Monodromy theorem for Artin kernels

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Joint work with J.I. Cogolludo and D. Matei

Definitions and examples

Definition

Γ simplicial graph, V_Γ set of vertices, $E_\Gamma \subset \{A \subset V_\gamma \mid \#A = 2\}$ edges

$$G_\Gamma = \langle g_v, v \in V_\Gamma \mid [g_v, g_w] = 1, \{v, w\} \in E_\Gamma \rangle$$

right-angled Artin group.

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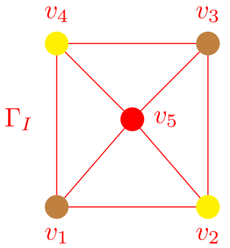
- Γ 1-skeleton of an n-simplex, $G_\Gamma = \mathbb{Z}^{n+1}$.
- Γ multipartite graph of $\Gamma_1, \ldots, \Gamma_n$, $G_\Gamma = \prod_{j=1}^n G_{\Gamma_j}$.

Test examples

Test Example I



$$G_{\Gamma_I} = \mathbb{F}_2 \times \mathbb{F}_2 \times \mathbb{Z}$$

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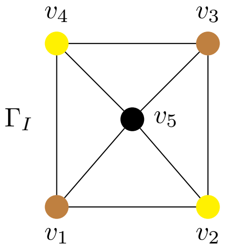

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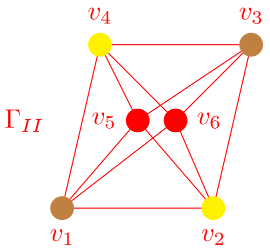
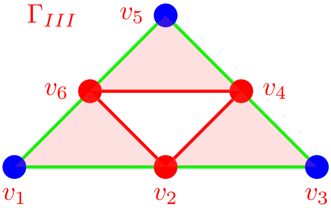
Test Example II



$G_{\tilde{\Gamma}_{II}} = \mathbb{F}_2 \times \mathbb{F}_2 \times \mathbb{F}_2$

More test examples

Test Example III



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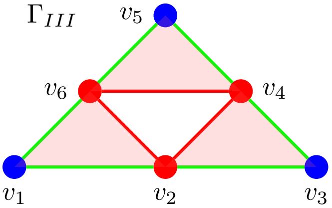
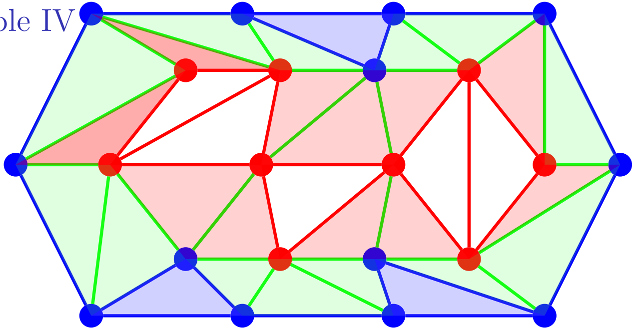


Test Example IV

Γ_{IV}



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Γ simplicial graph (V_Γ and E_Γ as before).

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Flag Complex of Γ (= its 1-skeleton).

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- The flag complex of Γ_{II} is the octahedron (2-dimensional).

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- $\mathbb{T}_{\mathbb{R}}(\mathbb{B}_\Gamma)$ CW-complex structure with zero differential.
- $\mathbb{T}_{\mathbb{C}}(\mathbb{B}_\Gamma)$ is a (singular) algebraic complex variety.

Toric complexes and Artin kernels

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Γ *simplicial graph*

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Definition

Γ simplicial graph, $\chi : V_\Gamma \to \mathbb{Z}^*$ $(n_v = \chi(t))$, $\gcd\{n_v \mid v \in V_\Gamma\} = 1$.
Denote by χ the induced epimorphism $G_\Gamma \twoheadrightarrow \mathbb{Z}$ $(g_v \notin \ker \chi, \forall v \in V_K)$.
The Artin kernel associated to χ is $A_\Gamma^\chi = \ker \chi$.

Goals

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- G_Γ

GEOMETRY

- $\mathbb{T}_{\mathbb{C}}(\mathbb{B}_\Gamma)$

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- $\mathbb{C}[t^{\pm 1}]$-module

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- $H_*(F_\chi;\mathbb{C})$
- Monodromy φ on $H_*(F_\chi;\mathbb{C})$

Differentials

$\tilde{C}_*(\mathbb{B}_\Gamma)$ $\qquad\qquad$ $C_{*+1}(\mathbb{T}^{\chi}_{\mathbb{R}}(\mathbb{B}_\Gamma))/\mathbb{C}[t^{\pm 1}]$

$$
\begin{array}{cccccc}
 & & & \sigma \in C_k & \\
 & & & \downarrow & \\
 & & \cdots & \vdots & \cdots \\
\tau_1 \in C_{k-1} & \rightarrow & \cdots & \pm 1 & \cdots \\
\tau_1 \subset \partial\sigma & & \cdots & \vdots & \cdots \\
\tau_2 \in C_{k-1} & \rightarrow & \cdots & 0 & \cdots \\
\tau_1 \not\subset \partial\sigma & & \cdots & \vdots & \cdots
\end{array}
\qquad
\begin{array}{ccccc}
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$\sigma = \tau_1 \cup \{v\}$ $\qquad\qquad$ $X_v = (t-1)^{n_v}$

Differentials

$C_{*+1}(\mathbb{T}^{\chi}_{\mathbb{R}}(\mathbb{B}_\Gamma)) \otimes_{\mathbb{C}[t^{\pm 1}]} \mathbb{C}(t)$

$$\begin{array}{ccccc} & & & \frac{\sigma}{X_\sigma} & \\ & & & \downarrow & \\ & & \dots & \vdots & \dots \\ \frac{\tau_1}{X_{\tau_1}} & \to & \dots & \pm 1 & \dots \\ \tau_1 \subset \partial\sigma & & \dots & \vdots & \dots \\ \frac{\tau_2}{X_{\tau_2}} & \to & \dots & 0 & \dots \\ \tau_1 \not\subset \partial\sigma & & \dots & \vdots & \dots \end{array}$$

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2. *For* $\ell \leq k$,

$$H_\ell(A_\Gamma^\xi;\mathbb{C}) \cong \prod_{k=1}^{r} \frac{\mathbb{C}[t^{\pm 1}]}{\Phi_{n_j}(t)^{m_j}},$$

where $\Phi_n(t)$ *is the* n^{th}*-cyclotomic polynomial and* $m_j \leq \ell + 1$.

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2. As a consequence, Monodromy Theorem holds for H_ℓ, $\ell \leq k$: eigenvalues are roots of unity and Jordan blocks are of size at most $\ell + 1$.

Idea of the proof

Assume $M = \mathbb{B}_\Gamma$ is $(n-1)$-connected and study $\partial : C_n(M) \to C_{n-1}(M)$, $\partial : C_{n+1}(\mathbb{T}^\chi_\mathbb{R}(M)) \to C_n(\mathbb{T}^\chi_\mathbb{R}(M))$, more precisely, the Fitting ideals of the corresponding matrices.

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A minor does not vanish if and only if (K, L) is acyclic. In that case its value is

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Remark

The statement about roots of unity is proved.

Jordan blocks I

To study the behavior for one cyclotomic, say Φ_d, fix a primitive d^{th}-root of unity ζ and consider the complex

$$C_*(\mathbb{T}^{\chi}_{\mathbb{R}}(K)) \otimes_{\mathbb{C}[t^{\pm 1}]} \mathbb{C}[[T]], \quad t = \zeta + T.$$

Jordan blocks I

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A minor associated to (K, L)

$$\frac{\left(\prod_{\sigma \in K \setminus M_{n-1}} X_\sigma\right)\left(\prod_{\tau \in L \setminus M_{n-2}} X_\tau\right)}{\prod_{\tau \in M_{n-1} \setminus M_{n-2}} X_\tau}. = \text{Unit} \cdot T^{\text{some power}}.$$

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The Fitting ideals will be a sequence $\mathbb{C}[t^{\pm 1}] \supset (T^N) \supset (T^{N-n_1}) \supset \cdots \supset (T^{N-(n_1+\cdots+n_k)}) \supset (0)$, $n_1 \geq \cdots \geq n_k$. The multiplicity of ζ in the characteristic polynomial is N and there are Jordan blocks of size $n_1, \ldots, n_k$.

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Strategy

1. Mark in red (resp. blue) the vertices v such that $d | n_v$ (resp. $d \not| n_v$)
2. The multiplicity of a simplex is the number of red vertices.

Jordan Blocks II

Strategy

Jordan Blocks II

Strategy

3. The multiplicity of $M_n \supset K \supset M_{n-1}$ (resp. $M_{n-1} \supset L \supset M_{n-2}$) is the sum of the multiplicities of the n-simplices (resp. $(n-1)$-simplices).

Jordan Blocks II

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6. Consider filtrations

$$M_n = M_n^{(n+1)} \supset M_n^{(n)} \supset \cdots \supset M_n^{(1)} \supset M_n^{(0)} \supset M_{n-1} \supset M_{n-1} = M_{n-1}^{(n)}$$

$$M_{n-1} = M_{n-1}^{(n)} \supset M_{n-1}^{(n-1)} \supset \cdots \supset M_{n-1}^{(1)} \supset M_{n-1}^{(0)} \supset M_{n-2}$$

Jordan Blocks III

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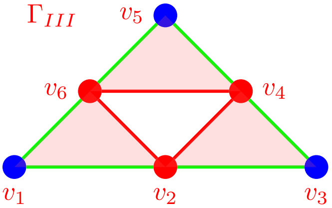
   - The multiplicity of K is independent of the choices.
8. For the next Fitting ideal, we may consider an acyclic (K_1, L_1) obtained from (K, L) by taking out an n-simplex of K (of multiplicity $\leq n+1$) and adding an $(n-1)$-simplex to L (of multiplicity ≥ 0). Hence the multiplicity decreases at most by $n+1$ and the second statement of Monodromy Theorem holds. The number of such blocks is:

$$\dim_{\mathbb{C}} \operatorname{Im}\left(\tilde{H}_{n-1}(M_{n-1}^{(0)}) \to \tilde{H}_{n-1}(M_n^{(n)})\right)$$

Computations

Test Example III



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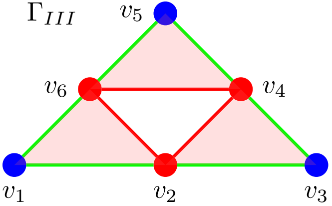
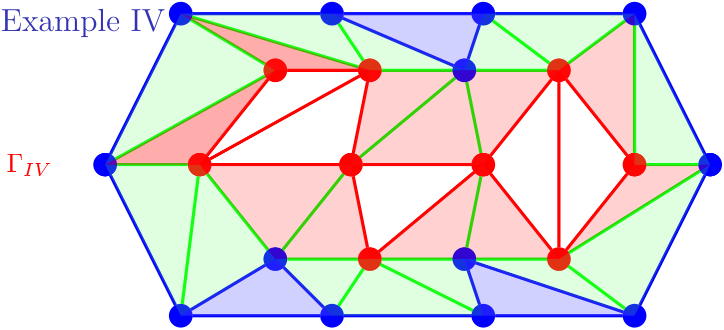
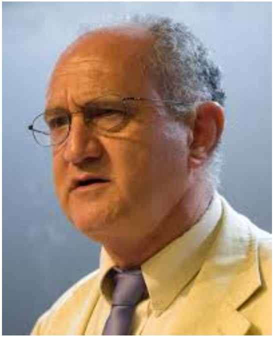


Test Example IV



PARABÉNS



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