilde{\rho} : G = \pi_1(\mathbb{S}^3 \setminus K) \to \mathrm{Iso}^+(\mathbb{H}^3) \equiv \mathrm{PSL}(2;\mathbb{C})$.
- (González-Acuña) It lifts to $\rho : G \to \mathrm{SL}(2;\mathbb{C})$,
$\rho, I(\rho), \bar{\rho}, I(\bar{\rho}) \in R(G)$, $[\rho], [I(\rho)], [\bar{\rho}], [I(\bar{\rho})] \in X(G)$.
- $\mathcal{K} \subset X(G)$ irreducible component such that either $[\rho], [I(\rho)], [\bar{\rho}], [I(\bar{\rho})]$ is in $\mathcal{K}$: excellent component. At most four components. It is a curve (Culler,Shalen,Thurston); $\mathcal{U}$ excellent curve is the union of excellent components.

Hyperbolic knots

- K is hyperbolic if $\mathbb{S}^3 \setminus K$ admits a finite volume complete hyperbolic metric.
- Holonomy: $\tilde{\rho} : G = \pi_1(\mathbb{S}^3 \setminus K) \to \mathrm{Iso}^+(\mathbb{H}^3) \equiv \mathrm{PSL}(2;\mathbb{C})$.
- (González-Acuña) It lifts to $\rho : G \to \mathrm{SL}(2;\mathbb{C})$, $\rho, I(\rho), \bar{\rho}, I(\bar{\rho}) \in R(G)$, $[\rho], [I(\rho)], [\bar{\rho}], [I(\bar{\rho})] \in X(G)$.
- $\mathcal{K} \subset X(G)$ irreducible component such that either $[\rho], [I(\rho)], [\bar{\rho}], [I(\bar{\rho})]$ is in $\mathcal{K}$: excellent component. At most four components. It is a curve (Culler,Shalen,Thurston); $\mathcal{U}$ excellent curve is the union of excellent components.
- $m \to \frac{1}{0} = \infty$, $\ell \to \frac{0}{1} = 0$, $V \sim pm + q\ell \to \frac{p}{q}$:

Hyperbolic knots

- K is hyperbolic if $\mathbb{S}^3 \setminus K$ admits a finite volume complete hyperbolic metric.
- Holonomy: $\tilde{\rho} : G = \pi_1(\mathbb{S}^3 \setminus K) \to \mathrm{Iso}^+(\mathbb{H}^3) \equiv \mathrm{PSL}(2; \mathbb{C})$.
- (González-Acuña) It lifts to $\rho : G \to \mathrm{SL}(2; \mathbb{C})$, $\rho, I(\rho), \bar{\rho}, I(\bar{\rho}) \in R(G)$, $[\rho], [I(\rho)], [\bar{\rho}], [I(\bar{\rho})] \in X(G)$.
- $\mathcal{K} \subset X(G)$ irreducible component such that either $[\rho], [I(\rho)], [\bar{\rho}], [I(\bar{\rho})]$ is in $\mathcal{K}$: excellent component. At most four components. It is a curve (Culler,Shalen,Thurston); $\mathcal{U}$ excellent curve is the union of excellent components.
- $m \to \frac{1}{0} = \infty$, $\ell \to \frac{0}{1} = 0$, $V \sim pm + q\ell \to \frac{p}{q}$:
- $t_{\frac{p}{q}} : \mathcal{K} \to \mathbb{C}$ algebraic morphism

Hyperbolic knots

- K is hyperbolic if $\mathbb{S}^3 \setminus K$ admits a finite volume complete hyperbolic metric.
- Holonomy: $\tilde{\rho} : G = \pi_1(\mathbb{S}^3 \setminus K) \to \mathrm{Iso}^+(\mathbb{H}^3) \equiv \mathrm{PSL}(2;\mathbb{C})$.
- (González-Acuña) It lifts to $\rho : G \to \mathrm{SL}(2;\mathbb{C})$, $\rho, I(\rho), \bar{\rho}, I(\bar{\rho}) \in R(G)$, $[\rho], [I(\rho)], [\bar{\rho}], [I(\bar{\rho})] \in X(G)$.
- $\mathcal{K} \subset X(G)$ irreducible component such that either $[\rho], [I(\rho)], [\bar{\rho}], [I(\bar{\rho})]$ is in $\mathcal{K}$: excellent component. At most four components. It is a curve (Culler,Shalen,Thurston); $\mathcal{U}$ excellent curve is the union of excellent components.
- $m \to \frac{1}{0} = \infty$, $\ell \to \frac{0}{1} = 0$, $V \sim pm + q\ell \to \frac{p}{q}$:
- $t_{\frac{p}{q}} : \mathcal{K} \to \mathbb{C}$ algebraic morphism
- $E = \left(V_{\frac{p_1}{q_1}}, V_{\frac{p_2}{q_2}}\right)$ (basis of $H_1(\tau;\mathbb{Z})$) yields $t_E : \mathcal{K} \to \mathcal{K}_E \subset \mathbb{C}^2$.

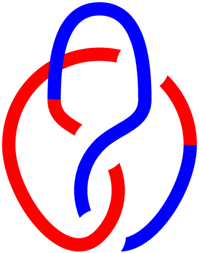
Hyperbolic knots

- K is hyperbolic if $\mathbb{S}^3 \setminus K$ admits a finite volume complete hyperbolic metric.
- Holonomy: $\tilde{\rho} : G = \pi_1(\mathbb{S}^3 \setminus K) \to \mathrm{Iso}^+(\mathbb{H}^3) \equiv \mathrm{PSL}(2;\mathbb{C})$.
- (González-Acuña) It lifts to $\rho : G \to \mathrm{SL}(2;\mathbb{C})$, $\rho, I(\rho), \bar{\rho}, I(\bar{\rho}) \in R(G)$, $[\rho], [I(\rho)], [\bar{\rho}], [I(\bar{\rho})] \in X(G)$.
- $\mathcal{K} \subset X(G)$ irreducible component such that either $[\rho], [I(\rho)], [\bar{\rho}], [I(\bar{\rho})]$ is in $\mathcal{K}$: excellent component. At most four components. It is a curve (Culler,Shalen,Thurston); $\mathcal{U}$ excellent curve is the union of excellent components.
- $m \to \frac{1}{0} = \infty$, $\ell \to \frac{0}{1} = 0$, $V \sim pm + q\ell \to \frac{p}{q}$:
- $t_{\frac{p}{q}} : \mathcal{K} \to \mathbb{C}$ algebraic morphism
- $E = \left(V_{\frac{p_1}{q_1}}, V_{\frac{p_2}{q_2}}\right)$ (basis of $H_1(\tau;\mathbb{Z})$) yields $t_E : \mathcal{K} \to \mathcal{K}_E \subset \mathbb{C}^2$.
- $T = \left(V_{\frac{p_1}{q_1}}, V_{\frac{p_2}{q_2}}, V_{\frac{p_3}{q_3}}\right)$ (3 basis of $H_1(\tau;\mathbb{Z})$) yields $t_T : \mathcal{K} \to \mathcal{K}_E \subset \mathbb{C}^3$.

Hyperbolic knots

- K is hyperbolic if $\mathbb{S}^3 \setminus K$ admits a finite volume complete hyperbolic metric.
- Holonomy: $\tilde{\rho} : G = \pi_1(\mathbb{S}^3 \setminus K) \to \mathrm{Iso}^+(\mathbb{H}^3) \equiv \mathrm{PSL}(2;\mathbb{C})$.
- (González-Acuña) It lifts to $\rho : G \to \mathrm{SL}(2;\mathbb{C})$, $\rho, I(\rho), \bar{\rho}, I(\bar{\rho}) \in R(G)$, $[\rho], [I(\rho)], [\bar{\rho}], [I(\bar{\rho})] \in X(G)$.
- $\mathcal{K} \subset X(G)$ irreducible component such that either $[\rho], [I(\rho)], [\bar{\rho}], [I(\bar{\rho})]$ is in $\mathcal{K}$: excellent component. At most four components. It is a curve (Culler,Shalen,Thurston); $\mathcal{U}$ excellent curve is the union of excellent components.
- $m \to \frac{1}{0} = \infty$, $\ell \to \frac{0}{1} = 0$, $V \sim pm + q\ell \to \frac{p}{q}$:
- $t_{\frac{p}{q}} : \mathcal{K} \to \mathbb{C}$ algebraic morphism
- $E = \left(V_{\frac{p_1}{q_1}}, V_{\frac{p_2}{q_2}}\right)$ (basis of $H_1(\tau;\mathbb{Z})$) yields $t_E : \mathcal{K} \to \mathcal{K}_E \subset \mathbb{C}^2$.
- $T = \left(V_{\frac{p_1}{q_1}}, V_{\frac{p_2}{q_2}}, V_{\frac{p_3}{q_3}}\right)$ (3 basis of $H_1(\tau;\mathbb{Z})$) yields $t_T : \mathcal{K} \to \mathcal{K}_E \subset \mathbb{C}^3$.
- E-peripheral polynomial $\wp_E$: equation of $\mathcal{K}_E$; total E-peripheral polynomial $\wp_E$: equation of $\mathcal{U}_E$

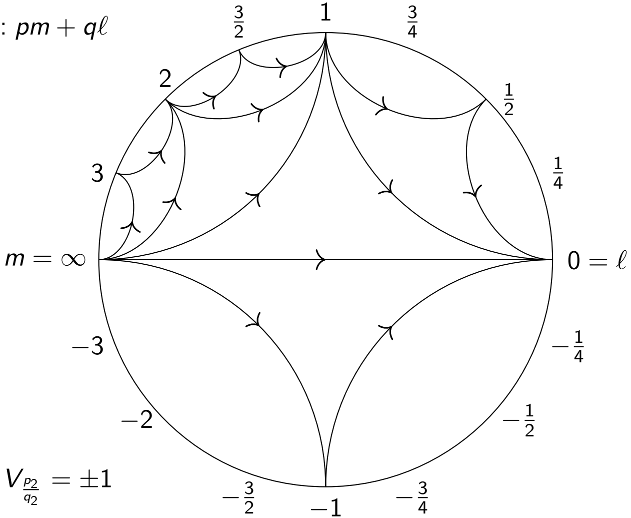
Figure Eight Knot II



- $G = \langle g_1, g_2 \mid g_2^{g_1 \cdot g_2^{-1}} = g_1^{g_2^{-1} \cdot g_1} \rangle$
- $y = t_{g_1} = t_{g_2}$, $z = t_{g_1 g_2}$,
  $X(G) = \{(1 - 2y^2 + z + y^2 z - z^2)(y^2 - z - 2) = 0\}$ (Whittemore)
- Meridian $m = g_2$, canonical longitude $\ell = g_1 g_2^{-1} g_1^{-1} g_2^2 g_1^{-1} g_2^{-1} g_1$
- $t_m = t_\infty = y$, $t_\ell = t_0 = -2 + 5y^2 - y^4$
- $\wp_{(\infty,0)} = -2 + 5t_\infty^2 - t_\infty^4 + t_0$ (zero locus)

Modular tessellation

Vertices $V_{\frac{p}{q}} : pm + q\ell$



Edges: $V_{\frac{p_1}{q_1}} \cdot V_{\frac{p_2}{q_2}} = \pm 1$

Triangles: $V_{\frac{p_1}{q_1}}, V_{\frac{p_2}{q_2}}, V_{\frac{p_3}{q_3}}$ yield 3 edges

Dehn surgeries and cone manifolds

- $E = (V_1, V_2)$, M_{V_1} obtained by V_1-Dehn surgery on K ($\bar{V}_2$ core of the new solid torus).

Dehn surgeries and cone manifolds

- $E = (V_1, V_2)$, M_{V_1} obtained by V_1-Dehn surgery on K ($\bar{V_2}$ core of the new solid torus).
- (limit of hyperbolicty) $\exists \alpha_0(V_1) > 0$ such that the cone manifold $Q_\alpha = (M_{V_1}, \bar{V_2}^{\alpha})$ is hyperbolic if $0 \leq \alpha < \alpha_0(V_1)$; one holonomy of Q_α is in $\mathcal{K}$.

Dehn surgeries and cone manifolds

- $E = (V_1, V_2)$, M_{V_1} obtained by V_1-Dehn surgery on K ($\bar{V}_2$ core of the new solid torus).
- (limit of hyperbolicty) $\exists \alpha_0(V_1) > 0$ such that the cone manifold $Q_\alpha = (M_{V_1}, \bar{V}_2{}^\alpha)$ is hyperbolic if $0 \leq \alpha < \alpha_0(V_1)$; one holonomy of Q_α is in $\mathcal{K}$.
- (HLM) Real and imaginary parts of a complex conjugate root of $\wp_E\left(2\cos\frac{\alpha}{2}, z\right)$ determine volume and Chern-Simon invariant of Q_α, respectively.

Dehn surgeries and cone manifolds

- $E = (V_1, V_2)$, M_{V_1} obtained by V_1-Dehn surgery on K ($\bar{V}_2$ core of the new solid torus).
- (limit of hyperbolicty) $\exists \alpha_0(V_1) > 0$ such that the cone manifold $Q_\alpha = (M_{V_1}, \bar{V}_2{}^{\alpha})$ is hyperbolic if $0 \le \alpha < \alpha_0(V_1)$; one holonomy of Q_α is in $\mathcal{K}$.
- (HLM) Real and imaginary parts of a complex conjugate root of $\wp_E\left(2\cos\frac{\alpha}{2}, z\right)$ determine volume and Chern-Simon invariant of Q_α, respectively.
- Steps:

Dehn surgeries and cone manifolds

- $E = (V_1, V_2)$, M_{V_1} obtained by V_1-Dehn surgery on K ($\bar{V}_2$ core of the new solid torus).
- (limit of hyperbolicty) $\exists \alpha_0(V_1) > 0$ such that the cone manifold $Q_\alpha = (M_{V_1}, \bar{V}_2{}^{\alpha})$ is hyperbolic if $0 \leq \alpha < \alpha_0(V_1)$; one holonomy of Q_α is in $\mathcal{K}$.
- (HLM) Real and imaginary parts of a complex conjugate root of $\wp_E\left(2\cos\frac{\alpha}{2}, z\right)$ determine volume and Chern-Simon invariant of Q_α, respectively.
- Steps:
  - Describe $\mathcal{U}$: done for rational knots (2-bridge knots), periodic knots, tunnel number 1, 4-nets: hard computations.

Dehn surgeries and cone manifolds

- $E = (V_1, V_2)$, M_{V_1} obtained by V_1-Dehn surgery on K ($\bar{V}_2$ core of the new solid torus).
- (limit of hyperbolicty) $\exists \alpha_0(V_1) > 0$ such that the cone manifold $Q_\alpha = (M_{V_1}, \bar{V}_2{}^\alpha)$ is hyperbolic if $0 \leq \alpha < \alpha_0(V_1)$; one holonomy of Q_α is in $\mathcal{K}$.
- (HLM) Real and imaginary parts of a complex conjugate root of $\wp_E\left(2\cos\frac{\alpha}{2}, z\right)$ determine volume and Chern-Simon invariant of Q_α, respectively.
- Steps:
  - Describe $\mathcal{U}$: done for rational knots (2-bridge knots), periodic knots, tunnel number 1, 4-nets: hard computations.
  - Describe t_V (not so hard)

Dehn surgeries and cone manifolds

- $E = (V_1, V_2)$, M_{V_1} obtained by V_1-Dehn surgery on K ($\bar{V}_2$ core of the new solid torus).
- (limit of hyperbolicty) $\exists \alpha_0(V_1) > 0$ such that the cone manifold $Q_\alpha = (M_{V_1}, \bar{V}_2{}^\alpha)$ is hyperbolic if $0 \leq \alpha < \alpha_0(V_1)$; one holonomy of Q_α is in $\mathcal{K}$.
- (HLM) Real and imaginary parts of a complex conjugate root of $\wp_E\left(2\cos\frac{\alpha}{2}, z\right)$ determine volume and Chern-Simon invariant of Q_α, respectively.
- Steps:
  - Describe $\mathcal{U}$: done for rational knots (2-bridge knots), periodic knots, tunnel number 1, 4-nets: hard computations.
  - Describe t_V (not so hard)
  - Find total E-peripheral polynomial (main reference)

Total E-peripheral polynomials

- $T = (V_1, V_2, V_3)$ 3-system. Knowledge $t_{V_1}, t_{V_2}, t_{V_3}$ implies knowledge of any t_V, and t_T is a birational isomorphism.

Total E-peripheral polynomials

- $T = (V_1, V_2, V_3)$ 3-system. Knowledge $t_{V_1}, t_{V_2}, t_{V_3}$ implies knowledge of any t_V, and t_T is a birational isomorphism.
- $T = (V_1, V_2, V_3)$ 3-system. Knowledge of $\wp_{V_1,V_2}, \wp_{V_2,V_3}$ implies knowledge of any $\wp_E$.

Total E-peripheral polynomials

- $T = (V_1, V_2, V_3)$ 3-system. Knowledge $t_{V_1}, t_{V_2}, t_{V_3}$ implies knowledge of any t_V, and t_T is a birational isomorphism.
- $T = (V_1, V_2, V_3)$ 3-system. Knowledge of $\wp_{V_1,V_2}, \wp_{V_2,V_3}$ implies knowledge of any $\wp_E$.
- E 2-system: t_E is of a degree at most 2 (at most one E with degree 2).

Total E-peripheral polynomials

- $T = (V_1, V_2, V_3)$ 3-system. Knowledge $t_{V_1}, t_{V_2}, t_{V_3}$ implies knowledge of any t_V, and t_T is a birational isomorphism.
- $T = (V_1, V_2, V_3)$ 3-system. Knowledge of $\wp_{V_1,V_2}, \wp_{V_2,V_3}$ implies knowledge of any $\wp_E$.
- E 2-system: t_E is of a degree at most 2 (at most one E with degree 2).
- K amphicheiral and $\mathcal{U}$ irreducible: $t_{\infty,0}$ is of degree 2.

Total E-peripheral polynomials

- $T = (V_1, V_2, V_3)$ 3-system. Knowledge $t_{V_1}, t_{V_2}, t_{V_3}$ implies knowledge of any t_V, and t_T is a birational isomorphism.
- $T = (V_1, V_2, V_3)$ 3-system. Knowledge of $\wp_{V_1,V_2}, \wp_{V_2,V_3}$ implies knowledge of any $\wp_E$.
- E 2-system: t_E is of a degree at most 2 (at most one E with degree 2).
- K amphicheiral and $\mathcal{U}$ irreducible: $t_{\infty,0}$ is of degree 2.
- $T = (V_1, V_2, V_3)$ 3-system: t_{V_1,V_2} is of degree 2 if and only if the zero locus of $\langle \wp_{V_1,V_2}(v_1, v_2), D(v_1, v_2, v_3) \rangle$ is irreducible.

Gracias Maite