Braid monodromy and conjugate curves

Enrique Artal (Universidad de Zaragoza)

Oberwolfach, September 2002

Fundamental Group in Geometry

Braid monodromy and conjugate curves

Enrique Artal (Universidad de Zaragoza)

Oberwolfach, September 2002

Fundamental Group in Geometry

Joint work [ACC02, ACC02a] with:
Jorge Carmona (Universidad Complutense)
José I. Cogolludo (Universidad de Zaragoza)

Contents

1. Startup problem

▷ $T_1, \ldots, T_r$ topological types of *singularities* of plane curves

1. Startup problem

- ▷ $T_1, \ldots, T_r$ topological types of *singularities* of plane curves

- ▷ $\Sigma := \Sigma(k_1T_1, \ldots, k_rT_r; d)$ space of plane projective curves of degree d with k_i singular points of topological type T_i

1. Startup problem

- $T_1, \ldots, T_r$ topological types of *singularities* of plane curves
- $\Sigma := \Sigma(k_1T_1, \ldots, k_rT_r; d)$ space of plane projective curves of degree d with k_i singular points of topological type T_i
- $\mathscr{M} := \mathscr{M}(k_1T_1, \ldots, k_rT_r; d) :=$

$$\Sigma(k_1T_1, \ldots, k_rT_r; d)/PGL(3; \mathbb{C})$$

1. Startup problem

- ▷ $T_1, \ldots, T_r$ topological types of *singularities* of plane curves
- ▷ $\Sigma := \Sigma(k_1T_1, \ldots, k_rT_r; d)$ space of plane projective curves of degree d with k_i singular points of topological type T_i
- ▷ $\mathscr{M} := \mathscr{M}(k_1T_1, \ldots, k_rT_r; d) :=$

$$\Sigma(k_1T_1, \ldots, k_rT_r; d)/PGL(3; \mathbb{C})$$

- ▷ Σ^{irr}: irreducible curves

$\mathscr{M} \neq \emptyset$?

$\mathscr{M} \neq \emptyset$?

Smoothness of Σ

$\mathscr{M} \neq \emptyset$?

Irreducibility of Σ

Smoothness of Σ

$\mathscr{M} \neq \emptyset$?

Irreducibility of Σ

Connectivity of $\mathscr{M}$

Smoothness of Σ

$\mathscr{M} \neq \emptyset$?

Irreducibility of Σ

Connectivity of $\mathscr{M}$

Adjacency: $\Sigma \subset \overline{\Sigma'}$?

$$\Sigma' := \Sigma(\boldsymbol{k}_1' \boldsymbol{T}_1', \ldots, \boldsymbol{k}_r' \boldsymbol{T}_r'; \boldsymbol{d})$$

Smoothness of Σ

$\mathscr{M} \neq \emptyset$?

Irreducibility of Σ

Connectivity of $\mathscr{M}$

Adjacency: $\Sigma \subset \overline{\Sigma'}$?

$$\Sigma' := \Sigma(\boldsymbol{k}_1' \boldsymbol{T}_1', \ldots, \boldsymbol{k}_r' \boldsymbol{T}_r'; \boldsymbol{d})$$

Local irreducibility of Σ around Σ'

Smoothness of Σ

$\mathscr{M} \neq \emptyset$?

Irreducibility of Σ

Connectivity of $\mathscr{M}$

Adjacency: $\Sigma \subset \overline{\Sigma'}$?

?

$$\Sigma' := \Sigma(\boldsymbol{k}'_1 \boldsymbol{T}'_1, \ldots, \boldsymbol{k}'_r \boldsymbol{T}'_r; \boldsymbol{d})$$

Local irreducibility of Σ around Σ'

Smoothness of Σ

$\tilde{\Sigma} \subset \Sigma$ connected component $\mathcal{C}_1, \mathcal{C}_2 \in \tilde{\Sigma} \Rightarrow \exists$ *oriented* isotopy h_t such that $h_0 = 1_{\mathbb{P}^2}$, $h_1(\mathcal{C}_1) = \mathcal{C}_2$.

$\tilde{\Sigma} \subset \Sigma$ connected component $\mathcal{C}_1, \mathcal{C}_2 \in \tilde{\Sigma} \Rightarrow \exists$ *oriented* isotopy h_t such that $h_0 = 1_{\mathbb{P}^2}$, $h_1(\mathcal{C}_1) = \mathcal{C}_2$.

What about the converse?

$\tilde{\Sigma} \subset \Sigma$ connected component $\mathcal{C}_1, \mathcal{C}_2 \in \tilde{\Sigma} \Rightarrow \exists$ *oriented* isotopy h_t such that $h_0 = 1_{\mathbb{P}^2}$, $h_1(\mathcal{C}_1) = \mathcal{C}_2$.

What about the converse?

If there exists an *oriented* isotopy (homeomorphism)

$$\Phi : \mathbb{P}^2 \longrightarrow \mathbb{P}^2$$

such that $\Phi(\mathcal{C}_1) = \mathcal{C}_2$, do they belong to the same connected component of Σ?

2. Previous results

- ▷ Works of Severi [SEV21],Zariski [ZAR29], Harris [HAR87], Greuel [GLS98, GLS98a, GL99, GLS99, GLS02], Shustin [SHU97, SHU97a], Lossen about irreducibility, smoothness, existence,. . .

2. Previous results

- ▷ Works of Severi [SEV21],Zariski [ZAR29], Harris [HAR87], Greuel [GLS98, GLS98a, GL99, GLS99, GLS02], Shustin [SHU97, SHU97a], Lossen about irreducibility, smoothness, existence,. . .
- ▷ Existence and connectedness have been solved for $d \leq 5$ by Namba [NMB86] and Degtyarev [DEG90], see here.

▷ $\Sigma(6\mathbb{A}_2; 6) = \Sigma^{\text{irr}}(6\mathbb{A}_2; 6)$ is reducible and not connected [ZAR29], [ZAR31], [ZAR37]

▷ $\Sigma(6\mathbb{A}_2;6) = \Sigma^{\text{irr}}(6\mathbb{A}_2;6)$ is reducible and not connected [ZAR29], [ZAR31], [ZAR37]

- $\Sigma^{\text{tor}}(6\mathbb{A}_2;6)$: cusps on a conic

▷ $\Sigma(6\mathbb{A}_2; 6) = \Sigma^{\text{irr}}(6\mathbb{A}_2; 6)$ is reducible and not connected [ZAR29], [ZAR31], [ZAR37]

- $\Sigma^{\text{tor}}(6\mathbb{A}_2; 6)$: cusps on a conic
- $\Sigma'(6\mathbb{A}_2; 6)$, $\Sigma''(6\mathbb{A}_2; 6)$,... other ones (at least one)

- $\Sigma(6\mathbb{A}_2;6) = \Sigma^{\text{irr}}(6\mathbb{A}_2;6)$ is reducible and not connected [ZAR29], [ZAR31], [ZAR37]
  - $\Sigma^{\text{tor}}(6\mathbb{A}_2;6)$: cusps on a conic
  - $\Sigma'(6\mathbb{A}_2;6)$, $\Sigma''(6\mathbb{A}_2;6)$,... other ones (at least one)
- Study the case $d = 6$, $T_i = \mathbb{A}_k, \mathbb{D}_l, \mathbb{E}_r$

3. Sextics with simple points

- $\mathcal{C} \in \Sigma$, $\pi : \widehat{Y} \longrightarrow \mathbb{P}^2$ double covering ramified along $\mathcal{C}$, $\tau : Y \longrightarrow \widehat{Y}$ minimal resolution, Y $K3$ surface (see Barth-Peters-Van de Ven [BPV84])

3. Sextics with simple points

- $\mathcal{C} \in \Sigma$, $\pi : \widehat{Y} \longrightarrow \mathbb{P}^2$ double covering ramified along $\mathcal{C}$, $\tau : Y \longrightarrow \widehat{Y}$ minimal resolution, Y $K3$ surface (see Barth-Peters-Van de Ven [BPV84])
- $\mu(\mathcal{C})$ sum of Milnor numbers of $\mathrm{Sing}(\mathcal{C})$, Y $K3 \Rightarrow \mu(\mathcal{C}) \leq 19$

3. Sextics with simple points

- $\mathcal{C} \in \Sigma$, $\pi : \widehat{Y} \longrightarrow \mathbb{P}^2$ double covering ramified along $\mathcal{C}$, $\tau : Y \longrightarrow \widehat{Y}$ minimal resolution, Y $K3$ surface (see Barth-Peters-Van de Ven [BPV84])
- $\mu(\mathcal{C})$ sum of Milnor numbers of $\mathrm{Sing}(\mathcal{C})$, Y $K3 \Rightarrow \mu(\mathcal{C}) \leq 19$
- Characterization of $\Sigma \neq \emptyset$ by Urabe, Yang [YA96] using Nikulin's results (intersection form lattice of a $K3$ surface)

3. Sextics with simple points

- $\mathcal{C} \in \Sigma$, $\pi : \widehat{Y} \longrightarrow \mathbb{P}^2$ double covering ramified along $\mathcal{C}$, $\tau : Y \longrightarrow \widehat{Y}$ minimal resolution, Y $K3$ surface (see Barth-Peters-Van de Ven [BPV84])
- $\mu(\mathcal{C})$ sum of Milnor numbers of $\mathrm{Sing}(\mathcal{C})$, Y $K3 \Rightarrow \mu(\mathcal{C}) \leq 19$
- Characterization of $\Sigma \neq \emptyset$ by Urabe, Yang [YA96] using Nikulin's results (intersection form lattice of a $K3$ surface)
  - $\mu(\mathcal{C}) = 19$ complete list + $\mu(\mathcal{C}) = 18$ supplementary list

3. Sextics with simple points

- $\mathcal{C} \in \Sigma$, $\pi : \widehat{Y} \longrightarrow \mathbb{P}^2$ double covering ramified along $\mathcal{C}$, $\tau : Y \longrightarrow \widehat{Y}$ minimal resolution, Y $K3$ surface (see Barth-Peters-Van de Ven [BPV84])
- $\mu(\mathcal{C})$ sum of Milnor numbers of $\mathrm{Sing}(\mathcal{C})$, Y $K3 \Rightarrow \mu(\mathcal{C}) \leq 19$
- Characterization of $\Sigma \neq \emptyset$ by Urabe, Yang [YA96] using Nikulin's results (intersection form lattice of a $K3$ surface)
  - $\mu(\mathcal{C}) = 19$ complete list + $\mu(\mathcal{C}) = 18$ supplementary list
  - $\Sigma \neq \emptyset$ if and only if the graph of singular points is a subgraph of a graph in one on the lists

3. Sextics with simple points

- $\mathcal{C} \in \Sigma$, $\pi : \widehat{Y} \longrightarrow \mathbb{P}^2$ double covering ramified along $\mathcal{C}$, $\tau : Y \longrightarrow \widehat{Y}$ minimal resolution, Y $K3$ surface (see Barth-Peters-Van de Ven [BPV84])
- $\mu(\mathcal{C})$ sum of Milnor numbers of $\mathrm{Sing}(\mathcal{C})$, Y $K3 \Rightarrow \mu(\mathcal{C}) \leq 19$
- Characterization of $\Sigma \neq \emptyset$ by Urabe, Yang [YA96] using Nikulin's results (intersection form lattice of a $K3$ surface)
  - $\mu(\mathcal{C}) = 19$ complete list + $\mu(\mathcal{C}) = 18$ supplementary list
  - $\Sigma \neq \emptyset$ if and only if the graph of singular points is a subgraph of a graph in one on the lists
  - Yang also studies $\Sigma(\Gamma)$: global irreducible components

4. Open problems about sextics with simple points

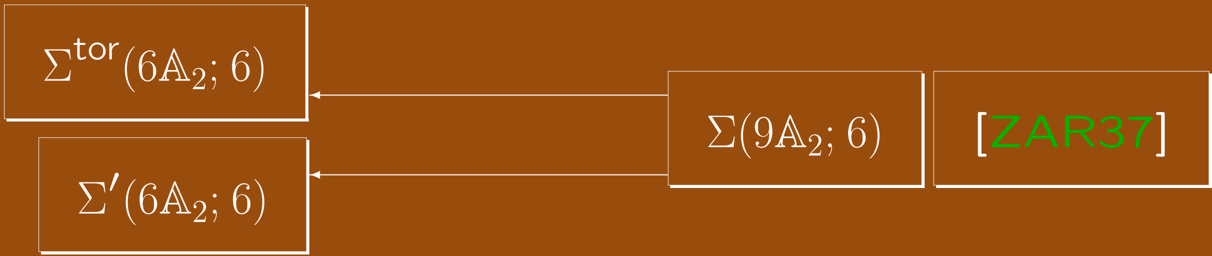
- If $\Sigma(\Gamma) \neq \emptyset$, how many connected components?

4. Open problems about sextics with simple points

- If $\Sigma(\Gamma) \neq \emptyset$, how many connected components?
- Understand adjacencies

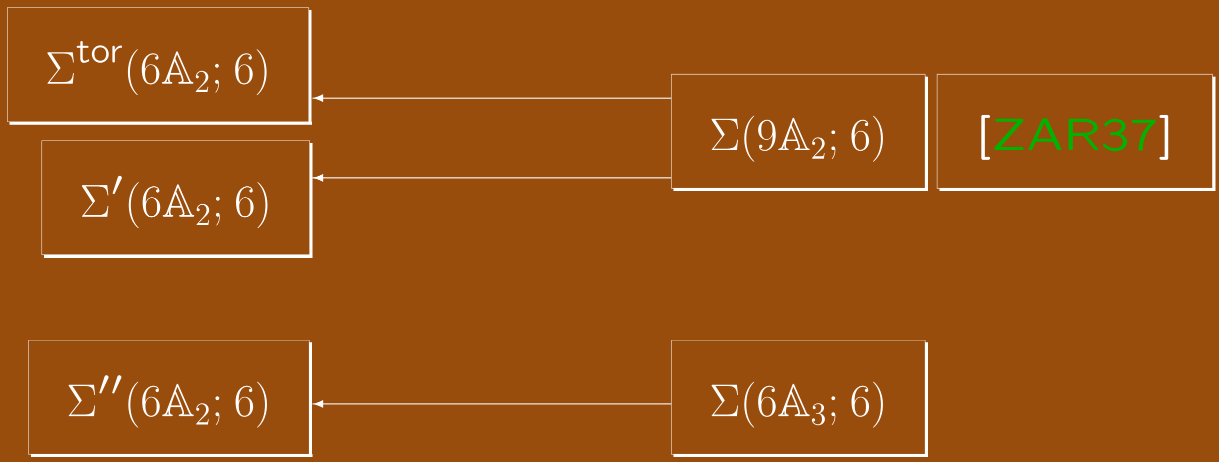
4. Open problems about sextics with simple points

- If $\Sigma(\Gamma) \neq \emptyset$, how many connected components?
- Understand adjacencies



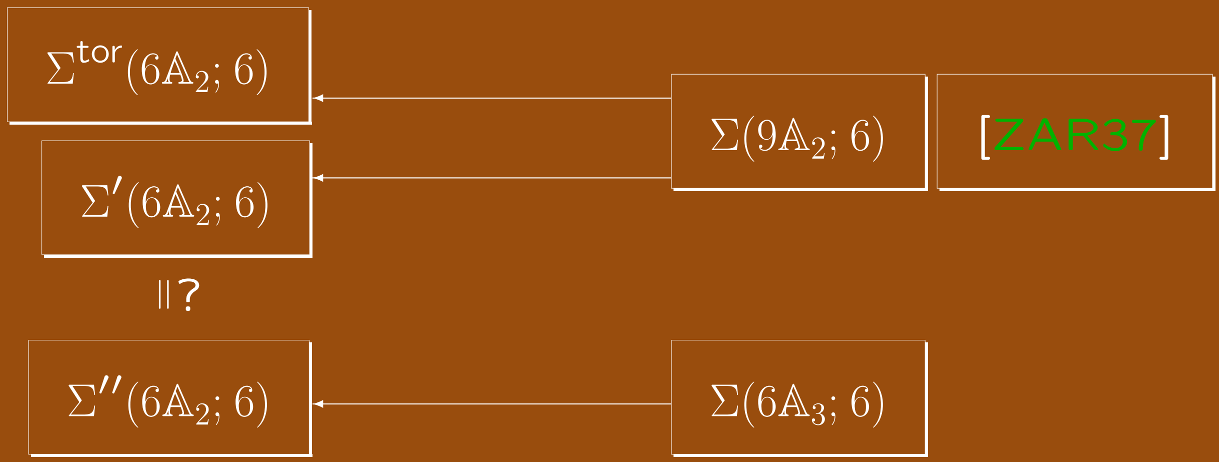
4. Open problems about sextics with simple points

- If $\Sigma(\Gamma) \neq \emptyset$, how many connected components?
- Understand adjacencies



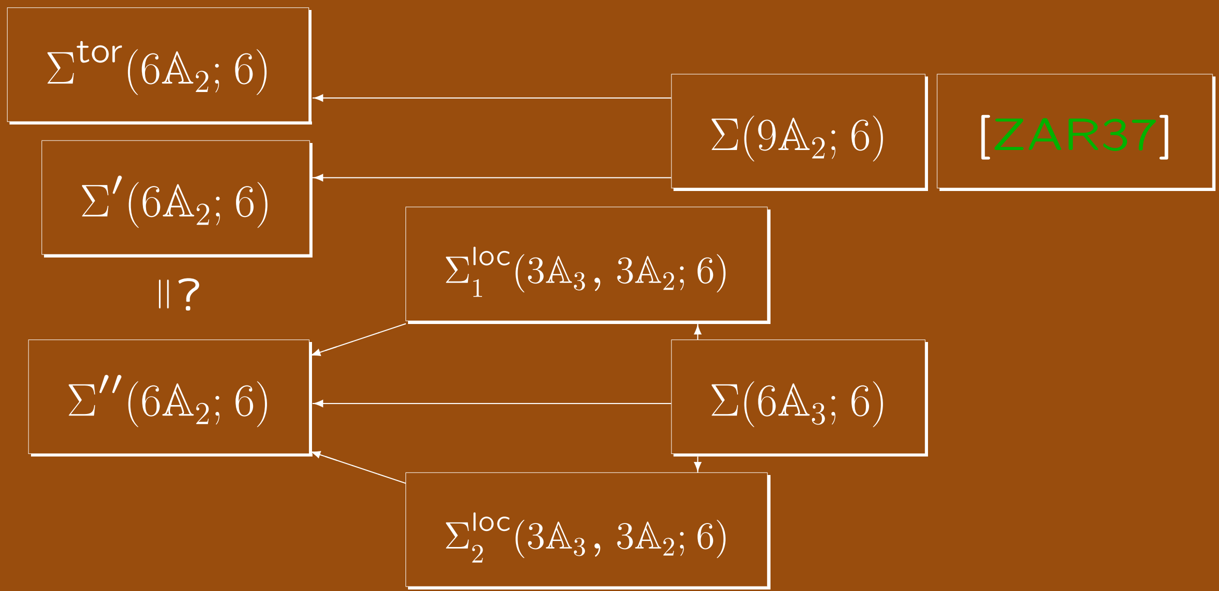
4. Open problems about sextics with simple points

- If $\Sigma(\Gamma) \neq \emptyset$, how many connected components?
- Understand adjacencies



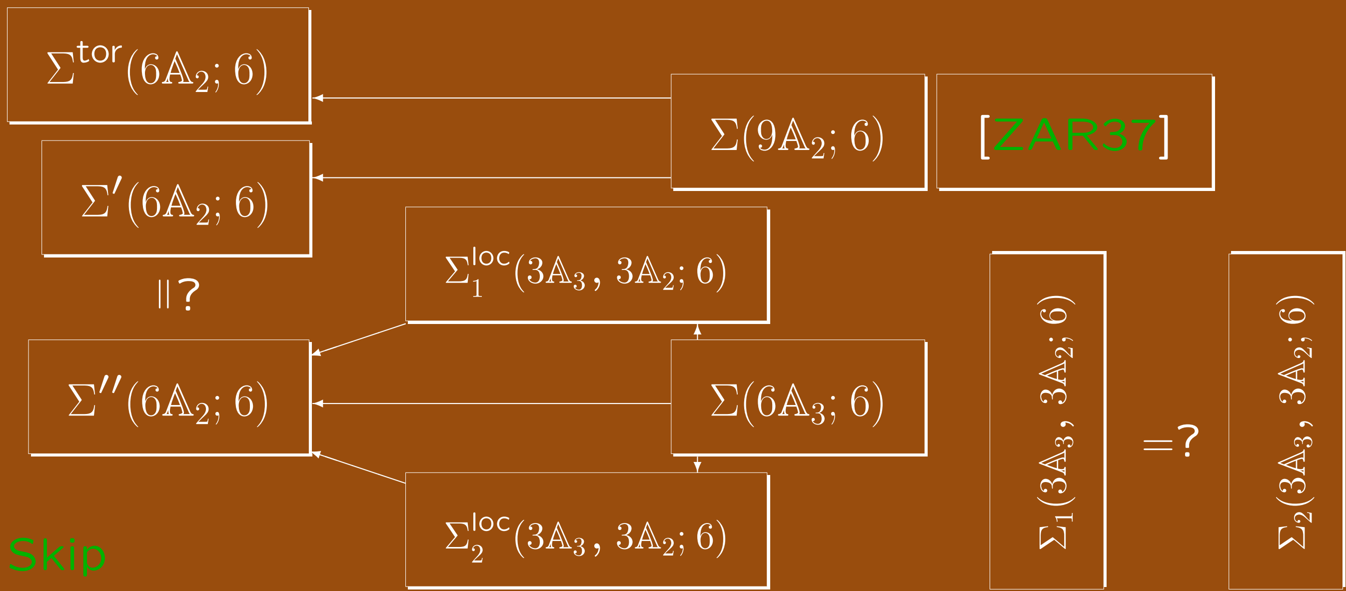
4. Open problems about sextics with simple points

- If $\Sigma(\Gamma) \neq \emptyset$, how many connected components?
- Understand adjacencies



4. Open problems about sextics with simple points

- If $\Sigma(\Gamma) \neq \emptyset$, how many connected components?
- Understand adjacencies



Skip

Consider $\Sigma(\mathbb{A}_{15}, \mathbb{A}_3; 6) \setminus \Sigma^{\mathsf{irr}}(\mathbb{A}_{15}, \mathbb{A}_3; 6)$

Consider $\Sigma(\mathbb{A}_{15}, \mathbb{A}_3; 6) \setminus \Sigma^{\mathrm{irr}}(\mathbb{A}_{15}, \mathbb{A}_3; 6)$

$\Sigma_1(\mathbb{A}_{15}, \mathbb{A}_3; 6)$

$\Sigma_2(\mathbb{A}_{15}, \mathbb{A}_3; 6)$

$\Sigma_3(\mathbb{A}_{15}, \mathbb{A}_3; 6)$

- Σ_1: tangent line at $\mathbb{A}_{15}$ pass through $\mathbb{A}_3$
- Σ_2: *generic*
- Σ_3: 4-fold tangent conic to $\mathbb{A}_{15}$ is tangent at $\mathbb{A}_3$

Consider $\Sigma(\mathbb{A}_{15}, \mathbb{A}_3; 6) \setminus \Sigma^{\mathrm{irr}}(\mathbb{A}_{15}, \mathbb{A}_3; 6)$

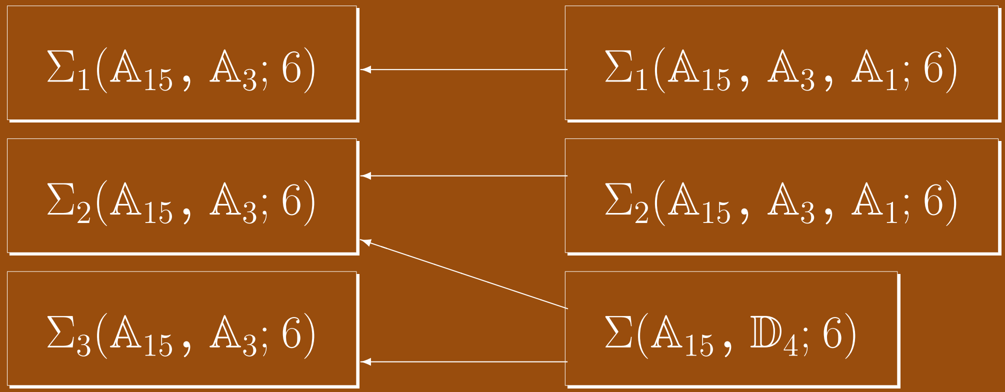
$\Sigma_1(\mathbb{A}_{15}, \mathbb{A}_3; 6) \leftarrow \Sigma_1(\mathbb{A}_{15}, \mathbb{A}_3, \mathbb{A}_1; 6)$

$\Sigma_2(\mathbb{A}_{15}, \mathbb{A}_3; 6) \leftarrow \Sigma_2(\mathbb{A}_{15}, \mathbb{A}_3, \mathbb{A}_1; 6)$

$\Sigma_3(\mathbb{A}_{15}, \mathbb{A}_3; 6)$

- Σ_1: tangent line at $\mathbb{A}_{15}$ pass through $\mathbb{A}_3$
- Σ_2: *generic*
- Σ_3: 4-fold tangent conic to $\mathbb{A}_{15}$ is tangent at $\mathbb{A}_3$

Consider $\Sigma(\mathbb{A}_{15}, \mathbb{A}_3; 6) \setminus \Sigma^{\mathrm{irr}}(\mathbb{A}_{15}, \mathbb{A}_3; 6)$



- Σ_1: tangent line at $\mathbb{A}_{15}$ pass through $\mathbb{A}_3$
- Σ_2: *generic*
- Σ_3: 4-fold tangent conic to $\mathbb{A}_{15}$ is tangent at $\mathbb{A}_3$

$$\Sigma^{\mathsf{tor,irr}}(\mathbb{A}_{17},\, \mathbb{A}_1; 6)$$

$$\Sigma_1^{\mathsf{irr}}(\mathbb{A}_{17},\, \mathbb{A}_1; 6)$$

$$\Sigma^{\mathsf{tor,irr}}(\mathbb{A}_{17}, \mathbb{A}_1; 6) \longleftarrow \Sigma(\mathbb{A}_{17}, \mathbb{A}_2; 6)$$

$$\Sigma_1^{\mathsf{irr}}(\mathbb{A}_{17}, \mathbb{A}_1; 6)$$

$\Sigma^{\text{tor,irr}}(\mathbb{A}_{17}, \mathbb{A}_1; 6) \longleftarrow \Sigma(\mathbb{A}_{17}, \mathbb{A}_2; 6)$

$\Sigma_1^{\text{irr}}(\mathbb{A}_{17}, \mathbb{A}_1; 6) \longleftarrow \Sigma_i(\mathbb{A}_{18}, \mathbb{A}_1; 6)$ $\quad i = 1, 2, 3$

$$\Sigma^{\mathsf{tor,irr}}(\mathbb{A}_{17},\, \mathbb{A}_1; 6) \longleftarrow \Sigma(\mathbb{A}_{17},\, \mathbb{A}_2; 6)$$

$$\Sigma_1^{\mathsf{irr}}(\mathbb{A}_{17},\, \mathbb{A}_1; 6) \longleftarrow \Sigma_i(\mathbb{A}_{18},\, \mathbb{A}_1; 6) \qquad i = 1,\, 2,\, 3$$

$$\Sigma_1^{\mathsf{irr}}(\mathbb{A}_{17},\, \mathbb{A}_1; 6) \longleftarrow \Sigma_j(\mathbb{A}_{19}; 6) \qquad j = 1,\, 2$$

$$\Sigma^{\mathrm{tor,irr}}(\mathbb{A}_{17}, \mathbb{A}_1; 6) \longleftarrow \Sigma(\mathbb{A}_{17}, \mathbb{A}_2; 6)$$

[ACC02b]

$$\Sigma_1^{\mathrm{irr}}(\mathbb{A}_{17}, \mathbb{A}_1; 6) \longleftarrow \Sigma_i(\mathbb{A}_{18}, \mathbb{A}_1; 6) \quad i = 1, 2, 3$$

$$\Sigma_1^{\mathrm{irr}}(\mathbb{A}_{17}, \mathbb{A}_1; 6) \longleftarrow \Sigma_j(\mathbb{A}_{19}; 6) \quad j = 1, 2$$

$\Sigma^{\text{tor,irr}}(\mathbb{A}_{17}, \mathbb{A}_1; 6) \leftarrow \Sigma(\mathbb{A}_{17}, \mathbb{A}_2; 6)$ [ACC02b]

$\Sigma_1^{\text{irr}}(\mathbb{A}_{17}, \mathbb{A}_1; 6) \leftarrow \Sigma_i(\mathbb{A}_{18}, \mathbb{A}_1; 6)$ $\quad i = 1, 2, 3$

$\Sigma_1^{\text{irr}}(\mathbb{A}_{17}, \mathbb{A}_1; 6) \leftarrow \Sigma_j(\mathbb{A}_{19}; 6)$ $\quad j = 1, 2$

- $\Sigma_i(\mathbb{A}_{18}, \mathbb{A}_1; 6)$: $\exists$ conjugate representatives with coefficents in $\mathbb{Q}(19s^3 + 50s^2 + 36s + 8)$
- $\Sigma_j(\mathbb{A}_{19}; 6)$: $\exists$ conjugate representatives in $\mathbb{Q}(\sqrt{5})$ (see [YOS79] for a more complicated extension)

- In Yang's list for $\mu(\mathcal{C}) = 19$, a lot of such examples appear

- In Yang's list for $\mu(\mathcal{C}) = 19$, a lot of such examples appear
- Many topological invariants come from algebraic properties

- In Yang's list for $\mu(\mathcal{C}) = 19$, a lot of such examples appear
- Many topological invariants come from algebraic properties
- Look for other invariants

5. Braid monodromy for affine curves

$\mathcal{C}^{\mathsf{aff}} := \{ f(x, y) = 0 \} \subset \mathbb{C}^2$ *horizontal of degree d*:

5. Braid monodromy for affine curves

$\mathcal{C}^{\mathsf{aff}} := \{f(x, y) = 0\} \subset \mathbb{C}^2$ *horizontal of degree d*:

$$f(x, y) = y^d + f_1(x)y^{d-1} + \cdots + f_{d-1}(x)y + f_d(x),$$

$$f_j(x) \in \mathbb{C}[x], \;\; j = 1, \ldots, d.$$

5. Braid monodromy for affine curves

$\mathcal{C}^{\mathsf{aff}} := \{f(x, y) = 0\} \subset \mathbb{C}^2$ *horizontal of degree* d:

$$f(x, y) = y^d + f_1(x)y^{d-1} + \cdots + f_{d-1}(x)y + f_d(x),$$

$$f_j(x) \in \mathbb{C}[x], \; j = 1, \ldots, d.$$

- $D(x) := \mathrm{Disc}_y(f(x, y))$
- $\mathscr{D} := \{x \in \mathbb{C} \mid D(x) = 0\} = \{x_1, \ldots, x_r\}$

5. Braid monodromy for affine curves

$\mathcal{C}^{\mathsf{aff}} := \{f(x, y) = 0\} \subset \mathbb{C}^2$ *horizontal of degree d*:

$$f(x, y) = y^d + f_1(x)y^{d-1} + \cdots + f_{d-1}(x)y + f_d(x),$$
$$f_j(x) \in \mathbb{C}[x], \quad j = 1, \ldots, d.$$

- $D(x) := \operatorname{Disc}_y(f(x, y))$
- $\mathscr{D} := \{x \in \mathbb{C} \mid D(x) = 0\} = \{x_1, \ldots, x_r\}$
- $V := \{p(t) \in \mathbb{C}[t] \mid p$ monic of degree $d\}$, D discriminant hypersurface
- $V \setminus D \equiv \{A \subset \mathbb{C} \mid \#A = d\}$

$$\begin{aligned} \tilde{f} : \mathbb{C} \setminus \mathscr{D} &\longrightarrow V \setminus D \\ x &\longmapsto f(x, t) \end{aligned}$$

$$* := R \text{ s. t. } \mathscr{D} \subset \{z \in \mathbb{C} \mid |z| < R\}, \ \mathbf{y}^* := \tilde{f}(*)$$

$$\begin{array}{rcl} \tilde{f} : \mathbb{C} \setminus \mathscr{D} & \longrightarrow & V \setminus D \\ x & \longmapsto & f(x, t) \end{array}$$

$$* := R \text{ s. t. } \mathscr{D} \subset \{z \in \mathbb{C} \mid |z| < R\}, \; \mathbf{y}^* := \tilde{f}(*)$$

Braid monodromy of $\mathcal{C}^{\mathsf{aff}}$:

$$\nabla := \tilde{f}_* : \pi_1(\mathbb{C} \setminus \mathscr{D}; *) \longrightarrow \pi_1(V \setminus D; \mathbf{y}^*) \overset{..}{=} B_{\mathbf{y}^*}$$

Geometric bases of the free group $\pi_1(\mathbb{C} \setminus \mathscr{D}; *)$

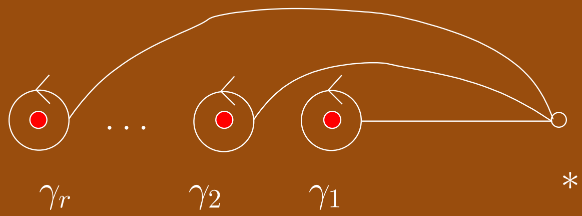


Figure 1: Geometric basis

Geometric bases of the free group $\pi_1(\mathbb{C} \setminus \mathscr{D}; *)$

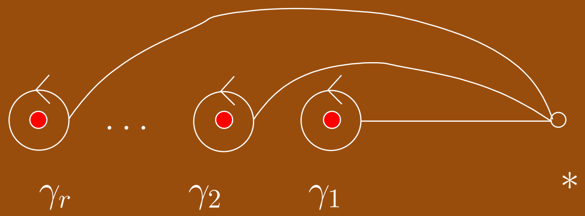


Figure 1: Geometric basis

- Each loop is meridian of a point of $\mathscr{D}$
- $c_\gamma := \gamma_r \cdot \ldots \cdot \gamma_1$ is the boundary of a big geometric disk; c_γ^{-1} is meridian of ∞
- $(\nabla(\gamma_1), \ldots, \nabla(\gamma_r)) \in (B_{\mathbf{y}^*})^r$

$$\mathbf{y}^0 := \{-1, \ldots, -d\}$$

$$B_{\mathbf{y}^0} \equiv B_d := \langle \sigma_1, \ldots, \sigma_{d-1} : [\sigma_i, \sigma_j] = 1, \ |i-j| \geq 2, \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \ i = 1, \ldots, d-2 \rangle$$

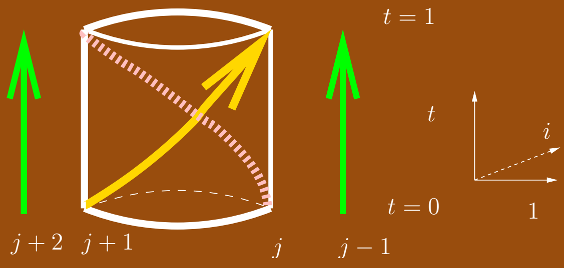


Figure 2: σ_j

- $\tau \in B(\mathbf{y}^*, \mathbf{y}^0)$ braid starting at $\mathbf{y}^*$ and ending at $\mathbf{y}^0$
- $\Phi_\tau : B_{\mathbf{y}^*} \longrightarrow B_d$, $\Phi_\tau(\sigma) := \tau \cdot \sigma \cdot \tau^{-1}$

- $\tau \in B(\mathbf{y}^*, \mathbf{y}^0)$ braid starting at $\mathbf{y}^*$ and ending at $\mathbf{y}^0$
- $\Phi_\tau : B_{\mathbf{y}^*} \longrightarrow B_d$, $\Phi_\tau(\sigma) := \tau \cdot \sigma \cdot \tau^{-1}$
- ∇, $(\gamma_1, \ldots, \gamma_r)$, τ, $\nabla_\tau := \Phi_\tau \circ \nabla$ determine

$$(\nabla_\tau(\gamma_1), \ldots, \nabla_\tau(\gamma_r)) \in (B_d)^r$$

- $\tau \in B(\mathbf{y}^*, \mathbf{y}^0)$ braid starting at $\mathbf{y}^*$ and ending at $\mathbf{y}^0$
- $\Phi_\tau : B_{\mathbf{y}^*} \longrightarrow B_d$, $\Phi_\tau(\sigma) := \tau \cdot \sigma \cdot \tau^{-1}$
- ∇, $(\gamma_1, \ldots, \gamma_r)$, τ, $\nabla_\tau := \Phi_\tau \circ \nabla$ determine

$$(\nabla_\tau(\gamma_1), \ldots, \nabla_\tau(\gamma_r)) \in (B_d)^r$$

Braid monodromy $+ \cdots$

$$\Downarrow$$

An element of $(B_d)^r$

- Choice of geometric basis

- Choice of geometric basis
  - $\mathscr{G} := \{$Geometric bases of $\pi_1(\mathbb{C} \setminus \mathscr{D}; *)\}$

- Choice of geometric basis
  - $\mathscr{G} := \{\text{Geometric bases of } \pi_1(\mathbb{C} \setminus \mathscr{D}; *)\}$
  - Right action of B_r on $\mathscr{G}$:

$$(\gamma_1, \ldots, \gamma_r)^{\sigma_i} :=$$
$$(\gamma_1, \ldots, \gamma_{i-1}, \gamma_{i+1}, \gamma_{i+1}\gamma_i\gamma_{i+1}^{-1}, \gamma_{i+2}, \ldots, \gamma_r)$$

- Choice of geometric basis
  - $\mathscr{G} := \{\text{Geometric bases of } \pi_1(\mathbb{C} \setminus \mathscr{D}; *)\}$
  - Right action of B_r on $\mathscr{G}$:

$$(\gamma_1, \ldots, \gamma_r)^{\sigma_i} := (\gamma_1, \ldots, \gamma_{i-1}, \gamma_{i+1}, \gamma_{i+1}\gamma_i\gamma_{i+1}^{-1}, \gamma_{i+2}, \ldots, \gamma_r)$$

  - It is a free and transitive action, [ARTIN47]

- Choice of geometric basis
  - $\mathscr{G} := \{$Geometric bases of $\pi_1(\mathbb{C} \setminus \mathscr{D}; *)\}$
  - Right action of B_r on $\mathscr{G}$:

$$(\gamma_1, \ldots, \gamma_r)^{\sigma_i} :=$$
$$(\gamma_1, \ldots, \gamma_{i-1}, \gamma_{i+1}, \gamma_{i+1}\gamma_i\gamma_{i+1}^{-1}, \gamma_{i+2}, \ldots, \gamma_r)$$

  - It is a free and transitive action, [ARTIN47]
- Choice of $\tau \in B(\mathbf{y}^*, \mathbf{y}^0)$ and base point $*$

- Choice of geometric basis
  - $\mathscr{G} := \{$Geometric bases of $\pi_1(\mathbb{C} \setminus \mathscr{D}; *)\}$
  - Right action of B_r on $\mathscr{G}$:

$$(\gamma_1, \ldots, \gamma_r)^{\sigma_i} :=$$
$$(\gamma_1, \ldots, \gamma_{i-1}, \gamma_{i+1}, \gamma_{i+1}\gamma_i\gamma_{i+1}^{-1}, \gamma_{i+2}, \ldots, \gamma_r)$$

  - It is a free and transitive action, [ARTIN47]
- Choice of $\tau \in B(\mathbf{y}^*, \mathbf{y}^0)$ and base point $*$
  - Right action of B_d on B_d^r by simultaneous conjugation.

- Choice of geometric basis
  - $\mathscr{G} := \{\text{Geometric bases of } \pi_1(\mathbb{C} \setminus \mathscr{D}; *)\}$
  - Right action of B_r on $\mathscr{G}$:

$$(\gamma_1, \ldots, \gamma_r)^{\sigma_i} := (\gamma_1, \ldots, \gamma_{i-1}, \gamma_{i+1}, \gamma_{i+1}\gamma_i\gamma_{i+1}^{-1}, \gamma_{i+2}, \ldots, \gamma_r)$$

  - It is a free and transitive action, [ARTIN47]
- Choice of $\tau \in B(\mathbf{y}^*, \mathbf{y}^0)$ and base point $*$
  - Right action of B_d on B_d^r by simultaneous conjugation.
  - Pseudogeometric basis of $\pi_1(\mathbb{C} \setminus \mathscr{D}; *)$: c_γ^{-1} is a meridian of the line at infinity

Right action of $B_r \times B_d$ on $(B_d)^r$:

Right action of $B_r \times B_d$ on $(B_d)^r$:

- B_r acts by Hurwitz moves.

Right action of $B_r \times B_d$ on $(B_d)^r$:

- B_r acts by Hurwitz moves.
- Both actions commute

Right action of $B_r \times B_d$ on $(B_d)^r$:

- B_r acts by Hurwitz moves.
- Both actions commute

Braid monodromy

$|||$

An element of $B_d^r/(B_r \times B_d)$

Right action of $B_r \times B_d$ on $(B_d)^r$:

- B_r acts by Hurwitz moves.
- Both actions commute

Braid monodromy

$$|||$$

An element of $B_d^r/(B_r \times B_d)$

Braid monodromy does not depend on Jung automorphisms as:

$$(x, y) \mapsto (ax + b, cy + p(x))$$

$a, c \in \mathbb{C}^*$, $b \in \mathbb{C}$, $p(x) \in \mathbb{C}[x]$

6. An example

$$\#\mathscr{M}(\mathbb{E}_6\,,\,\mathbb{A}_7\,,\,\mathbb{A}_3\,,\,\mathbb{A}_2\,,\,\mathbb{A}_1;\,6) = 2$$

6. An example

$\#\mathscr{M}(\mathbb{E}_6, \mathbb{A}_7, \mathbb{A}_3, \mathbb{A}_2, \mathbb{A}_1; 6) = 2$
Representantatives $\mathcal{C}_\beta$, $\beta^2 = 2$, with equations

$$f_\beta(x, y, z) g_\beta(x, y, z) = 0$$

having coefficients in $\mathbb{Q}(\sqrt{2})$

6. An example

$\#\mathscr{M}(\mathbb{E}_6, \mathbb{A}_7, \mathbb{A}_3, \mathbb{A}_2, \mathbb{A}_1; 6) = 2$
Representantatives $\mathcal{C}_\beta$, $\beta^2 = 2$, with equations

$$f_\beta(x, y, z) g_\beta(x, y, z) = 0$$

having coefficients in $\mathbb{Q}(\sqrt{2})$

$$\begin{aligned} f_\beta(x, y, z) :=& y^2 z^3 + (303 - 216\,\beta)\, y z^2 x^2 + \\ & + (-636 + 450\,\beta)\, y z x^3 + \\ & + (-234\,\beta + 331)\, y x^4 + (-18\,\beta + 27)\, z x^4 + \\ & + (18\,\beta - 26)\, x^5, \\ g_\beta(x, y, z) :=& y + \left(\frac{10449}{196} - \frac{3645}{98}\beta\right) z + \\ & + \left(-\frac{432}{7} + \frac{297}{7}\beta\right) x. \end{aligned} \tag{1}$$

- Take affine curves for $z = 1$

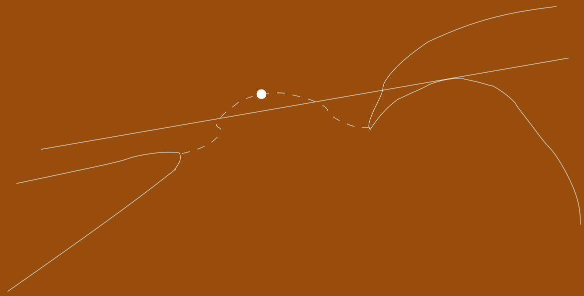
- Take affine curves for $z = 1$
- Line at infinity: tangent line at $\mathbb{E}_6$

- Take affine curves for $z = 1$
- Line at infinity: tangent line at $\mathbb{E}_6$
- $\mathbb{E}_6$, point at infinity of vertical lines

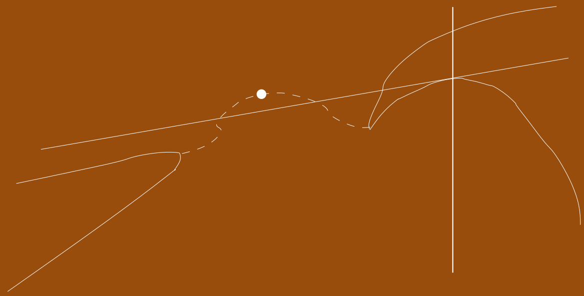
- Take affine curves for $z = 1$
- Line at infinity: tangent line at $\mathbb{E}_6$
- $\mathbb{E}_6$, point at infinity of vertical lines
- Corresponding affine curves of horizontal degree 3

- Take affine curves for $z = 1$
- Line at infinity: tangent line at $\mathbb{E}_6$
- $\mathbb{E}_6$, point at infinity of vertical lines
- Corresponding affine curves of horizontal degree 3
- Corresponding $\mathscr{D}$ are subsets of $\mathbb{R}$

Curve $C_{\sqrt{2}}$



Curve $C_{\sqrt{2}}$



$\gamma_1^{\sqrt{2}} \longmapsto$ σ_2^8

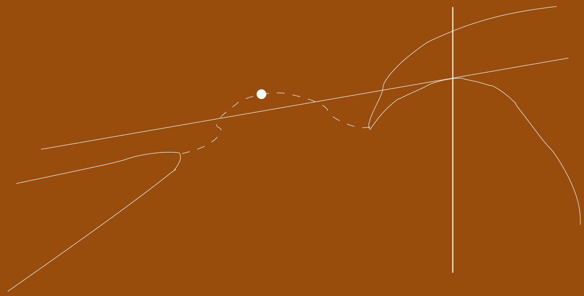
$\gamma_2^{\sqrt{2}} \longmapsto$

$\gamma_3^{\sqrt{2}} \longmapsto$

$\gamma_4^{\sqrt{2}} \longmapsto$

$\gamma_5^{\sqrt{2}} \longmapsto$

Curve $C_{\sqrt{2}}$



$\gamma_1^{\sqrt{2}} \longmapsto \sigma_2^8$

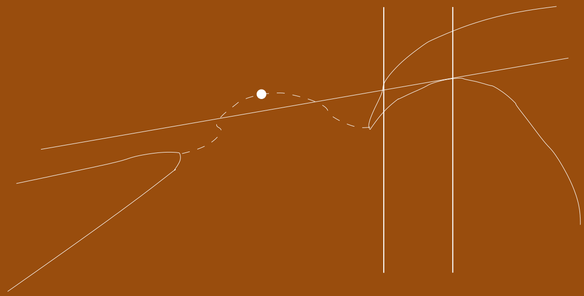
$\gamma_2^{\sqrt{2}} \longmapsto \sigma_2^4$

$\gamma_3^{\sqrt{2}} \longmapsto$

$\gamma_4^{\sqrt{2}} \longmapsto$

$\gamma_5^{\sqrt{2}} \longmapsto$

Curve $C_{\sqrt{2}}$



$\gamma_1^{\sqrt{2}} \longmapsto \sigma_2^8$

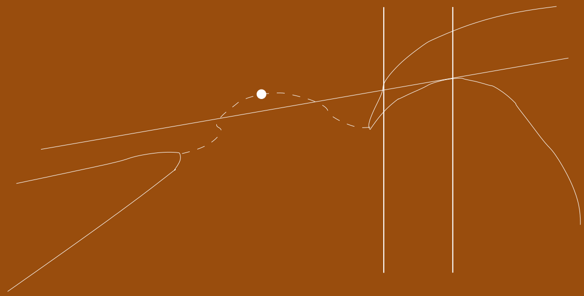
$\gamma_2^{\sqrt{2}} \longmapsto \sigma_2^4 * \sigma_1^2$

$\gamma_3^{\sqrt{2}} \longmapsto$

$\gamma_4^{\sqrt{2}} \longmapsto$

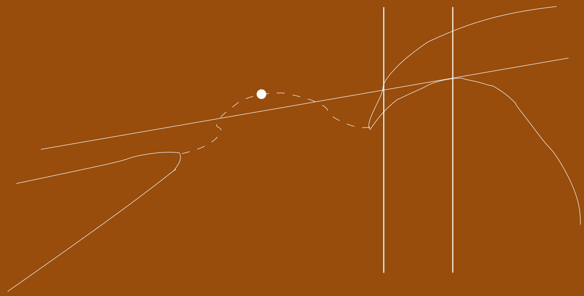
$\gamma_5^{\sqrt{2}} \longmapsto$

Curve $C_{\sqrt{2}}$



$$\gamma_1^{\sqrt{2}} \mapsto \sigma_2^8$$
$$\gamma_2^{\sqrt{2}} \mapsto \sigma_2^4 * \sigma_1^2$$
$$\gamma_3^{\sqrt{2}} \mapsto \sigma_2^4$$
$$\gamma_4^{\sqrt{2}} \mapsto$$
$$\gamma_5^{\sqrt{2}} \mapsto$$

Curve $C_{\sqrt{2}}$



$$\gamma_1^{\sqrt{2}} \longmapsto \sigma_2^{8}$$

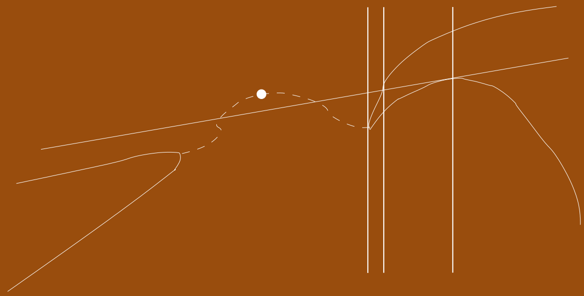
$$\gamma_2^{\sqrt{2}} \longmapsto \sigma_2^{4} * \sigma_1^{2}$$

$$\gamma_3^{\sqrt{2}} \longmapsto \sigma_2^{4}\sigma_1$$

$$\gamma_4^{\sqrt{2}} \longmapsto$$

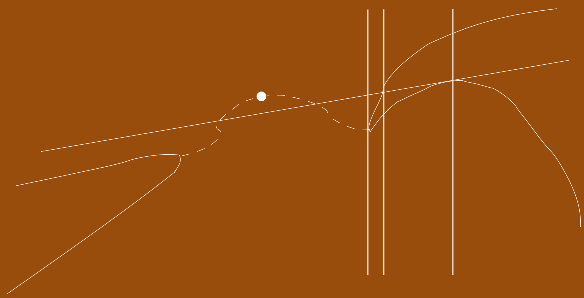
$$\gamma_5^{\sqrt{2}} \longmapsto$$

Curve $C_{\sqrt{2}}$



$$\gamma_1^{\sqrt{2}} \mapsto \sigma_2^8$$
$$\gamma_2^{\sqrt{2}} \mapsto \sigma_2^4 * \sigma_1^2$$
$$\gamma_3^{\sqrt{2}} \mapsto \sigma_2^4\sigma_1 * \sigma_2^3$$
$$\gamma_4^{\sqrt{2}} \mapsto$$
$$\gamma_5^{\sqrt{2}} \mapsto$$

Curve $C_{\sqrt{2}}$



$$\gamma_1^{\sqrt{2}} \mapsto \sigma_2^8$$

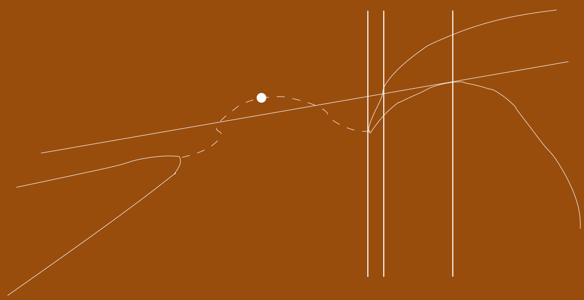
$$\gamma_2^{\sqrt{2}} \mapsto \sigma_2^4 * \sigma_1^2$$

$$\gamma_3^{\sqrt{2}} \mapsto \sigma_2^4\sigma_1 * \sigma_2^3$$

$$\gamma_4^{\sqrt{2}} \mapsto \sigma_2^4\sigma_1$$

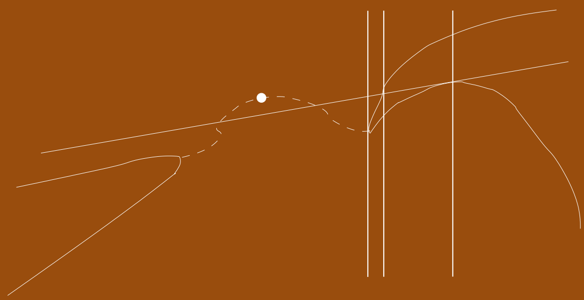
$$\gamma_5^{\sqrt{2}} \mapsto$$

Curve $C_{\sqrt{2}}$



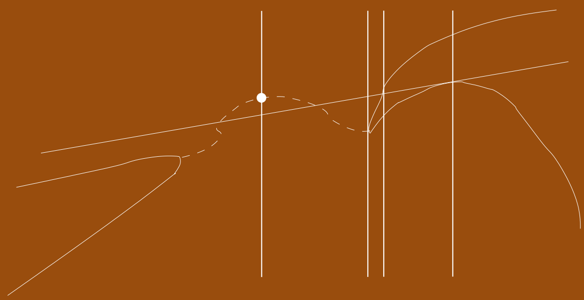
$$\gamma_1^{\sqrt{2}} \mapsto \sigma_2^8$$
$$\gamma_2^{\sqrt{2}} \mapsto \sigma_2^4 * \sigma_1^2$$
$$\gamma_3^{\sqrt{2}} \mapsto \sigma_2^4\sigma_1 * \sigma_2^3$$
$$\gamma_4^{\sqrt{2}} \mapsto \sigma_2^4\sigma_1\sigma_2$$
$$\gamma_5^{\sqrt{2}} \mapsto$$

Curve $C_{\sqrt{2}}$



$$
\begin{aligned}
\gamma_1^{\sqrt{2}} &\mapsto \sigma_2^8 \\
\gamma_2^{\sqrt{2}} &\mapsto \sigma_2^4 * \sigma_1^2 \\
\gamma_3^{\sqrt{2}} &\mapsto \sigma_2^4\sigma_1 * \sigma_2^3 \\
\gamma_4^{\sqrt{2}} &\mapsto \sigma_2^4\sigma_1\sigma_2\sigma_1^{-1}\sigma_2 \\
\gamma_5^{\sqrt{2}} &\mapsto
\end{aligned}
$$

Curve $C_{\sqrt{2}}$



$$\gamma_1^{\sqrt{2}} \mapsto \sigma_2^8$$

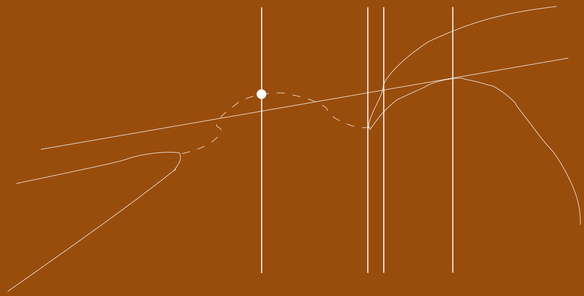
$$\gamma_2^{\sqrt{2}} \mapsto \sigma_2^4 * \sigma_1^2$$

$$\gamma_3^{\sqrt{2}} \mapsto \sigma_2^4\sigma_1 * \sigma_2^3$$

$$\gamma_4^{\sqrt{2}} \mapsto \sigma_2^4\sigma_1\sigma_2\sigma_1^{-1}\sigma_2 * \sigma_1^4$$

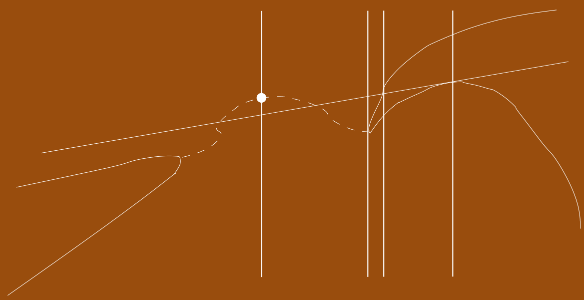
$$\gamma_5^{\sqrt{2}} \mapsto$$

Curve $C_{\sqrt{2}}$



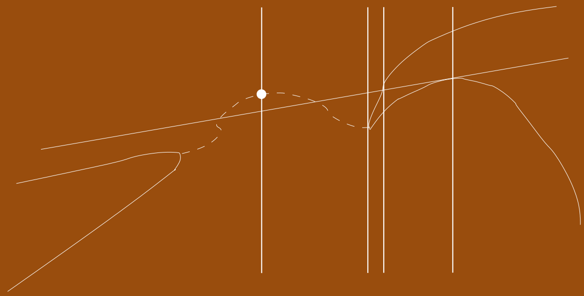
$$\gamma_1^{\sqrt{2}} \longmapsto \sigma_2^8$$
$$\gamma_2^{\sqrt{2}} \longmapsto \sigma_2^4 * \sigma_1^2$$
$$\gamma_3^{\sqrt{2}} \longmapsto \sigma_2^4\sigma_1 * \sigma_2^3$$
$$\gamma_4^{\sqrt{2}} \longmapsto \sigma_2^4\sigma_1\sigma_2\sigma_1^{-1}\sigma_2 * \sigma_1^4$$
$$\gamma_5^{\sqrt{2}} \longmapsto \sigma_2^4\sigma_1\sigma_2\sigma_1^{-1}\sigma_2$$

Curve $C_{\sqrt{2}}$



$$\gamma_1^{\sqrt{2}} \mapsto \sigma_2^8$$
$$\gamma_2^{\sqrt{2}} \mapsto \sigma_2^4 * \sigma_1^2$$
$$\gamma_3^{\sqrt{2}} \mapsto \sigma_2^4\sigma_1 * \sigma_2^3$$
$$\gamma_4^{\sqrt{2}} \mapsto \sigma_2^4\sigma_1\sigma_2\sigma_1^{-1}\sigma_2 * \sigma_1^4$$
$$\gamma_5^{\sqrt{2}} \mapsto \sigma_2^4\sigma_1\sigma_2\sigma_1^{-1}\sigma_2\sigma_1^2$$

Curve $C_{\sqrt{2}}$



$$\gamma_1^{\sqrt{2}} \mapsto \sigma_2^8$$

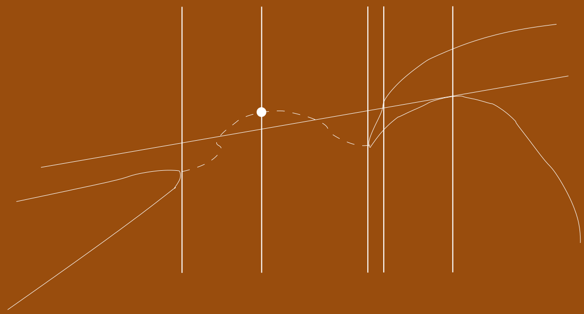
$$\gamma_2^{\sqrt{2}} \mapsto \sigma_2^4 * \sigma_1^2$$

$$\gamma_3^{\sqrt{2}} \mapsto \sigma_2^4\sigma_1 * \sigma_2^3$$

$$\gamma_4^{\sqrt{2}} \mapsto \sigma_2^4\sigma_1\sigma_2\sigma_1^{-1}\sigma_2 * \sigma_1^4$$

$$\gamma_5^{\sqrt{2}} \mapsto \sigma_2^4\sigma_1\sigma_2\sigma_1^{-1}\sigma_2\sigma_1^2\sigma_1\sigma_2^{-1}$$

Curve $C_{\sqrt{2}}$



$$\gamma_1^{\sqrt{2}} \mapsto \sigma_2^8$$

$$\gamma_2^{\sqrt{2}} \mapsto \sigma_2^4 * \sigma_1^2$$

$$\gamma_3^{\sqrt{2}} \mapsto \sigma_2^4\sigma_1 * \sigma_2^3$$

$$\gamma_4^{\sqrt{2}} \mapsto \sigma_2^4\sigma_1\sigma_2\sigma_1^{-1}\sigma_2 * \sigma_1^4$$

$$\gamma_5^{\sqrt{2}} \mapsto \sigma_2^4\sigma_1\sigma_2\sigma_1^{-1}\sigma_2\sigma_1^2\sigma_1\sigma_2^{-1} * \sigma_2$$

Curve $C_{-\sqrt{2}}$

Curve $C_{-\sqrt{2}}$

$$\gamma_1^{-\sqrt{2}} \mapsto \sigma_2^3$$
$$\gamma_2^{-\sqrt{2}} \mapsto \left(\sigma_2\sigma_1^{-1}\sigma_2\right) * \sigma_1$$
$$\gamma_3^{-\sqrt{2}} \mapsto \sigma_2 * \sigma_1^8$$
$$\gamma_4^{-\sqrt{2}} \mapsto \sigma_1^{-2} * \sigma_2^4$$
$$\gamma_5^{-\sqrt{2}} \mapsto \sigma_1^{-3} * \sigma_2^2.$$

7. Braid monodromy of projective curves

- $(\mathcal{C}, L, P)$ triple: $\mathcal{C} \subset \mathbb{P}^2$ projective curve, $L \not\subset \mathcal{C}$ line , $P \in L$

7. Braid monodromy of projective curves

- $(\mathcal{C}, L, P)$ triple: $\mathcal{C} \subset \mathbb{P}^2$ projective curve, $L \not\subset \mathcal{C}$ line , $P \in L$
- Homogeneous coordinates $[x : y : z]$: $L = \{z = 0\}$, $P = [0 : 1 : 0]$
- $\mathbb{C}^2 := \mathbb{P}^2 \setminus L$, affine coordinates (x, y), $\mathcal{C}^{\text{aff}} := \mathcal{C} \cap \mathbb{C}^2$

7. Braid monodromy of projective curves

- $(\mathcal{C}, L, P)$ triple: $\mathcal{C} \subset \mathbb{P}^2$ projective curve, $L \not\subset \mathcal{C}$ line , $P \in L$
- Homogeneous coordinates $[x : y : z]$: $L = \{z = 0\}$, $P = [0 : 1 : 0]$
- $\mathbb{C}^2 := \mathbb{P}^2 \setminus L$, affine coordinates (x, y), $\mathcal{C}^{\text{aff}} := \mathcal{C} \cap \mathbb{C}^2$
- $(\mathcal{C}, L, P)$ is *horizontal of degree d* if $\mathcal{C}^{\text{aff}}$ is
- Braid monodromy of $(\mathcal{C}, L, P)$: the one of $\mathcal{C}^{\text{aff}}$

7. Braid monodromy of projective curves

- $(\mathcal{C}, L, P)$ triple: $\mathcal{C} \subset \mathbb{P}^2$ projective curve, $L \not\subset \mathcal{C}$ line , $P \in L$
- Homogeneous coordinates $[x : y : z]$: $L = \{z = 0\}$, $P = [0 : 1 : 0]$
- $\mathbb{C}^2 := \mathbb{P}^2 \setminus L$, affine coordinates (x, y), $\mathcal{C}^{\text{aff}} := \mathcal{C} \cap \mathbb{C}^2$
- $(\mathcal{C}, L, P)$ is *horizontal of degree d* if $\mathcal{C}^{\text{aff}}$ is
- Braid monodromy of $(\mathcal{C}, L, P)$: the one of $\mathcal{C}^{\text{aff}}$
- Classic case: generic choice of L and P
- Generalization because of computing reasons: use real curves, or programs (Carmona, Bessis)

In the example,

- P singular point $\mathbb{E}_6$
- L tangent line at P

In the example,

- P singular point $\mathbb{E}_6$
- L tangent line at P

Theorem 1 ([ACC02a]). *Braid monodromies of the triples* $(\mathcal{C}_{\sqrt{2}}, L, P)$ *and* $(\mathcal{C}_{-\sqrt{2}}, L, P)$ *are not equivalent*

In the example,

- P singular point $\mathbb{E}_6$
- L tangent line at P

Theorem 1 ([ACC02a]). *Braid monodromies of the triples* $(\mathcal{C}_{\sqrt{2}}, L, P)$ *and* $(\mathcal{C}_{-\sqrt{2}}, L, P)$ *are not equivalent*

Look for topological consequences

- Zariski-Van Kampen theorem [ZAR29] [VK33]: fundamental group of the complement of the curve (braid monodromy appears implicitely)

- Zariski-Van Kampen theorem [ZAR29] [VK33]: fundamental group of the complement of the curve (braid monodromy appears implicitely)
- Explicited by O. Chisini (1937) [CHI37]: fascio charatteristico

- Zariski-Van Kampen theorem [ZAR29] [VK33]: fundamental group of the complement of the curve (braid monodromy appears implicitely)
- Explicited by O. Chisini (1937) [CHI37]: fascio charatteristico
- Developed by B. Moishezon (1981) [MOI81] and in a series of papers with M. Teicher, [MoTeI] a [MoTeV]

- A. Libgober (1986) [LIB86]: homotopy type of the complement of the curve

- A. Libgober (1986) [LIB86]: homotopy type of the complement of the curve
- V. Kulikov, M. Teicher (2000) [KT00]: embedding of the curve in the projective plane (generic case and the curve only has ordinary nodes y cusps)

- A. Libgober (1986) [LIB86]: homotopy type of the complement of the curve
- V. Kulikov, M. Teicher (2000) [KT00]: embedding of the curve in the projective plane (generic case and the curve only has ordinary nodes y cusps)
- J. Carmona (2002) [CAR02]: Same result without the restrictions on the types of singularities

$$\mathcal{C}^{\varphi} := \mathcal{C} \cup \bigcup_{j=1}^{r} L_i, \; L_i := \{x = x_i z\},$$ *fibered curve*

$$\mathcal{C}^{\varphi} := \mathcal{C} \cup \bigcup_{j=1}^{r} L_i,\ L_i := \{x = x_i z\},$$ *fibered curve*

Theorem 2 ([ACC02]).

- $\mathcal{C}_1, \mathcal{C}_2 \subset \mathbb{P}^2$ *curves,* $L \not\subset \mathcal{C}_1 \cup \mathcal{C}_2$

$$\mathcal{C}^{\varphi} := \mathcal{C} \cup \bigcup_{j=1}^{r} L_i,\ L_i := \{x = x_i z\},$$ *fibered curve*

Theorem 2 ([ACC02]).

- $\mathcal{C}_1, \mathcal{C}_2 \subset \mathbb{P}^2$ *curves,* $L \not\subset \mathcal{C}_1 \cup \mathcal{C}_2$
- $P \in L$ *such that* $(\mathcal{C}_1, L, P)$ *and* $(\mathcal{C}_2, L, P)$ *are horizontal triples of the same degree*

$$\mathcal{C}^{\varphi} := \mathcal{C} \cup \bigcup_{j=1}^{r} L_i,\ L_i := \{x = x_i z\},$$ *fibered curve*

Theorem 2 ([ACC02]).

- $\mathcal{C}_1, \mathcal{C}_2 \subset \mathbb{P}^2$ *curves*, $L \not\subset \mathcal{C}_1 \cup \mathcal{C}_2$
- $P \in L$ *such that* $(\mathcal{C}_1, L, P)$ *and* $(\mathcal{C}_2, L, P)$ *are horizontal triples of the same degree*

$F : \mathbb{P}^2 \longrightarrow \mathbb{P}^2$ *orientation-preserving homeomorphism*

$$\mathcal{C}^{\varphi} := \mathcal{C} \cup \bigcup_{j=1}^{r} L_i,\ L_i := \{x = x_i z\},$$ *fibered curve*

Theorem 2 ([ACC02]).

- $\mathcal{C}_1, \mathcal{C}_2 \subset \mathbb{P}^2$ *curves,* $L \not\subset \mathcal{C}_1 \cup \mathcal{C}_2$
- $P \in L$ *such that* $(\mathcal{C}_1, L, P)$ *and* $(\mathcal{C}_2, L, P)$ *are horizontal triples of the same degree*

$F : \mathbb{P}^2 \longrightarrow \mathbb{P}^2$ *orientation-preserving homeomorphism*

(i) $F(P) = P$, $F(L) = L$ *preserving orientations*

$$\mathcal{C}^{\varphi} := \mathcal{C} \cup \bigcup_{j=1}^{r} L_i,\ L_i := \{x = x_i z\},$$ *fibered curve*

Theorem 2 ([ACC02]).

- $\mathcal{C}_1, \mathcal{C}_2 \subset \mathbb{P}^2$ *curves,* $L \not\subset \mathcal{C}_1 \cup \mathcal{C}_2$
- $P \in L$ *such that* $(\mathcal{C}_1, L, P)$ *and* $(\mathcal{C}_2, L, P)$ *are horizontal triples of the same degree*

$F : \mathbb{P}^2 \longrightarrow \mathbb{P}^2$ *orientation-preserving homeomorphism*

(i) $F(P) = P$, $F(L) = L$ *preserving orientations*

(ii) $F(\mathcal{C}_1^{\varphi}) = \mathcal{C}_2^{\varphi}$ *preserving orientations.*

$$\mathcal{C}^{\varphi} := \mathcal{C} \cup \bigcup_{j=1}^{r} L_i,\ L_i := \{x = x_i z\},$$ *fibered curve*

Theorem 2 ([ACC02]).

- $\mathcal{C}_1, \mathcal{C}_2 \subset \mathbb{P}^2$ *curves,* $L \not\subset \mathcal{C}_1 \cup \mathcal{C}_2$
- $P \in L$ *such that* $(\mathcal{C}_1, L, P)$ *and* $(\mathcal{C}_2, L, P)$ *are horizontal triples of the same degree*

$F : \mathbb{P}^2 \longrightarrow \mathbb{P}^2$ *orientation-preserving homeomorphism*

(i) $F(P) = P$, $F(L) = L$ *preserving orientations*

(ii) $F(\mathcal{C}_1^{\varphi}) = \mathcal{C}_2^{\varphi}$ *preserving orientations.*

Then, braid monodromies of the triples are equal.

Corollary 3. *$\mathcal{C}^{\varphi}_{\sqrt{2}} \cup L$ and $\mathcal{C}^{\varphi}_{-\sqrt{2}} \cup L$ are non-homeomorphic curves, conjugated in $\mathbb{Q}(\sqrt{2})$*

Sketch of the proof of Theorem 2   Skip

Sketch of the proof of Theorem 2 Skip

$\pi : \mathbb{C}^2 \setminus \mathcal{C}^\varphi \longrightarrow \mathbb{C} \setminus \mathscr{D}$, $\pi(x, y) := x$ locally trivial fiber bundle with fiber $\mathbb{C} \setminus \{d$ points$\}$

Sketch of the proof of Theorem 2 Skip

$\pi : \mathbb{C}^2 \setminus \mathcal{C}^{\varphi} \longrightarrow \mathbb{C} \setminus \mathscr{D}$, $\pi(x, y) := x$ locally trivial fiber bundle with fiber $\mathbb{C} \setminus \{d \text{ points}\}$

Long exact sequence of homotopy

$$1 \longrightarrow \pi_1(\mathbb{C} \setminus \mathbf{y}^*; M) \longrightarrow \pi_1(\mathbb{C}^2 \setminus \mathcal{C}^{\varphi}; (*, M)) \xrightarrow{\pi_*} \pi_1(\mathbb{C}_{\mathcal{C}}; *) \longrightarrow 1 \tag{2}$$

Sketch of the proof of Theorem 2 Skip

$\pi : \mathbb{C}^2 \setminus \mathcal{C}^\varphi \longrightarrow \mathbb{C} \setminus \mathscr{D}$, $\pi(x, y) := x$ locally trivial fiber bundle with fiber $\mathbb{C} \setminus \{d$ points$\}$

Long exact sequence of homotopy

$$1 \longrightarrow \pi_1(\mathbb{C} \setminus \mathbf{y}^*; M) \longrightarrow \pi_1(\mathbb{C}^2 \setminus \mathcal{C}^\varphi; (*, M)) \xrightarrow{\pi_*} \pi_1(\mathbb{C}_{\mathcal{C}}; *) \longrightarrow 1 \qquad (2)$$

Look for a presentation

- $M \gg 0$ such that $f(x, y) = 0$ and $|x| \leq R$ $\Rightarrow |y| < M$

- $M \gg 0$ such that $f(x, y) = 0$ and $|x| \leq R$ $\Rightarrow |y| < M$
- Given $\tau \in B(\mathbf{y}^*, \mathbf{y}^0)$ we have an automorphism $\Psi_\tau : \pi_1(\mathbb{C} \setminus \mathbf{y}^*; M) \longrightarrow \pi_1(\mathbb{C} \setminus \mathbf{y}^0; M)$; the standard basis $\mu_1^0, \ldots, \mu_d^0$ of $\pi_1(\mathbb{C} \setminus \mathbf{y}^0; M)$, see Figure 6, determines a geometric basis $\mu_1, \ldots, \mu_d$ of $\pi_1(\mathbb{C} \setminus \mathbf{y}^*; M)$, $\Psi_\tau(\mu_j) = \mu_j^0$.

- $M \gg 0$ such that $f(x, y) = 0$ and $|x| \leq R$ $\Rightarrow |y| < M$
- Given $\tau \in B(\mathbf{y}^*, \mathbf{y}^0)$ we have an automorphism $\Psi_\tau : \pi_1(\mathbb{C} \setminus \mathbf{y}^*; M) \longrightarrow \pi_1(\mathbb{C} \setminus \mathbf{y}^0; M)$; the standard basis $\mu_1^0, \ldots, \mu_d^0$ of $\pi_1(\mathbb{C} \setminus \mathbf{y}^0; M)$, see Figure 6, determines a geometric basis $\mu_1, \ldots, \mu_d$ of $\pi_1(\mathbb{C} \setminus \mathbf{y}^*; M)$, $\Psi_\tau(\mu_j) = \mu_j^0$.
- Natural right actions of B_d on $\pi_1(\mathbb{C} \setminus \mathbf{y}^0; M)$ and of $B_{\mathbf{y}^*}$ on $\pi_1(\mathbb{C} \setminus \mathbf{y}^*; M)$, see Figure 7

$$\mu_i^{\sigma_i} = \mu_{i+1} \qquad \mu_{i+1}^{\sigma_i} = \mu_{i+1} * \mu_i \qquad a * b := a b a^{-1}$$

■ Actions of $\sigma \in B_{\mathbf{y}^*}$ and $\Phi_\tau(\sigma) \in B_d$

$$\begin{array}{ccc}
\pi_1(\mathbb{C} \setminus \mathbf{y}^*; M) & \stackrel{\sigma}{\longrightarrow} & \pi_1(\mathbb{C} \setminus \mathbf{y}^*; M) \\
\Psi_\tau \downarrow & & \downarrow \Psi_\tau \\
\pi_1(\mathbb{C} \setminus \mathbf{y}^0; M) & \stackrel{\Phi_\tau(\sigma)}{\longrightarrow} & \pi_1(\mathbb{C} \setminus \mathbf{y}^0; M)
\end{array}$$

- Actions of $\sigma \in B_{\mathbf{y}^*}$ and $\Phi_\tau(\sigma) \in B_d$

$$\begin{array}{ccc} \pi_1(\mathbb{C} \setminus \mathbf{y}^*; M) & \xrightarrow{\sigma} & \pi_1(\mathbb{C} \setminus \mathbf{y}^*; M) \\ \Psi_\tau \downarrow & & \downarrow \Psi_\tau \\ \pi_1(\mathbb{C} \setminus \mathbf{y}^0; M) & \xrightarrow{\Phi_\tau(\sigma)} & \pi_1(\mathbb{C} \setminus \mathbf{y}^0; M) \end{array}$$

- Recall (2). Lift a pseudo-geometric basis $\gamma_1, \ldots, \gamma_r$ of $\pi_1(\mathbb{C} \setminus \mathscr{D}; *)$ to $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_r$ in $\mathbb{C} \times \{M\}$, see Figure 8
- $\mu_i^{\tilde{\gamma}_j} = ?$

$$\pi_1(\mathbb{C}^2 \backslash \mathcal{C}^{\varphi}; (*, M)) = \Big\langle \mu_1, \ldots, \mu_d, \tilde{\gamma}_1, \ldots, \tilde{\gamma}_r \; : \; \mu_i^{\tilde{\gamma}_j} = \mu_i^{\nabla(\gamma_j)}, i = 1, \ldots, d, \; j = 1, \ldots, r \Big\rangle \cong$$

$$\Big\langle \mu_1^0, \ldots, \mu_d^0, \tilde{\gamma}_1, \ldots, \tilde{\gamma}_r \; : \; (\mu_i^0)^{\tilde{\gamma}_j} = (\mu_i^0)^{\nabla_\tau(\gamma_j)}, i = 1, \ldots, d, \; j = 1, \ldots, r \Big\rangle \tag{3}$$

- $\nabla_\tau(\gamma_j) \in B_d$ is determined by the presentation

$$\pi_1(\mathbb{C}^2\setminus\mathcal{C}^\varphi;(*,M)) = \Big\langle \mu_1,\ldots,\mu_d,\tilde\gamma_1,\ldots,\tilde\gamma_r \;:\; \mu_i^{\tilde\gamma_j} = \mu_i^{\nabla(\gamma_j)}, i=1,\ldots,d,\; j=1,\ldots,r\Big\rangle \cong \Big\langle \mu_1^0,\ldots,\mu_d^0,\tilde\gamma_1,\ldots,\tilde\gamma_r \;:\; (\mu_i^0)^{\tilde\gamma_j} = (\mu_i^0)^{\nabla_\tau(\gamma_j)}, i=1,\ldots,d,\; j=1,\ldots,r\Big\rangle \tag{3}$$

- $\nabla_\tau(\gamma_j) \in B_d$ is determined by the presentation
- *A priori* these data are not topological invariants
- The goal is to prove that the *oriented topology* of $(\mathcal{C}^\varphi, L, P)$ does determine these data.

Skip

Step 1. *Meridians of $\mathcal{C}$ are determined by the oriented topology of* $(\mathcal{C}^{\varphi}, L, P)$

Step 1. *Meridians of $\mathcal{C}$ are determined by the oriented topology of $(\mathcal{C}^{\varphi}, L, P)$*

Step 2. $K := \pi_1(\mathbb{C} \setminus \mathbf{y}^*; M)$ *is the subgroup generated by the meridians of $\mathcal{C}$. In particular, the short exact sequence (2) does not depend on π_**

Step 1. *Meridians of* $\mathcal{C}$ *are determined by the oriented topology of* $(\mathcal{C}^{\varphi}, L, P)$

Step 2. $K := \pi_1(\mathbb{C}\setminus \mathbf{y}^*; M)$ *is the subgroup generated by the meridians of* $\mathcal{C}$*. In particular, the short exact sequence (2) does not depend on* π_*

Step 3. *Let us choose* $*$ *near one* x_i*; the element* $c := \mu_d \cdot \ldots \cdot \mu_1$ *is well-defined by the oriented topology of* $(\mathcal{C}^{\varphi}, L, P)$

Step 4. *An ordered family* $\hat{\mu}_1, \ldots, \hat{\mu}_d$ *of meridians of* $\mathcal{C}$ *such that* $c = \hat{\mu}_d \cdot \ldots \cdot \hat{\mu}_1$ *is a geometric basis of* K

Step 4. *An ordered family* $\hat{\mu}_1, \ldots, \hat{\mu}_d$ *of meridians of* $\mathcal{C}$ *such that* $c = \hat{\mu}_d \cdot \ldots \cdot \hat{\mu}_1$ *is a geometric basis of* K

Step 5. *The element* $\tilde{\gamma}_j$ *is the unique lift of* $\gamma_j \in H$ *such that:*

Step 4. *An ordered family* $\hat{\mu}_1, \ldots, \hat{\mu}_d$ *of meridians of* $\mathcal{C}$ *such that* $c = \hat{\mu}_d \cdot \ldots \cdot \hat{\mu}_1$ *is a geometric basis of* K

Step 5. *The element* $\tilde{\gamma}_j$ *is the unique lift of* $\gamma_j \in H$ *such that:*

- $\tilde{\gamma}_j$ *is a meridian of the line* $x = x_j z$

Step 4. *An ordered family* $\hat{\mu}_1, \ldots, \hat{\mu}_d$ *of meridians of* $\mathcal{C}$ *such that* $c = \hat{\mu}_d \cdot \ldots \cdot \hat{\mu}_1$ *is a geometric basis of* K

Step 5. *The element* $\tilde{\gamma}_j$ *is the unique lift of* $\gamma_j \in H$ *such that:*

- $\tilde{\gamma}_j$ *is a meridian of the line* $x = x_j z$
- *Conjugation by* $\tilde{\gamma}_j$ *induces on* K *a braid-like automorphism with respect to the family of geometric bases of* K

Step 4. *An ordered family* $\hat{\mu}_1, \ldots, \hat{\mu}_d$ *of meridians of* $\mathcal{C}$ *such that* $c = \hat{\mu}_d \cdot \ldots \cdot \hat{\mu}_1$ *is a geometric basis of* K

Step 5. *The element* $\tilde{\gamma}_j$ *is the unique lift of* $\gamma_j \in H$ *such that:*

- $\tilde{\gamma}_j$ *is a meridian of the line* $x = x_j z$
- *Conjugation by* $\tilde{\gamma}_j$ *induces on* K *a braid-like automorphism with respect to the family of geometric bases of* K

Step 6. *The product* $(\tilde{\gamma}_r \cdot \ldots \cdot \tilde{\gamma}_1)^{-1}$ *is a meridian of the line* L *in* $\pi_1(\mathbb{P}^2 \setminus (L_1 \cup \cdots \cup L_r \cup L); (*, M))$

Sketch of the proof of Corollary 3

Sketch of the proof of Corollary 3

▲ Let us suppose there exists a homeomorphism $\Phi : \mathbb{P}^2 \longrightarrow \mathbb{P}^2$ such that $\Phi(\mathcal{C}^{\varphi}_{\sqrt{2}} \cup L) = \mathcal{C}^{\varphi}_{-\sqrt{2}} \cup L$

Sketch of the proof of Corollary 3

- Let us suppose there exists a homeomorphism $\Phi : \mathbb{P}^2 \longrightarrow \mathbb{P}^2$ such that $\Phi(\mathcal{C}^{\varphi}_{\sqrt{2}} \cup L) = \mathcal{C}^{\varphi}_{-\sqrt{2}} \cup L$
- It is easily seen that $\Phi(P) = P$, $\Phi(L) = L$ and $\Phi(\mathcal{C}^{\varphi}_{\sqrt{2}}) = \mathcal{C}^{\varphi}_{-\sqrt{2}}$

Sketch of the proof of Corollary 3

- Let us suppose there exists a homeomorphism $\Phi : \mathbb{P}^2 \longrightarrow \mathbb{P}^2$ such that $\Phi(\mathcal{C}^{\varphi}_{\sqrt{2}} \cup L) = \mathcal{C}^{\varphi}_{-\sqrt{2}} \cup L$
- It is easily seen that $\Phi(P) = P$, $\Phi(L) = L$ and $\Phi(\mathcal{C}^{\varphi}_{\sqrt{2}}) = \mathcal{C}^{\varphi}_{-\sqrt{2}}$
- By orientation properties of algebraic knots, the homeomorphism Φ preserves the orientation of $\mathbb{P}^2$

Sketch of the proof of Corollary 3

- Let us suppose there exists a homeomorphism $\Phi : \mathbb{P}^2 \longrightarrow \mathbb{P}^2$ such that $\Phi(\mathcal{C}^{\varphi}_{\sqrt{2}} \cup L) = \mathcal{C}^{\varphi}_{-\sqrt{2}} \cup L$
- It is easily seen that $\Phi(P) = P$, $\Phi(L) = L$ and $\Phi(\mathcal{C}^{\varphi}_{\sqrt{2}}) = \mathcal{C}^{\varphi}_{-\sqrt{2}}$
- By orientation properties of algebraic knots, the homeomorphism Φ preserves the orientation of $\mathbb{P}^2$
- Since curves have real equations, eventually applying complex conjugation, we may suppose that Φ preserves the orientations of the quintics in $\mathcal{C}_{\sqrt{2}}$ and $\mathcal{C}_{-\sqrt{2}}$

- From the relationship of intersection and linking numbers, we deduce that Φ preserves the orientations of L, $\mathcal{C}^{\varphi}_{\sqrt{2}}$ and $\mathcal{C}^{\varphi}_{-\sqrt{2}}$

- From the relationship of intersection and linking numbers, we deduce that Φ preserves the orientations of L, $\mathcal{C}^{\varphi}_{\sqrt{2}}$ and $\mathcal{C}^{\varphi}_{-\sqrt{2}}$
- Φ verifies the conditions stated in Theorem 2

- From the relationship of intersection and linking numbers, we deduce that Φ preserves the orientations of L, $\mathcal{C}^{\varphi}_{\sqrt{2}}$ and $\mathcal{C}^{\varphi}_{-\sqrt{2}}$
- Φ verifies the conditions stated in Theorem 2
- Contradiction with Theorem 1

References

[ACC02] E. Artal, J. Carmona and J.I. Cogolludo, *Braid monodromy and topology of plane curves*, accepted in Duke Math. J., 2002.

[ACC02a] E. Artal, J. Carmona and J.I. Cogolludo, *Effective invariants of braid monodromy*, Preprint, 2002.

[ACC02b] E. Artal, J. Carmona, and J. I. Cogolludo, *On sextic curves with big Milnor number*, Trends in Singularities (A. Libgober and M. Tibăr, eds.), Trends in Mathematics, Birkhäuser Verlag Basel/Switzerland, 2002, pp. 1–29.

[ARTIN47] ________, *Theory of braids*, Ann. of Math. (2) **48** (1947), 101–126.

[BPV84] W. Barth, C. Peters, and A. Van de Ven, *Compact complex surfaces*, Erg. der Math. und ihrer Grenz., A Series of Modern Surveys in Math., 3, vol. 4, Springer-Verlag, Berlin, 1984.

[CAR02] J. Carmona, *Ph.D. thesis*.

[CHI37] O. Chisini, *Una suggestiva rappresentazione reale per le curve algebriche piane*, Ist. Lombardo, Rend., II. Ser. **66** (1933), 1141–1155.

[DEG90] A. I. Degtyarёv, *Isotopic classification of complex plane projective curves of degree* 5, Leningrad Math. J. **1** (1990), no. 4, 881–904.

[GLS98] Gert-Martin Greuel, Christoph Lossen and Eugenii Shustin, *New asymptotics in the geometry of equisingular families of curves*, Internat. Math. Res. Notices (1997), no. 13, 595–611. MR 98g:14039

[GLS98a] ______, *Geometry of families of nodal curves on the blown-up projective plane*, Trans. Amer. Math. Soc. **350** (1998), no. 1, 251–274. MR 98j:14034

[GLS99] ______, *Plane curves of minimal degree with prescribed singularities*, Invent. Math. **133** (1998), no. 3, 539–580. MR 99g:14035

[GLS02] ______, *The variety of plane curves with ordinary singularities is not irreducible*, Internat. Math. Res. Notices (2001), no. 11, 543–550. MR 2002e:14042

[GL99] Gert-Martin Greuel and Eugenii Shustin, *Geometry of equisingular families of curves*, Singularity theory (Liverpool, 1996), Cambridge Univ. Press, Cambridge, 1999, pp. xvi, 79–108. MR 2000e:14036

[HAR87] J. Harris, *On the Severi problem*, Invent. Math. **84** (1986), no. 3, 445–461. MR 87f:14012

[VK33] E.R. van Kampen, *On the fundamental group of an algebraic curve*, Amer. J. Math. **55** (1933), 255–260.

[KT00] Vik. S. Kulikov and M. Teicher, *Braid monodromy factorizations and diffeomorphism types*, Izv. Ross. Akad. Nauk Ser. Mat. **64** (2000), no. 2, 89–120.

[LIB86] A. Libgober, *On the homotopy type of the complement to plane algebraic curves*, J. Reine Angew. Math. **367** (1986), 103–114.

[MOI81] B. G. Moishezon, *Stable branch curves and braid monodromies*, L.N.M. 862, Algebraic geometry (Chicago, Ill., 1980), Springer, Berlin, 1981, pp. 107–192.

[MoTeI] B. Moishezon and M. Teicher, *Braid group technique in complex geometry. I. Line arrangements in* $\mathbf{c}\mathrm{p}^2$, Braids (Santa Cruz, CA, 1986), Amer. Math. Soc., Providence, RI, 1988, pp. 425–555. MR 90f:32014

[MoTeII] ______, *Braid group technique in complex geometry. II. From arrangements of lines and conics to cuspidal curves*, Algebraic geometry (Chicago, IL, 1989), Springer, Berlin, 1991, pp. 131–180. MR 93j:32045

[MoTeIII] ______, *Braid group techniques in complex geometry. III. Projective degeneration of* V_3, Classification of algebraic varieties (L'Aquila, 1992), Amer. Math. Soc., Providence, RI, 1994, pp. 313–332. MR 95k:14050

[MoTeIV] ______, *Braid group techniques in complex geometry. IV. Braid monodromy of the branch curve* S_3 *of* $V_3 \longrightarrow \mathbf{C}\mathrm{p}^2$ *and application to* $\pi_1(\mathbf{C}\mathrm{p}^2 - S_3, *)$, Classification of algebraic varieties (L'Aquila, 1992), Amer. Math. Soc., Providence, RI, 1994, pp. 333–358. MR 95k:14051

[MoTeV] ______, *Braid group technique in complex geometry. V. The fundamental group of a complement of a branch curve of a Veronese generic projection*, Comm. Anal. Geom. **4** (1996), no. 1-2, 1–120. MR 97j:14041

[NMB86] M. Namba, *Geometry of projective algebraic curves*, Marcel Dekker Inc., New York, 1984. MR 86d:14021

[SEV21] F. Severi, *Vorlesungen uber algebraische geometrie*, Teubner, Leipzig, 1921.

[SHU97] Eugenii Shustin, *Geometry of equisingular families of plane algebraic curves*, J. Algebraic Geom. **5** (1996), no. 2, 209–234. MR 97g:14025

[SHU97a] ________, *Smoothness of equisingular families of plane algebraic curves*, Internat. Math. Res. Notices (1997), no. 2, 67–82. MR 97j:14031

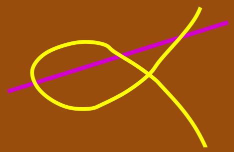
[YA96] J.-G. Yang, *Sextic curves with simple singularities*, Tohoku Math. J. (2) **48** (1996), no. 2, 203–227.

[YOS79] H. Yoshihara, *On plane rational curves*, Proc. Japan Acad. Ser. A Math. Sci. **55** (1979), no. 4, 152–155.

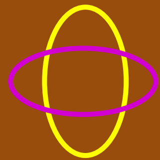
[ZAR29] O. Zariski, *On the problem of existence of algebraic functions of two variables possessing a given branch curve*, Amer. J. Math. **51** (1929), 305–328.

[ZAR31] ________, *On the irregularity of cyclic multiple planes*, Ann. Math. **32** (1931), 445–489.

[ZAR37] ________, *The topological discriminant group of a riemann surface of genus p*, Amer. J. Math. **59** (1937), 335–358.



(a) Nodal cubic and line

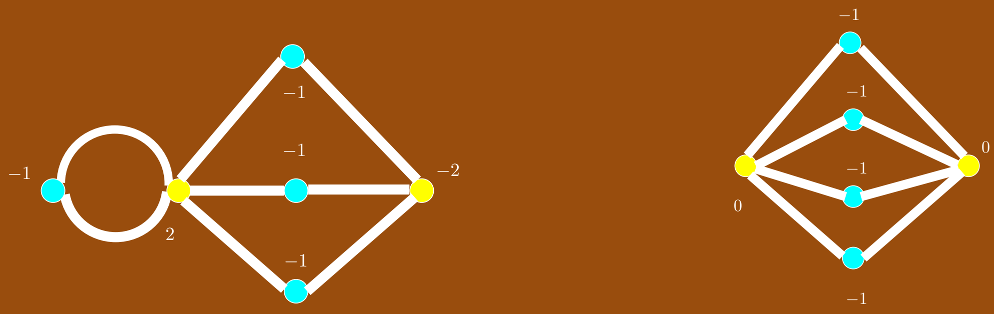


(b) Two conics

Figure 3: $\Sigma(4\mathbb{A}_1; 4)$

Define $\Sigma(\Gamma)$ and $\mathscr{M}(\Gamma)$ where Γ is:

- A weighted bi-coloured graph, which is dual to $\sigma^{-1}(\mathcal{C})$, $\sigma : Y \longrightarrow \mathbb{P}^2$, minimal embedded resolution of $\operatorname{Sing}(\mathcal{C})$.
- Weight $\equiv$ self-intersection number
- Vertices $\alpha \equiv$ exceptional divisor of σ
- Vertices $\beta \equiv$ strict transform of $\mathcal{C}$



(a) Nodal cubic and line

(b) Two conics

Figure 4: Graphs

If $d \leq 5$ and $\Sigma(\Gamma) \neq \emptyset$, $\Sigma(\Gamma)$ is irreducible

Go back

Definition of meridian

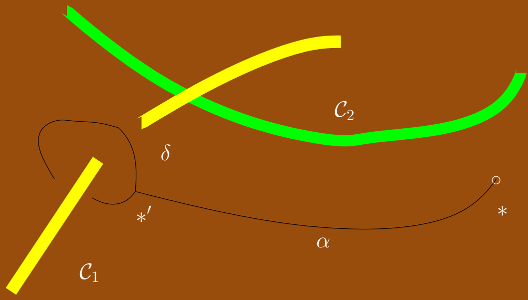


Figure 5: Meridian

- X surface, $\mathcal{C} \subset X$ curve, $\mathcal{C}_1 \subset \mathcal{C}$ irreducible component, $* \in X \setminus \mathcal{C}$, $G := \pi_1(X \setminus \mathcal{C}; *)$

Definition of meridian

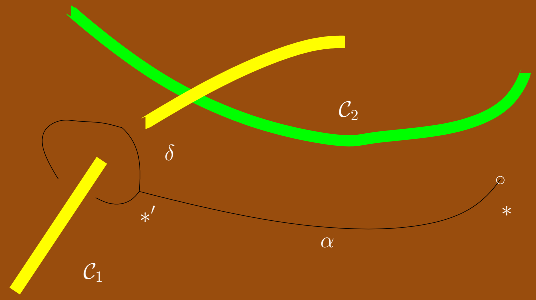


Figure 5: Meridian

- X surface, $\mathcal{C} \subset X$ curve, $\mathcal{C}_1 \subset \mathcal{C}$ irreducible component, $* \in X \setminus \mathcal{C}$, $G := \pi_1(X \setminus \mathcal{C}; *)$
- Δ small analytic disk $\pitchfork \mathcal{C}_1$, $*' \in \partial\Delta$, α path from $*$ to $*'$, δ loop en $*'$ running once and counterclockwise $\partial\Delta$

Definition of meridian

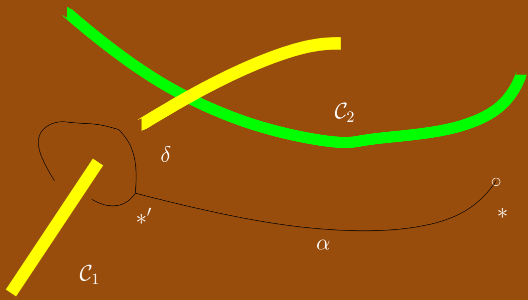


Figure 5: Meridian

- X surface, $\mathcal{C} \subset X$ curve, $\mathcal{C}_1 \subset \mathcal{C}$ irreducible component, $* \in X \setminus \mathcal{C}$, $G := \pi_1(X \setminus \mathcal{C}; *)$
- Δ small analytic disk $\pitchfork \mathcal{C}_1$, $*' \in \partial\Delta$, α path from $*$ to $*'$, δ loop en $*'$ running once and counterclockwise $\partial\Delta$
- $\alpha \cdot \delta \cdot \alpha^{-1}$ is a *meridian* of $\mathcal{C}_1$ in G. The set of meridians of $\mathcal{C}_1$ is a conjugation class. Go back

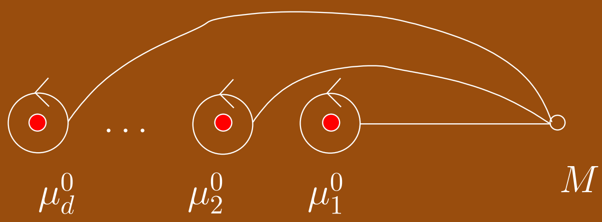


Figure 6: Geometric basis in the fiber

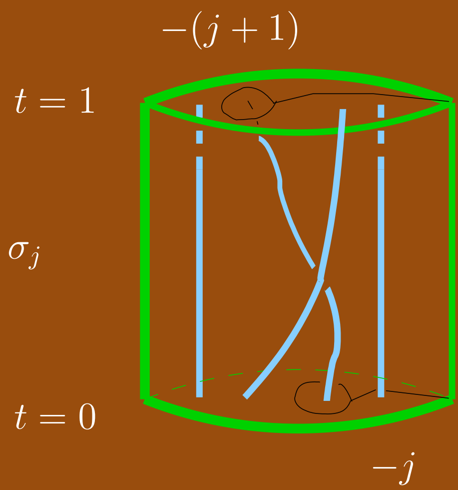


Figure 7: Action of σ_j

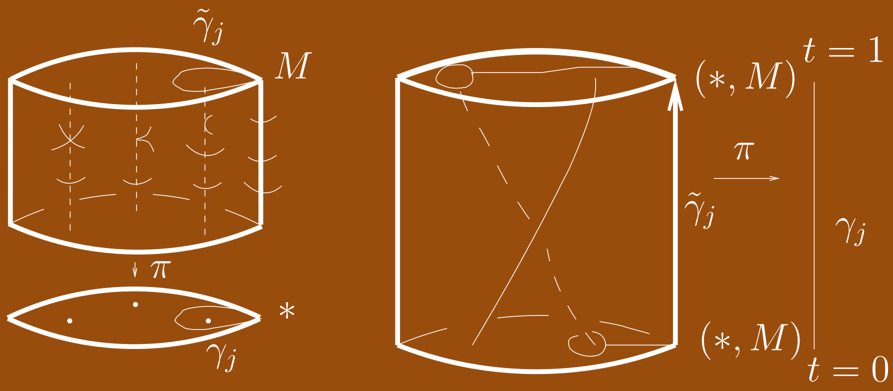


Figure 8: Adapted polydisks and conjugation