On fundamental groups of Milnor fibres of rational homology disk smoothings of surface singularities

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Singularities in the Midwest
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Joint work in progress with J.M. Wahl

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- f $\mathbb{Q}$HD-smoothing: $\mu = 0$

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Theorem (Laufer, Wahl, Bhupal, Stipsicz, Fowler)

The weighted homogeneous surface singularities admitting a $\mathbb{Q}$HD *smoothing are the following families:*

- $\mathcal{G}_{n,q}$ *(previous example)*
- $\mathcal{W}(p,q,r)$
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- $\mathcal{B}_2^3(p)$
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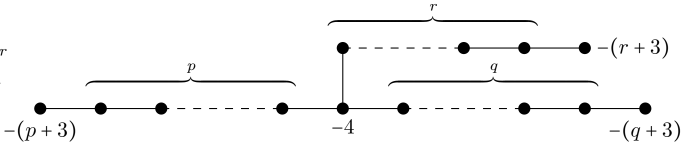
$\mathcal{W}_{p,q,r}$
rigid



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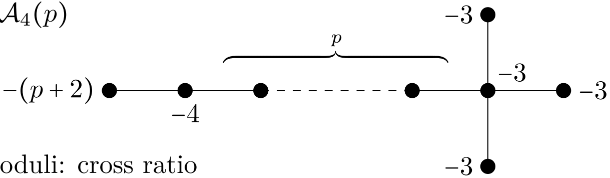

Analytic moduli: cross ratio

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Fundamental group of the Milnor fibers

- $\mathcal{G}_{n,q}$ cyclic
- $\mathcal{W}(p,q,r)$, $\mathcal{N}(p,q,r)$, $\mathcal{M}(p,q,r)$: abelian (Fowler, Wahl)
- $\mathcal{A}^4(p)$, $\mathcal{B}^4(p)$, $\mathcal{C}^4(p)$: non-abelian (Wahl)
- $\mathcal{B}^3_2(p)$, $\mathcal{C}^3_2(p)$, $\mathcal{C}^3_3(p)$: unknown

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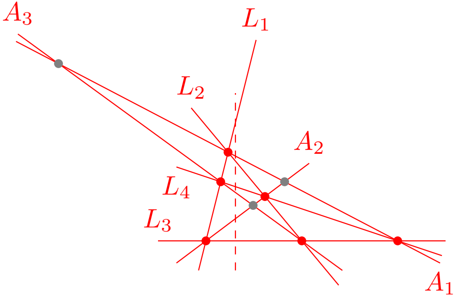
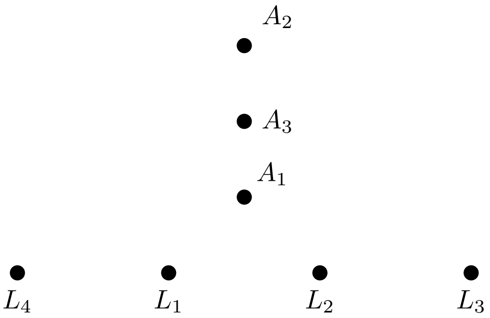
Basic model for $\mathcal{B}_3^2(p)$ and $\mathcal{C}_3^3(p)$



Curves D', D'' for $\mathcal{B}_3^2(p)$



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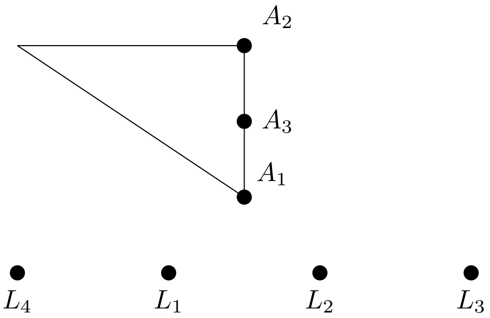

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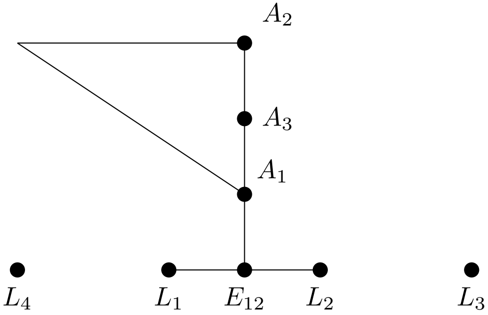

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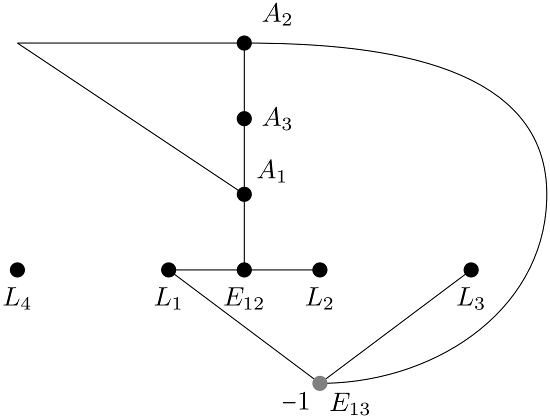

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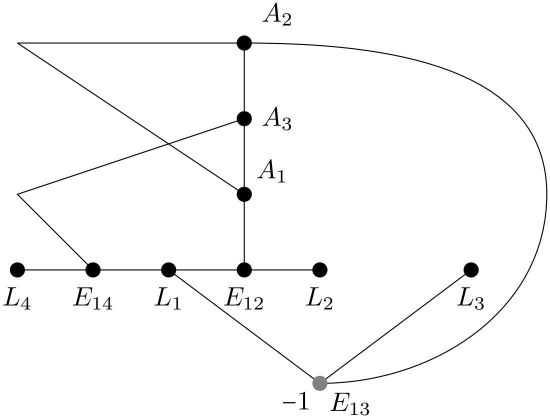

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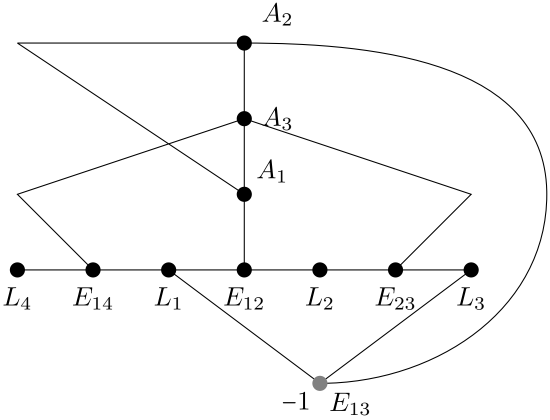

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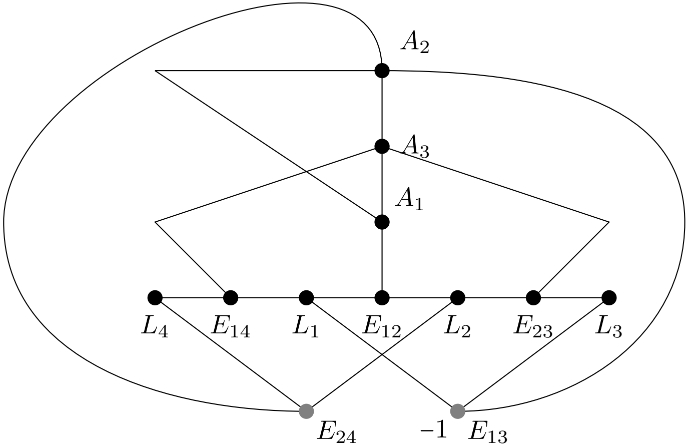

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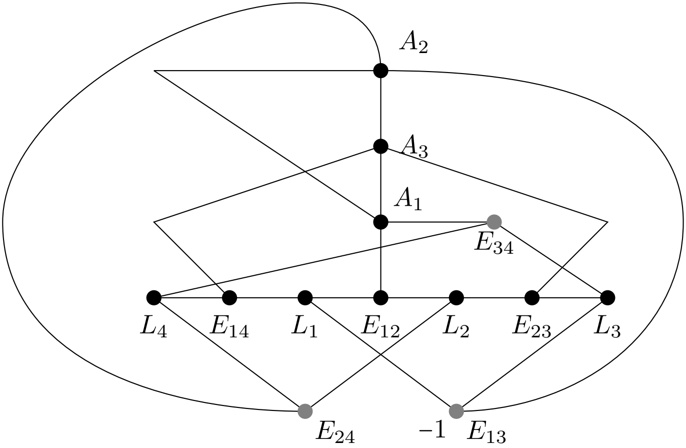

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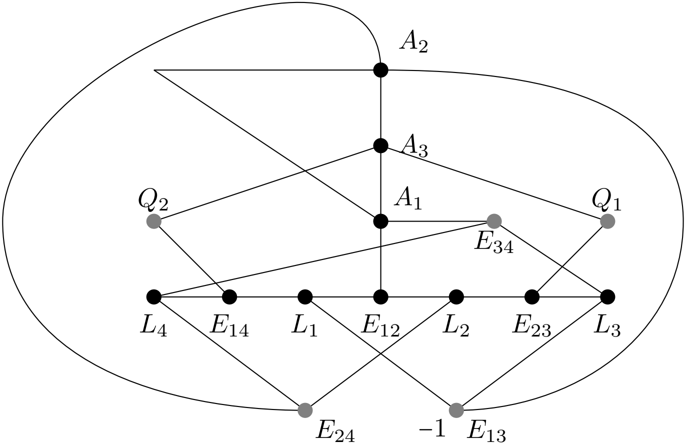

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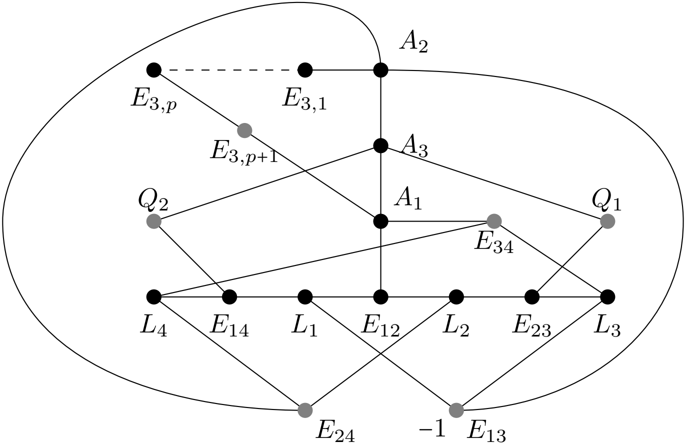

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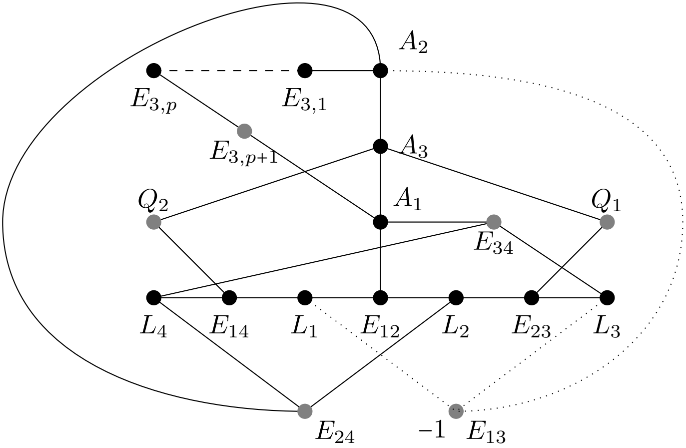

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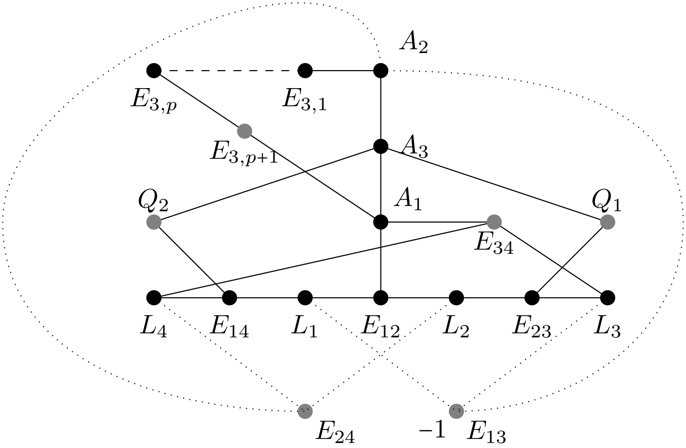

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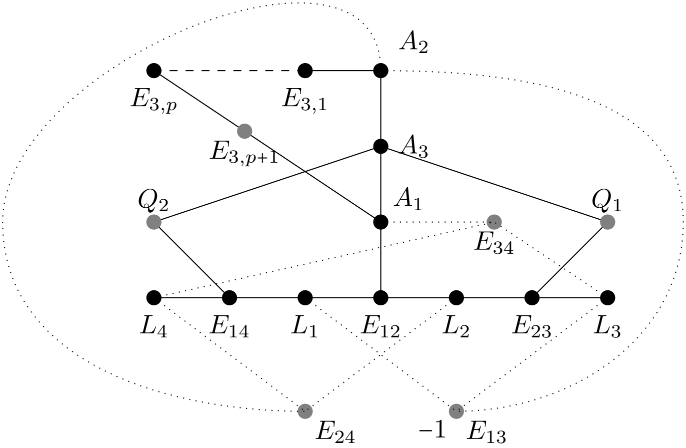

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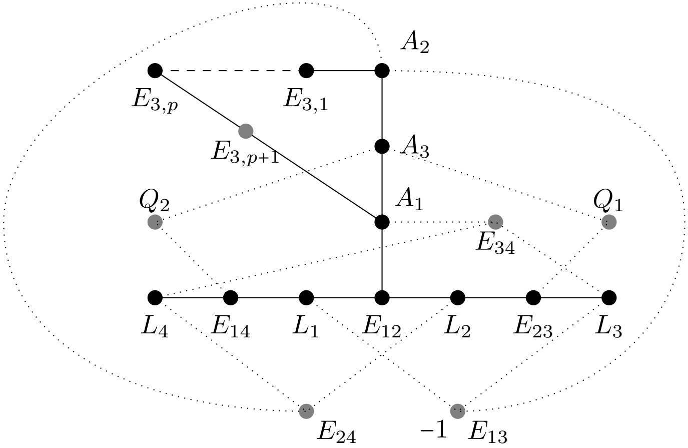

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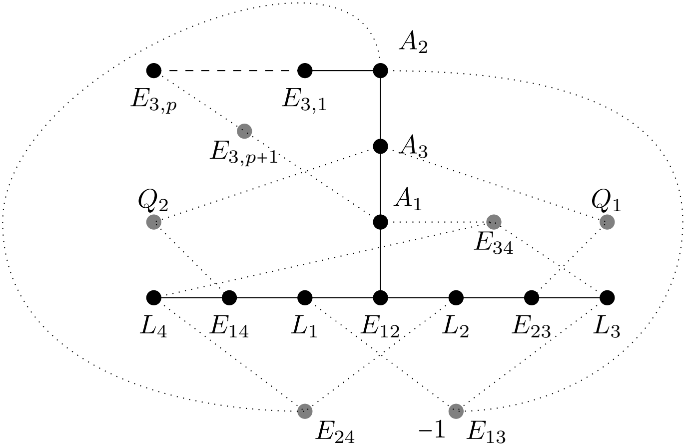

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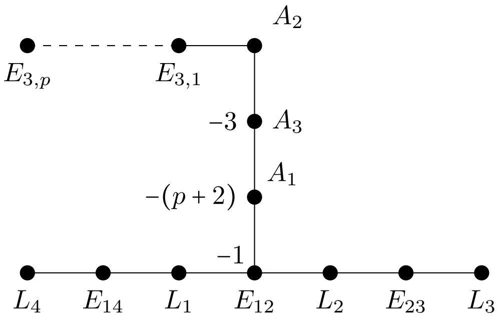

Wahl's basic model

Fowler's model for $\mathcal{C}_2^3(p)$

McLane line arrangement with a line joining two triple points.
No real picture!

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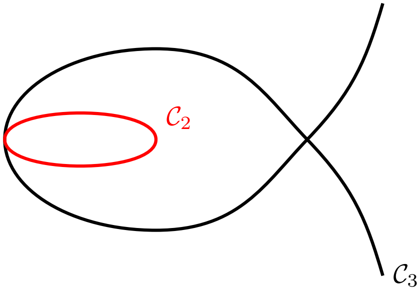

$$\mathcal{C}_3 : y^2 z = x^2(x+z)$$

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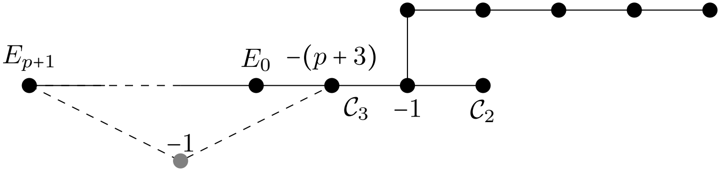

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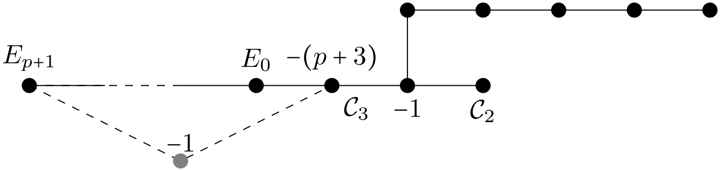
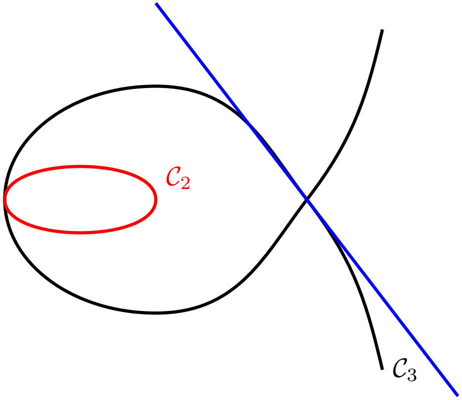


Theorem

$\pi_1(\mathbb{P}^2 \setminus D)$ *abelian* $\Longrightarrow$ $\pi_1(F)$ *abelian for a* $\mathbb{Q}$HD *smoothing of* $\mathcal{C}_3^3(p)$.

Modified Wahl's basic model

Model for $\mathcal{C}_2^3(p)$



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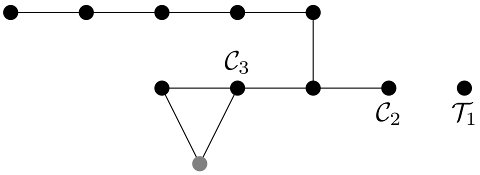

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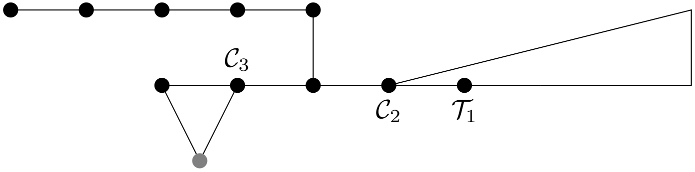

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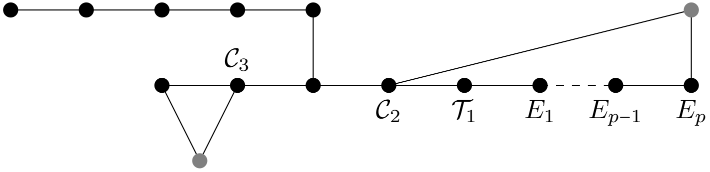

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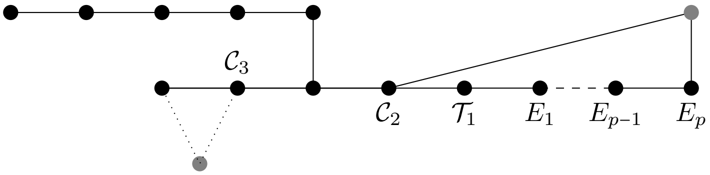

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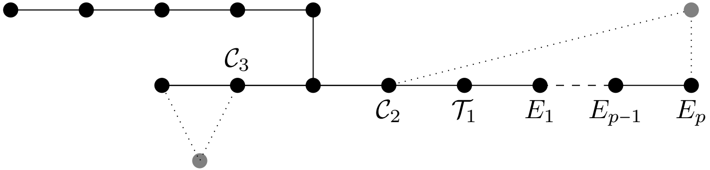

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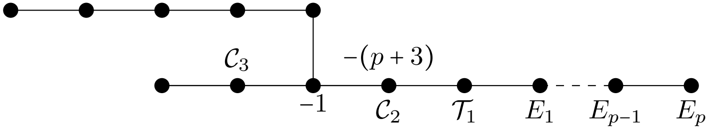

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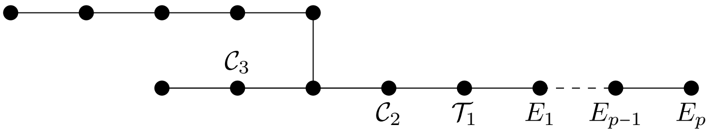

- $\pi_1(\mathbb{P}^2 \setminus D)$ abelian $\Longrightarrow$ $\pi_1(F)$ abelian.

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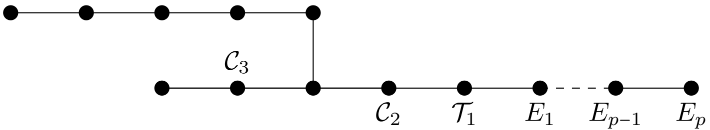

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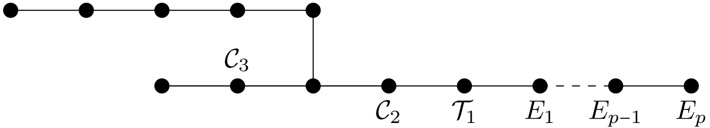

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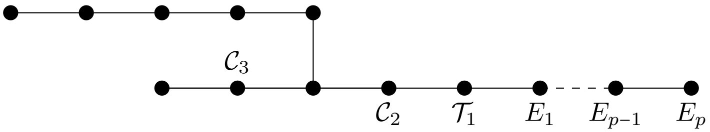

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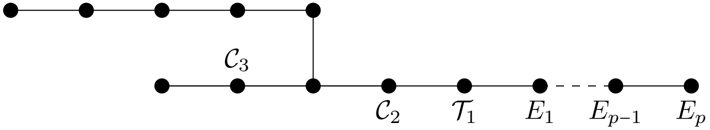

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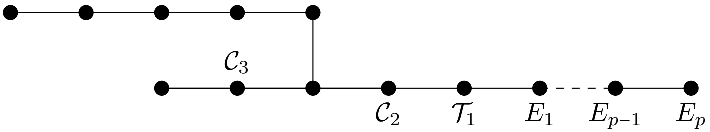

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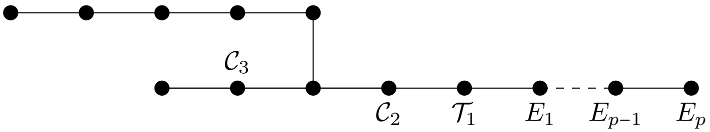

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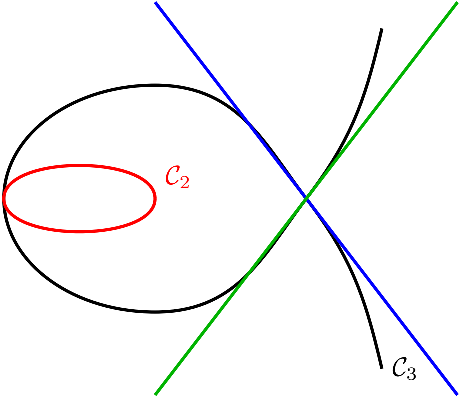
Birational morphism to Σ_3



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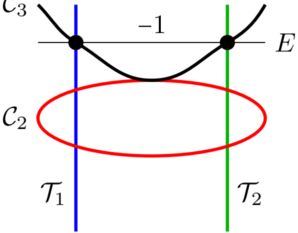
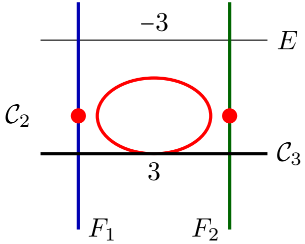
Birational morphism to Σ_3

$\sigma : \Sigma_1 \to \mathbb{P}^2$



Birational morphism to Σ_3

$\sigma_1 : \Sigma_3 \dashrightarrow \Sigma_1$



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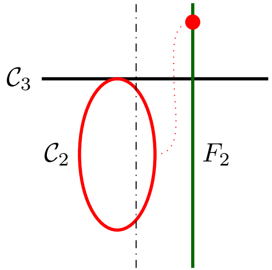

Non-abelian fundamental group

Theorem

For $\mathcal{B}_2^3(p)$*, let* $N := 2(p+2)(p+3)$

$$\pi_1(F) = \langle S, T \mid S^N = 1, T \cdot S \cdot T^{-1} = S^{-(2p+5)}, T^2 = S^{p+2} \rangle$$

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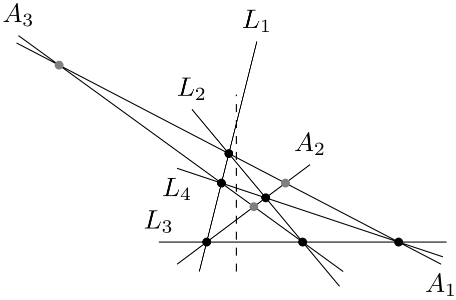

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There is a linear representation (ω primitive N-root of unity):

$$\pi_1(F) \xrightarrow{\Phi} G \subset \mathrm{GL}(4;\mathbb{C})$$

$$S \longmapsto \begin{pmatrix} \omega & 0 & 0 & 0 \\ 0 & \omega^{-(2p+5)} & 0 & 0 \\ 0 & 0 & \omega^{2p+5} & 0 \\ 0 & 0 & 0 & \omega^{-1} \end{pmatrix}$$

$$T \longmapsto \begin{pmatrix} 0 & \omega^{p+2} & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \omega^{-(p+2)} & 0 \end{pmatrix}$$

The action is free outside $\mathbf{0}$.

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- $h^{-1}(0) \cong \mathcal{B}_2^3(p)$.

Thanks for your attention!!