Zariski pairs of rational arrangements

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- Bijection between resolution graphs (respecting strict transforms of $\mathcal{C}_i^j$).

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$\mathcal{C}_1, \mathcal{C}_2 \subset \mathbb{P}^2$ form a *Zariski pair* if

1. Same combinatorics
2. $\nexists \psi : (\mathbb{P}^2, \mathcal{C}_1) \to (\mathbb{P}^2, \mathcal{C}_2)$ homeomorphism.

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Sextics with six cusps (lying or not on a conic).

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- $\pi_1(\mathbb{P}^2 \setminus \mathcal{C}_1) \not\cong \pi_1(\mathbb{P}^2 \setminus \mathcal{C}_2)$
- Alexander polynomial are different.

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- Construct enough examples

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Important

It can be computed algebraically (Zariski,Libgober, Esnault,______ , Degtyarev).

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- Galois coverings by algebraic properties (Tokunaga).

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- In case of sextics with simple singular points, lattice theory (Degtyarev-Shimada).

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Remark

In the original Zariski pair: two connected components (Degtyarev, Shimada).

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- Trivial Alexander polynomial but different characteristic varieties

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  - 11 lines, conjugate equations in $\mathbb{Q}(\sqrt{2})$.

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  - 11 lines, conjugate equations in $\mathbb{Q}(\sqrt{2})$.
  - Distinguished by braid monodromy ▸ Fig..

- Combinatorics $\equiv$ intersection patterns.
- First Zariski pair (Rybnikov '94):
  - 13 lines with non-real equations.
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- Moduli space of $\mathscr{C}_0$ is connected Fig..

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Proposition (______ -Cogolludo-Tokunaga)

$\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4$ is in $\mathscr{C}_1$
$P_3 = P_4 \iff$ there exists a conic through the eight tacnodes

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- Different number of positive dimensional irreducible components of characteristic varieties $\Rightarrow$ non-isomorphic fundamental groups (direct proof by Namba-Tsuchihashi).

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- One special example in torus type curves (Oka-Eyral)

Oka-Eyral example

- Moduli space of sextic curves with $\mathbb{E}_6, \mathbb{E}_6, \mathbb{A}_5, \mathbb{A}_2$ disconnected: equations in $\mathbb{Q}(\sqrt{3})$.

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- Fundamental groups are isomorphic.
- Braid monodromies of the figure are not equivalent.
- Adding lines, we obtain Zariski pairs with homeomorphic complement.

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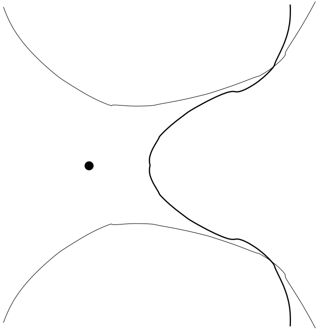

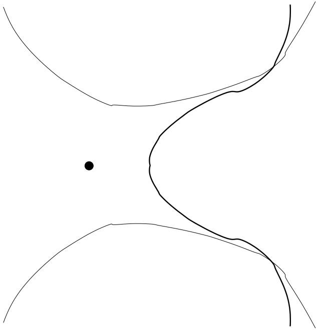

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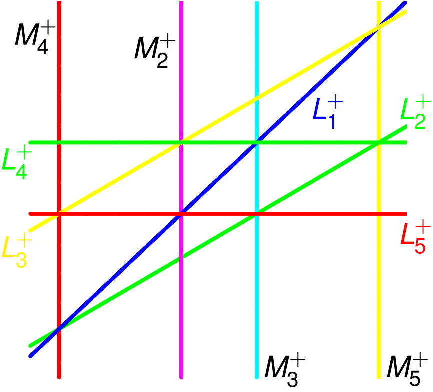
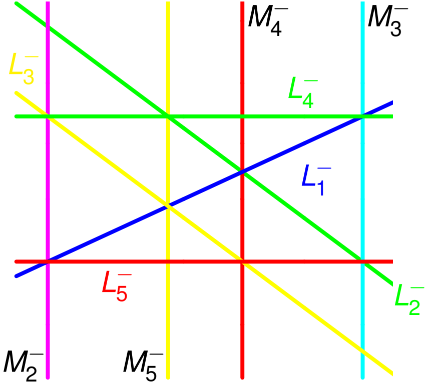


Figure: $\mathscr{A}^+$



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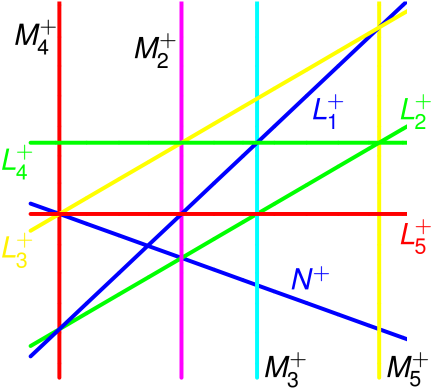
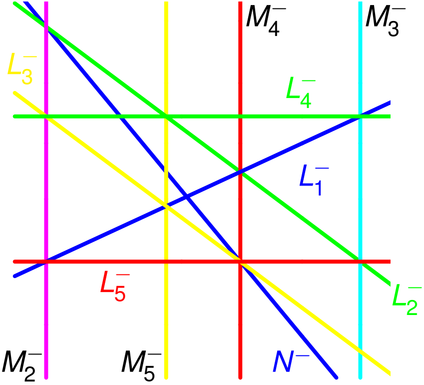


Figure: $\mathscr{B}^+$



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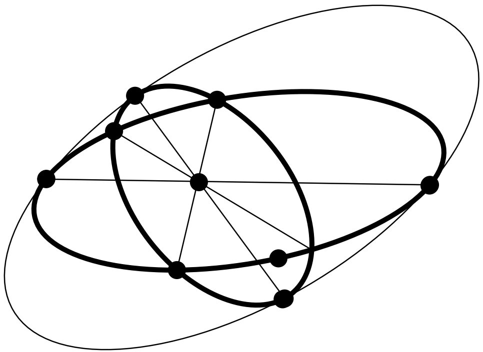

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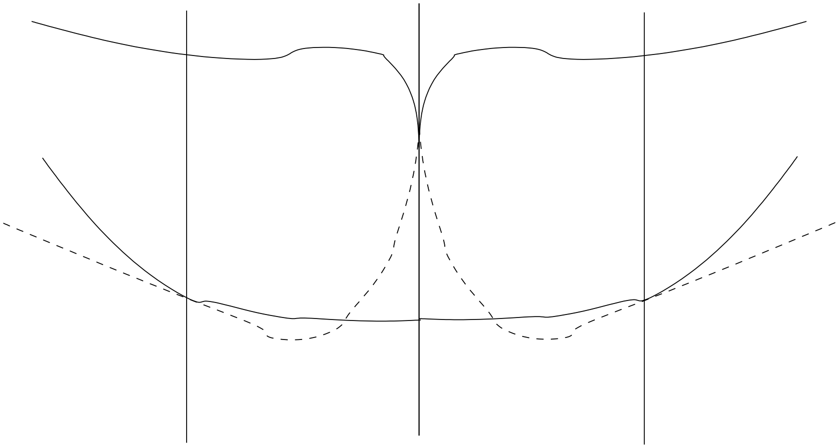

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Real picture of Oka-Eyral examples

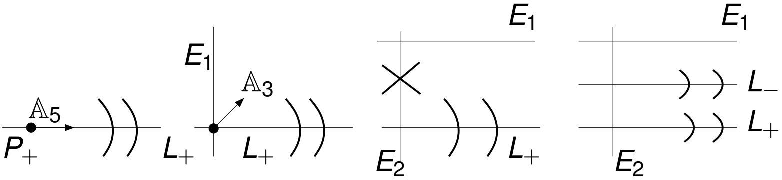

Figure: Sequence of blowups

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