

Topology of line arrangements, braids and symmetries

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Artin groups, braids and mapping class groups:
Celebrating the work of Luis París
Cáceres, June 18th 2025

Funded by MCIN/AEI/10.13039/501100011033 (PID2020-114750GB-C31) and Gobierno de Aragón

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- ▶ Hyperplane arrangement: $\mathcal{A} := \{H_1, \dots, H_r\}$ linear hyperplanes in $\mathbb{K}^{n+1}, \mathbb{K}\mathbb{P}^n$ or affine hyperplanes in $X = \mathbb{K}^n$, $\mathbb{K}$ field.



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Line Arrangements (and algebraic curves)

Theorem (Zariski-Lefschetz)

$V \subset \mathbb{P}^n$ hypersurface, H *general* hyperplane, $i : H \setminus V \hookrightarrow \mathbb{P}^n \setminus V$
determines isomorphisms on π_k , $k < n - 1$.



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$V \subset \mathbb{P}^n$ hypersurface, H *general* hyperplane, $i : H \setminus V \hookrightarrow \mathbb{P}^n \setminus V$
determines epimorphism on π_{n-1} .

In particular, for π_1 , it is enough to study $\mathbb{P}^2 \setminus C$ and $\pi_1(L \setminus C)$ (free group) generates.



Line Arrangements (and algebraic curves)

Zariski-van Kampen method

$C = \{f = 0\} \subset \mathbb{C}^2$ curve, $f(x, y) = y^d + \sum_{j=1}^n a_j(x)y^{d-j} \in \mathbb{C}[x, y]$

square-free.

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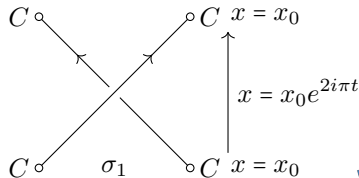
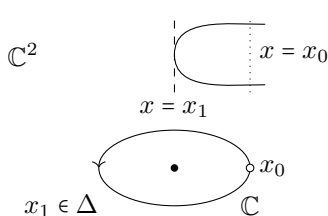
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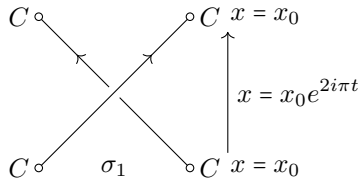
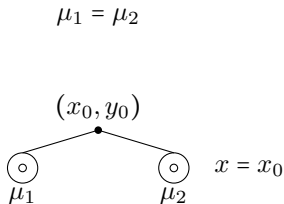
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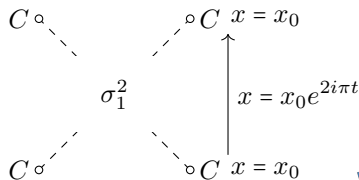
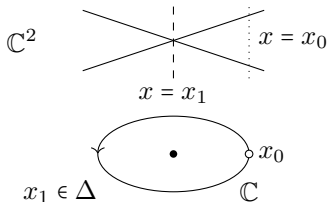
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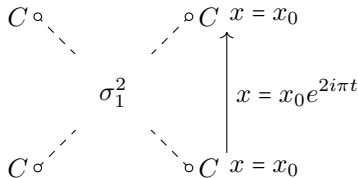
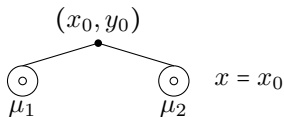
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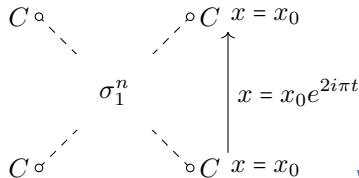
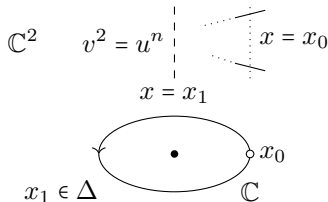
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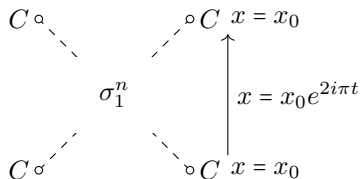
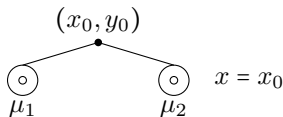
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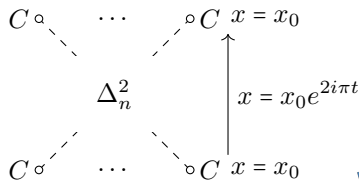
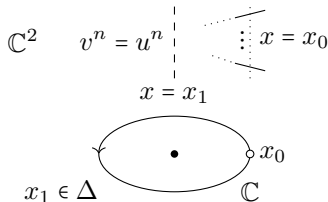
Line Arrangements (and algebraic curves)

Zariski-van Kampen method

$C = \{f = 0\} \subset \mathbb{C}^2$ curve, $f(x, y) = y^d + \sum_{j=1}^n a_j(x)y^{d-j} \in \mathbb{C}[x, y]$

square-free.

- ▶ $\Delta := \{t \in \mathbb{C} \mid \text{disc}_y f(t, y) = 0\} \subset \mathbb{C}$ finite.
- ▶ $x_0 \in \mathbb{C} \setminus \Delta$, $f(x_0, y_0) \neq 0$: a presentation of $\pi_1(\mathbb{C}^2 \setminus C; (x_0, y_0))$ is obtained by *moving* the lines $x = \gamma(t)$, γ x_0 -based loop in $\mathbb{C} \setminus \Delta$.
- ▶ $\overline{C} \subset \mathbb{P}^2$ smooth of degree d , $\pi_1(\mathbb{P}^2 \setminus \overline{C}) \cong \mathbb{Z}/d$.
- ▶ $\mathcal{A}$ generic $(r+1)$ -line arrangement in $\mathbb{P}^2$ (only double points), $\pi_1(\mathbb{P}^2 \setminus \bigcup \mathcal{A}) \cong \mathbb{Z}^r$.
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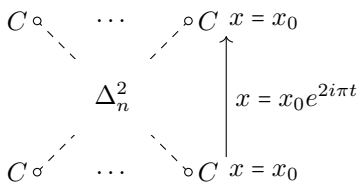
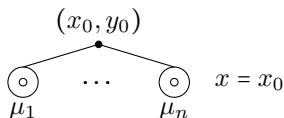
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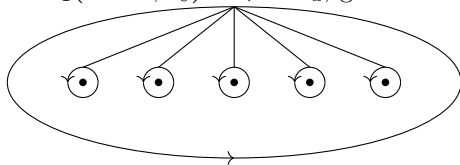
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$$[\mu_1 \dots \mu_n, \mu_i] = 1$$



Braid monodromy

- $\nabla : \pi_1(\mathbb{C} \setminus \Delta; x_0) \cong \mathbb{F}_r \rightarrow \mathbb{B}_d$, geometric basis.



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- ▶ Pseudo-Coxeter element: $c(\boldsymbol{\tau}) := \tau_1 \cdot \dots \cdot \tau_r$.
- ▶ Monodromy group: $M(\boldsymbol{\tau}) := \langle \tau_1, \dots, \tau_r \rangle$.

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- ▶ Line arrangements: replace $\mathbb{B}_d$ by $\mathbb{P}_d$ (pure braid group).
- ▶ Compatible actions of G and $\mathbb{B}_r$ on G^r .



Braid monodromy, fundamental groups and topology

Action of the braid group on the free group

$$\mathbb{F}_d = \langle x_1, \dots, x_d \mid _ \rangle$$

$$\mathbb{F}_d \times \mathbb{B}_d \longrightarrow \mathbb{F}_d, \quad x_i^{\sigma_j} := \begin{cases} x_j x_{j+1} x_j^{-1} & \text{if } i = j, \\ x_j & \text{if } i = j + 1, \\ x_i & \text{otherwise.} \end{cases}$$



Braid monodromy, fundamental groups and topology

Moishezon groups

$$\tau \in (\mathbb{B}_d)^r, c(\tau) = \Delta_d^2.$$

$$G^{\text{aff}} := \langle x_1, \dots, x_d \mid x_i = x_i^{\tau_j} \rangle$$

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- ▶ if $\overline{C} \subset \mathbb{P}^2$, L **generic**, $\mathbb{C}^2 = \mathbb{P}^2 \setminus L$, $C = \overline{C} \setminus L$, and τ comes from C :

$$G^{\mathbb{P}}(\tau) \cong \pi_1(\mathbb{P}^2 \setminus \overline{C}).$$



Braid monodromy, fundamental groups and topology

Theorem (Kulikov-Teicher, Carmona)

If $C_1, C_2 \subset \mathbb{P}^2$ have equivalent braid monodromies there is an orientation preserving homeomorphism $(\mathbb{P}^2, C_1) \rightarrow (\mathbb{P}^2, C_2)$.

Braid monodromy, fundamental groups and topology

Theorem (A_∞-Carmona-Cogolludo)

- ▶ $C_1, C_2 \subset \mathbb{C}^2$ (*y*-monic), τ_i braid monodromies.
- ▶ $\mathcal{A}_i$ arrangements of affine lines containing the vertical lines $\nparallel C_i$
- ▶ There is an orientation preserving homeomorphism

$$\left(\mathbb{P}^2, \overline{C}_1, \bigcup_{L \in \mathcal{A}_1} \overline{L}, L_\infty \right) \rightarrow \left(\mathbb{P}^2, \overline{C}_2, \bigcup_{L \in \mathcal{A}_2} \overline{L}, L_\infty \right)$$

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Variants

One can replace $\mathbb{B}_r$ by the pure braid group in the action of braid monodromies.



Braid monodromy, fundamental groups and topology

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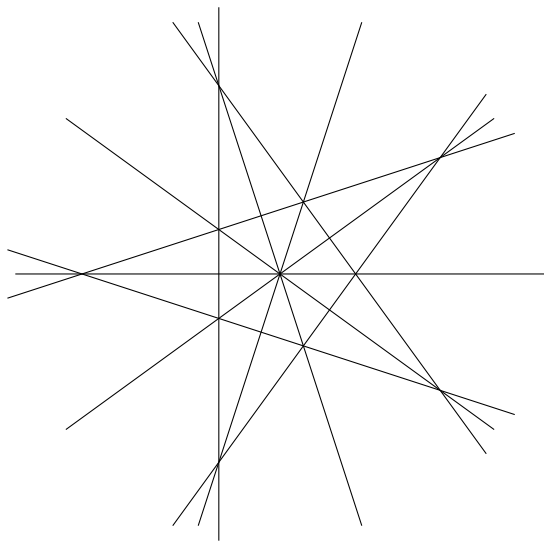
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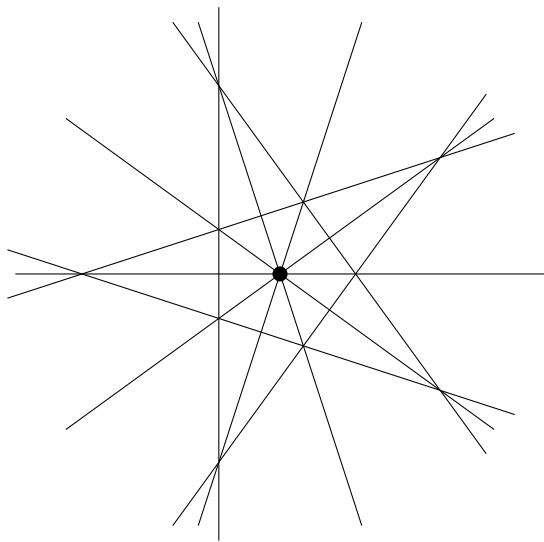
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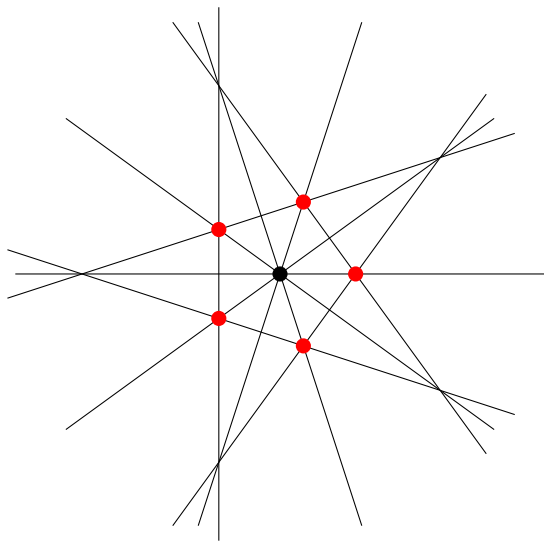
Line arrangements associated to an odd polygon



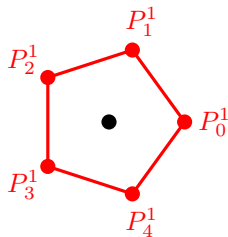
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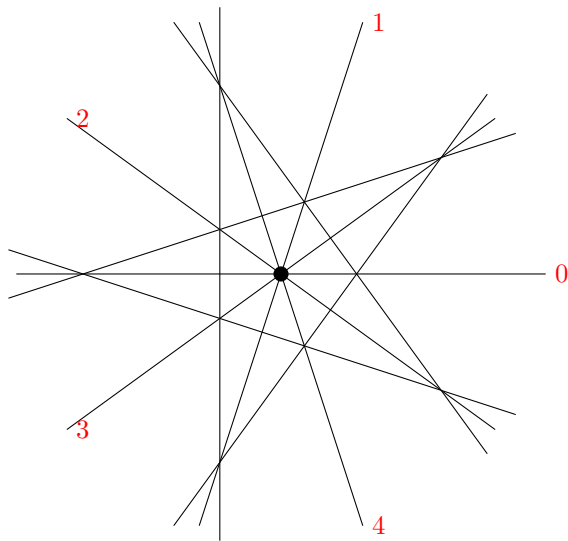
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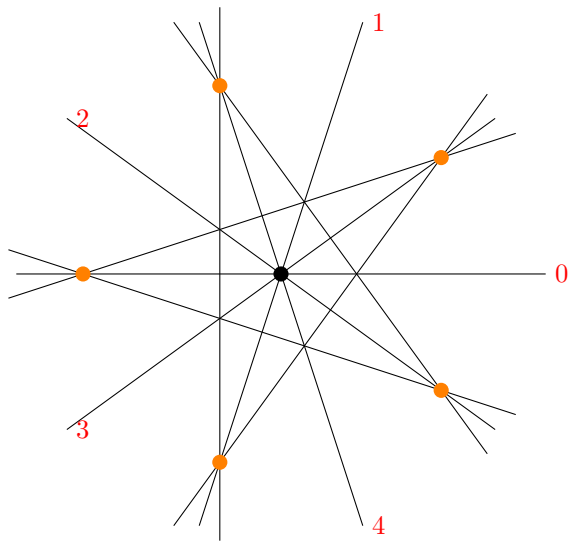
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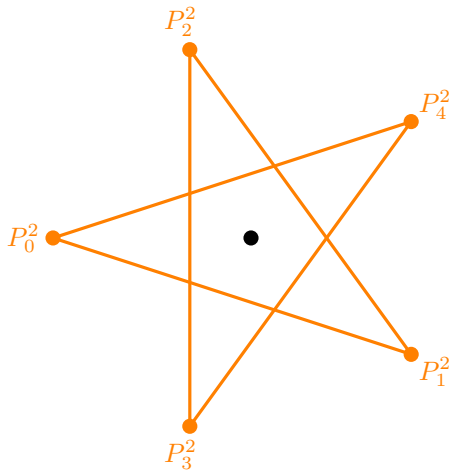
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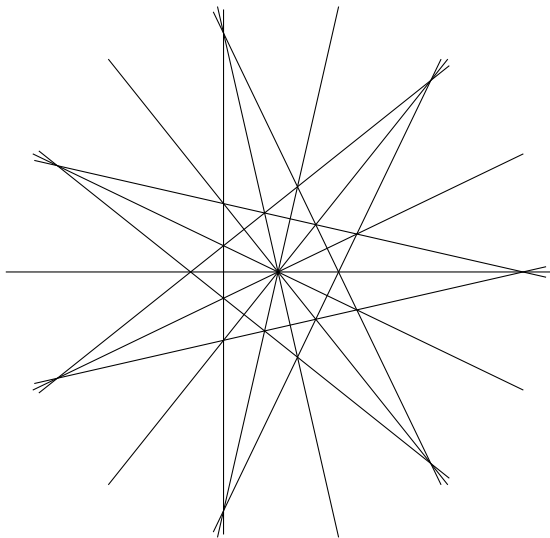
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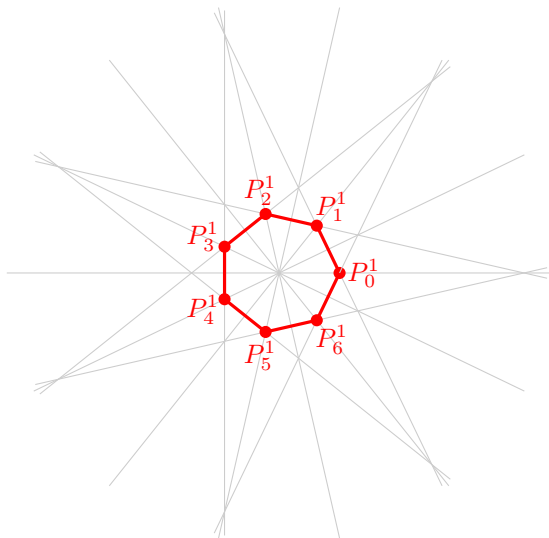
$$1 \longrightarrow \mathbb{D}_{10} \rightarrow \text{Aut} \rightarrow \text{Gal}\left(\mathbb{Q}\left[\cos \frac{2\pi}{5}\right] / \mathbb{Q}\right) \rightarrow 1$$



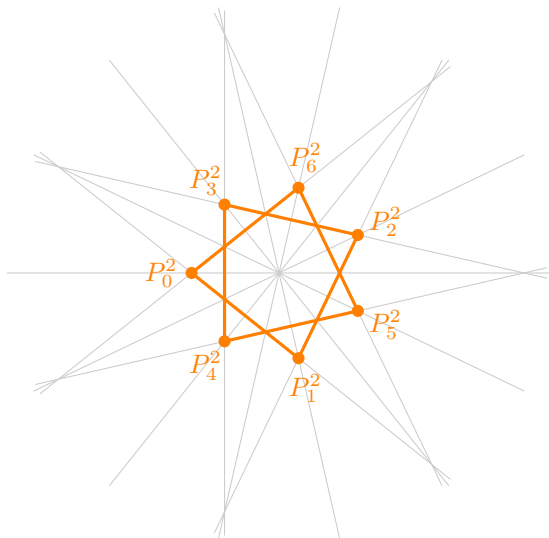
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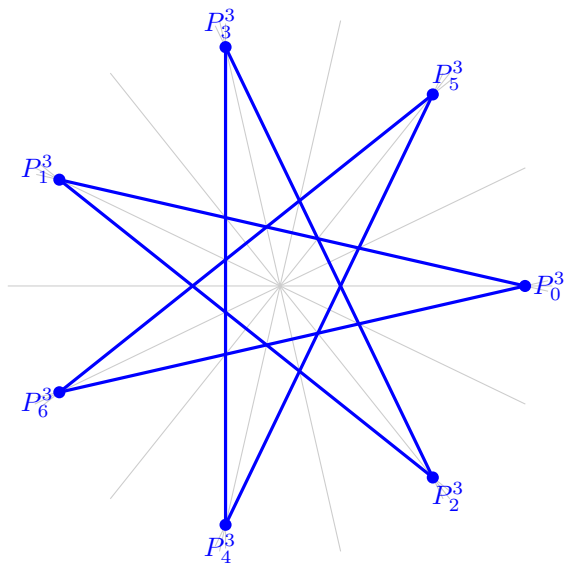
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What happens with odd polygon arrangements?

- ▶ Braid monodromy can be computed using orbifold techniques
- ▶ Computational problems to distinguish braid monodromies.



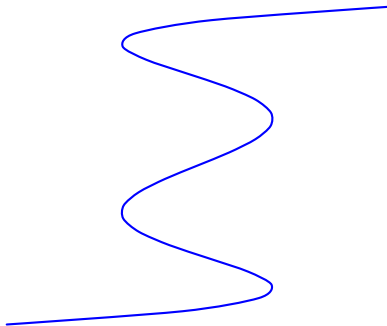
Chebyshev curves

Definition

Chebyshev curve

$$x - T_{2n+1}(y) = 0$$

where $T_{2n+1}(t)$ is the T -Chebyshev polynomial.



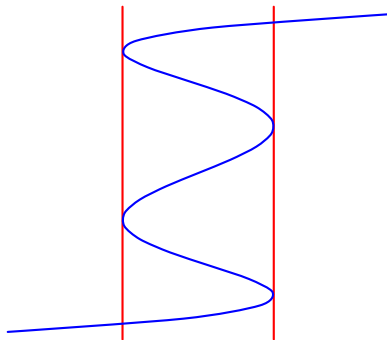
Snake curves

Definition

Constrained snake curve

$$z(x^2 - z^2) \left(xz^{2n} - z^{2n+1} T_{2n+1} \left(\frac{y}{z} \right) \right) = 0$$

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Snake curves

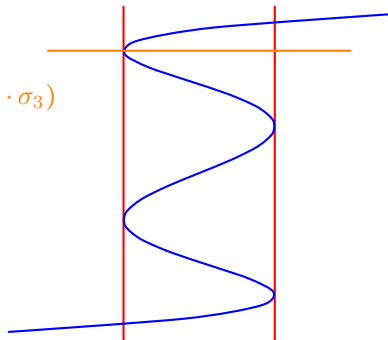
Definition

Stabbed constrained snake curve

$$D_n^1 : (y - \zeta_1 z) z (x^2 - z^2) \left(x z^{2n} - z^{2n+1} T_{2n+1} \left(\frac{y}{z} \right) \right) = 0$$

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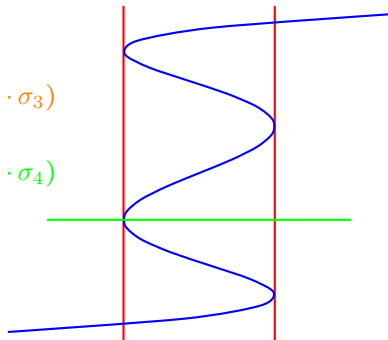
Stabbed constrained snake curve

$$D_n^2 : (y - \zeta_2 z) z (x^2 - z^2) \left(x z^{2n} - z^{2n+1} T_{2n+1} \left(\frac{y}{z} \right) \right) = 0$$

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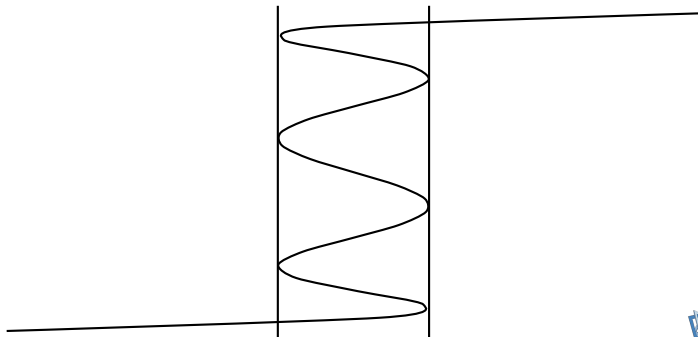
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Snake curves

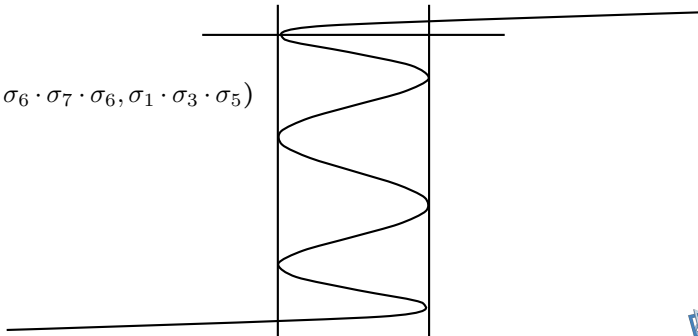
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Snake curves

Definition

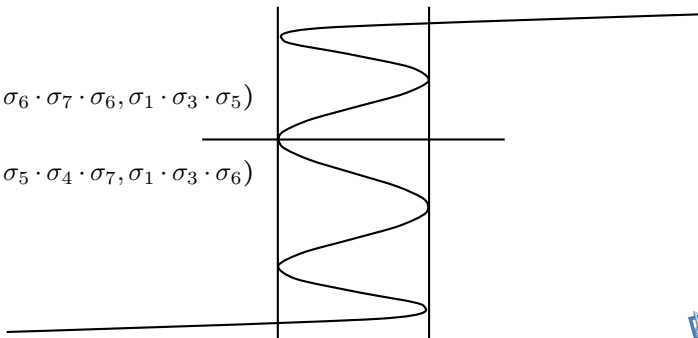
Stabbed constrained snake curve

$$xz^{2n} - z^{2n+1}T_{2n+1}\left(\frac{y}{z}\right) = 0$$

where $T_{2n+1}(t)$ is the T -Chebyshev polynomial in $\mathbb{Q}\left[\cos \frac{2\pi}{2n+1}\right]$.

$$\tau_1 = (\sigma_2 \cdot \sigma_4 \cdot \sigma_6 \cdot \sigma_7 \cdot \sigma_6, \sigma_1 \cdot \sigma_3 \cdot \sigma_5)$$

$$\tau_2 = (\sigma_2 \cdot \sigma_4 \cdot \sigma_5 \cdot \sigma_4 \cdot \sigma_7, \sigma_1 \cdot \sigma_3 \cdot \sigma_6)$$



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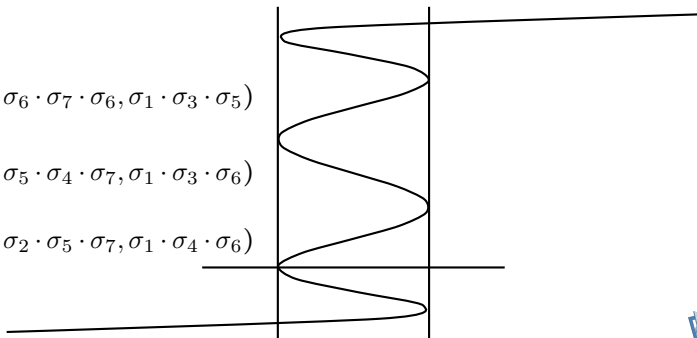
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$$\tau_1 \sim \tau_2 \iff \exists \delta \in C_{\mathbb{B}_6}(\gamma) \text{ s.t. } \beta_2 = \beta_1^\delta \stackrel{?}{\iff} \exists n \in \mathbb{Z} \text{ s.t. } \beta_2 = \beta_1^{\gamma^n}$$



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- ▶ Standard representation $\rho: \mathbb{B}_d \rightarrow \mathfrak{S}_d$ is **too small**.
- ▶ R finite ring, $\mathbb{Z}[t^{\pm 1}] \rightarrow R$, $t \mapsto t_0 \in R^\times$:

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¡¡¡Feliz cumpleaños,
Joven Luis!!!

