Quasi-projectivity of fundamental groups of algebraic links

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Topology and Geometry
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Joint work with J.I. Cogolludo and D. Matei

Questions& Answers

Ş. Papadima, *Global versus local algebraic fundamental groups*, Oberwolfach Rep. **4** (2007), no. 3, 2340–2342.

How many plane curve singularity groups (besides those coming from quasi-homogeneous curves) are global?

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Answer A.-Cogolludo-Matei (Work in progress)

None

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S. Friedl and A.I. Suciu, *Kähler groups, quasi-projective groups and 3-manifold groups.* J. London Math. Soc. (2) **89** (2014) 151–168

If a 3-manifold group is quasi-projective, any prime component is a graph manifold. For which graph manifolds is the fundamental group a quasi-projective group?

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Seifert fibrations (and exceptional cases)

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Seifert fibrations (and exceptional cases)

Goals

- Answer these questions using obstructions from characteristic varieties: Dimca-Papadima-Suci, A-Cogolludo-Matei
- Study other properties of graph manifold groups

Definitions

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Example

If f is quasihomogeneous, the complement of the germ $f^{-1}(0)$ is homeomorphic to a quasi-projective manifold ($\varepsilon = \infty$ basically works).

Graph manifolds and resolution graphs

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- Vertices $v \in \mathcal{V}$ (with weights $([g_v], e_v)$):
  Σ_v oriented surface of genus g_v with val_v boundary components
  $\pi_v : M_v \to \Sigma_v$ oriented $\mathbb{S}^1$-fibration with Euler number e_v
  prescribed sections in the boundary .

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- $\pi : (X, D) \to (\mathbb{C}^2, 0)$ (minimal) embedded resolution of C, $\pi^*(C) = \sum_{j=1}^{r} \hat{C}_j + \sum_{k=1}^{n} m_k D_k$ is SNC.

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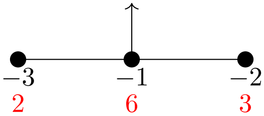
- Γ is the graph of the exterior of the algebraic link (plumbing tubular neighbourhoods)

Examples



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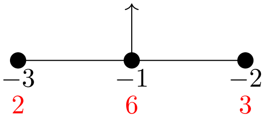

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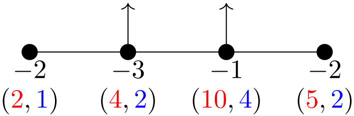

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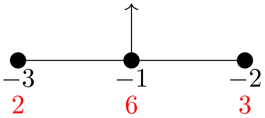

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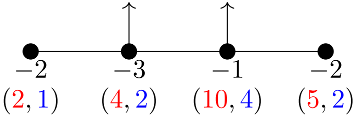

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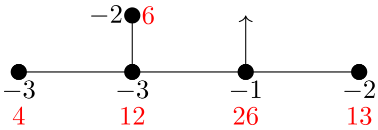

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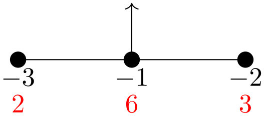

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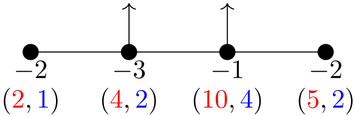

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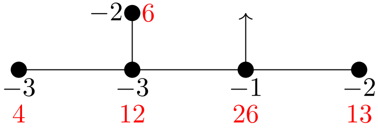

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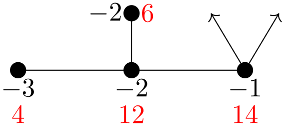

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- $\operatorname{Char}_k(G)$ are algebraic subvarieties defined over $\mathbb{Q}$.

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- $\xi \in \Sigma_1 \cap \Sigma_2$ distinct irreducible components of $\Sigma_{k_1}(G), \Sigma_{k_2}(G)$, respectively, $\Longrightarrow \xi \in \Sigma_{k_1+k_2}(G)$

Consequences

Corollary (Dimca-Papadima-Suciu)

Let $L \subset \mathbb{S}^3$ be a link with $r \geq 3$ connected components. If $\pi_1(\mathbb{S}^3 \setminus L)$ is quasi-projective, then the multivariable Alexander polynomial of L is expressed as $p(u)$ where u is a monomial in $x_1, \ldots, x_r$

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Corollary (A-Cogolludo-Matei)

Let G be a quasi-projective group, $A \subset \mathcal{V}_k(G)$, $B \subset \mathcal{V}_\ell(G)$ such that $k < \ell$ and $B \subset \overline{A}$. Then B is finite.

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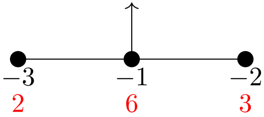
Let G be a quasi-projective group, $A \subset \mathcal{V}_k(G)$, $B \subset \mathcal{V}_\ell(G)$ such that $k < \ell$ and $B \subset \overline{A}$. Then B is finite.

Remark

A finite index subgroup of a quasi-projective group is also quasi-projective.

Alexander polynomials

A'Campo's formula for algebraic links (Eisenbud-Neumann)



$$y^2 - x^3 = 0$$

$$\Delta(t) = \frac{(t^6-1)(t-1)}{(t^2-1)(t^3-1)} = t^2 - t + 1$$

Alexander polynomials

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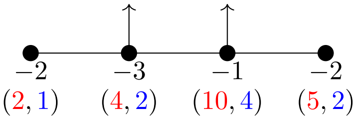

$$(y - x^2)(y^2 - x^5) = 0$$

$$\Delta(t_1, t_2) = \frac{(t_1^4 t_2^2 - 1)(t_1^{10} t_2^4 - 1)}{(t_1^2 t_2 - 1)(t_1^5 t_2^2 - 1)} = (t_1^2 t_2 + 1)(t_1^5 t_2^2 + 1)$$

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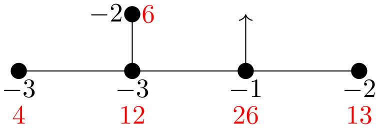

$$(y^2 - x^3)^2 - x^2 y^3 = 0$$

$$\Delta(t) = \frac{(t^{12}-1)(t^{26}-1)(t-1)}{(t^{13}-1)(t^4-1)(t^6-1)} = \frac{t^6+1}{t^2+1} \cdot \frac{t^{13}+1}{t+1}$$

Alexander polynomials

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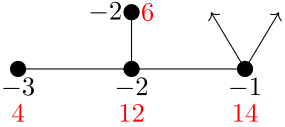
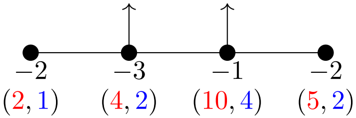


$$(y^2 - x^3)(y^2 - x^3 - x^2 y) = 0$$

$$\Delta(t_1, t_2) = \frac{((t_1t_2)^6 - 1)((t_1t_2)^7 - 1)}{((t_1t_2)^2 - 1)((t_1t_2)^3 - 1)} = ((t_1t_2)^2 - (t_1t_2) + 1)\frac{(t_1t_2)^7 - 1}{(t_1t_2) - 1}$$

A direct computation



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- $G = \pi_1(\mathbb{S}^3 \setminus L) = \langle t, s \mid [t^2, s^2] = 1 \rangle.$

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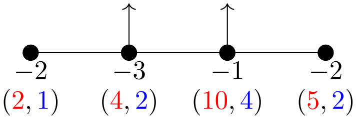

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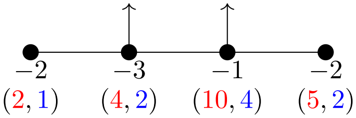

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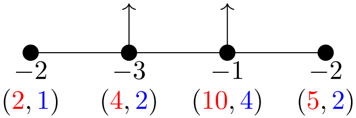

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- G is not quasi-projective.

The results

Theorem (A-Cogolludo-Matei)

Let M be a graph manifold, Γ its plumbing graph (with oriented bases and trivial cocycle). Assume that the intersection matrix of Γ is non-degenerated. If Γ has at least two vertices with positive genus, then $\pi_1(M)$ is not quasiprojective.

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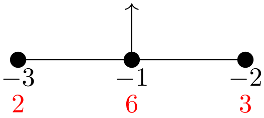
Corollary

If the minimal resolution graph of an algebraic link L has more than one branching vertex then $\pi_1(\mathbb{S}^3 \setminus L)$ is not quasi-projective

Coverings



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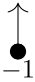
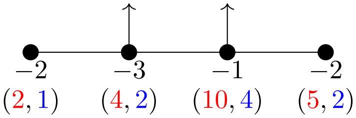
$$\pi_1 : M \to \mathbb{S}^3, \qquad 6:1$$



[1]

Coverings



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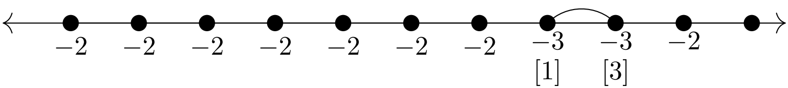
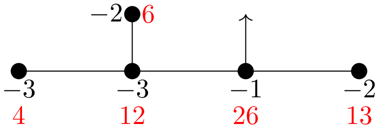


Coverings



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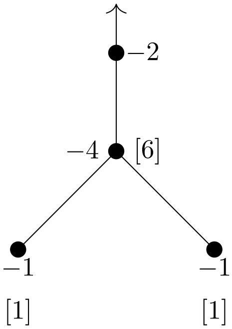

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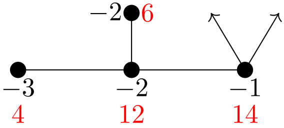

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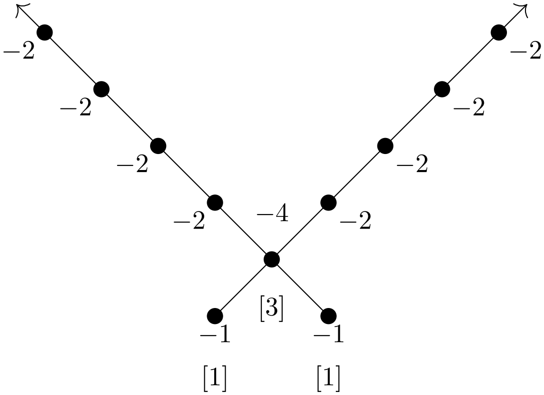

Thanks, Ştefan