Existence of Special Pencils with given Dicritical Set

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Joint work with S. Abhyankar, W. Heinzer and D. Shannon

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- ϕ is not well-defined at P; P is a base point of Φ.
- $m(\phi) := I_P(f, g) = I_P(f_u, f_v)$, $u, v \in \mathbb{P}^1$ different.

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Theorem

Such a resolution exists. Moreover, there exists a unique minimal one: any other resolution factorizes through the minimal one.

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Theorem (Abhyankar-Heinzer)

$\pi : \hat{X} \to X$ a sequence of blowing-ups, $\emptyset \neq \mathcal{U} \subset \mathcal{D}$. Then, there exists a pencil such that $\mathcal{U}$ is the set of dicritic components of this pencil.

Definitions and properties

Definition

A pencil is *special* if g can be chosen as δ^m, where $m \in \mathbb{Z}_{>0}$ and $\delta \in \mathfrak{m} \setminus \mathfrak{m}^2$. $\Phi := \{f_t := f - t\delta^m\}_{t \in \mathbb{C} \cup \{\infty\}}$.

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- (S3) $\hat{\phi}_{|D}$ is constant to ∞: D is of ∞-type.

Motivation

$p \in \mathbb{C}[x,y] \setminus \mathbb{C}$ defines a rational map $\Pi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$:

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- $p(x,y) = p_0 + p_1(x,y) + \cdots + p_d(x,y)$ ($d = \deg p$) and

$$\tilde{p}(x,y,z) := z^d p\left(\frac{x}{z}, \frac{y}{z}\right) = \sum_{j=0}^{d} z^{d-j} p_j(x,y).$$

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- For $P \in \mathcal{B}$, the germ of Π at P defines a special pencil.

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Definition

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$$m\nu_E(\delta) = \sum_{D \in \mathcal{U}} n_D\, c(D,E), \tag{1}$$

where $\nu_E(\delta)$ is the valuation at E of $\pi^*(\delta)$.

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- The desired pencil is (f, δ^m).

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- No other component is a dicritic: no connected component of the preimage of 0 can be compact.

Happy Birthday, Ram