Monodromy theorem for Artin kernels

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Congreso RSME 2015
Granada, Febrero 2015

Joint work with J.I. Cogolludo and D. Matei

Definitions and examples

Definition

Γ simplicial graph, V_Γ set of vertices, $E_\Gamma \subset \{A \subset V_\gamma \mid \#A = 2\}$ edges

$$G_\Gamma = \langle g_v, v \in V_\Gamma \mid [g_v, g_w] = 1, \{v, w\} \in E_\Gamma \rangle$$

right-angled Artin group.

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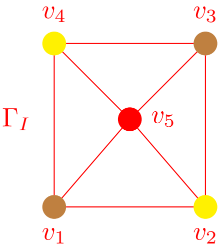
- Γ 1-skeleton of an n-simplex, $\qquad G_\Gamma = \mathbb{Z}^{n+1}$.
- Γ multipartite graph of $\Gamma_1, \ldots, \Gamma_n$, $\qquad G_\Gamma = \prod_{j=1}^n G_{\Gamma_j}$.

Test examples

Test Example I



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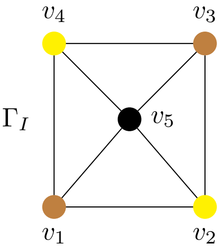

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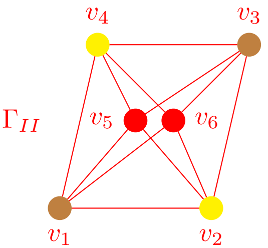
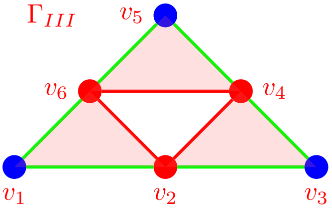
Test Example II



$G_{\tilde{\Gamma}_{II}} = \mathbb{F}_2 \times \mathbb{F}_2 \times \mathbb{F}_2$

More test examples

Test Example III



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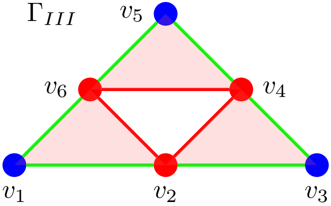
Test Example III

Γ_{III}



Test Example IV

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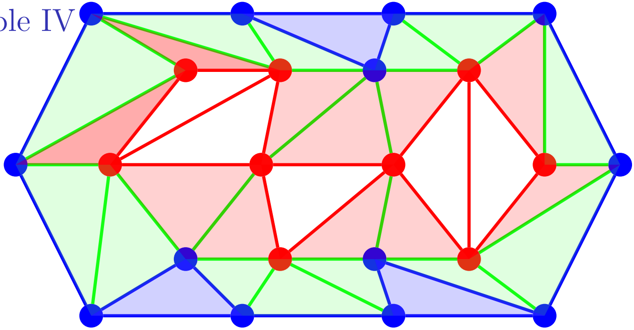

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Γ simplicial graph (V_Γ and E_Γ as before).

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- The flag complex of Γ_{II} is the octahedron (2-dimensional).

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- $\pi_1(\mathbb{T}_{\mathbb{R}}(\mathbb{B}_\Gamma)) = \pi_1(\mathbb{T}_{\mathbb{C}}(\mathbb{B}_\Gamma)) = G_\Gamma$ Eilenberg-McLane $K(G_\Gamma, 1)$ spaces.

Artin kernels

Definition

- Γ simplicial graph
- $\chi : V_\Gamma \to \mathbb{Z}^*$, $n_v = \chi(t)$, $\gcd\{n_v \mid v \in V_\Gamma\} = 1$.
- χ the induced epimorphism $G_\Gamma \twoheadrightarrow \mathbb{Z}$
- $g_v \notin \ker \chi, \forall v \in V_K$.
- The Artin kernel associated to χ is $A_\Gamma^\chi = \ker \chi$.

Goal

Study $H_*(A_\Gamma^\chi; \mathbb{C})$.

Goals

ALGEBRA

- G_Γ

GEOMETRY

- $\mathbb{T}_\mathbb{C}(\mathbb{B}_\Gamma)$

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- $\mathbb{T}_{\mathbb{C}}(\mathbb{B}_\Gamma)$
- $\mathbb{T}_{\mathbb{C}}(\mathbb{B}_\Gamma) \xrightarrow{f} \mathbb{C}^*$, $\underset{v \in V_\Gamma}{(x_v)} \mapsto \prod_{v \in V_\Gamma} x_v^{n_v}$

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- $\mathbb{C}[t^{\pm 1}]$-module

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- $F_\chi = f^{-1}(\varepsilon)$, $0 < \varepsilon \ll 1$
- $H_*(F_\chi; \mathbb{C})$
- Monodromy φ on $H_*(F_\chi; \mathbb{C})$

Differentials

$\tilde{C}_*(\mathbb{B}_\Gamma)$ $\qquad$ $C_{*+1}(\mathbb{T}^\chi_\mathbb{R}(\mathbb{B}_\Gamma))/\mathbb{C}[t^{\pm 1}]$

$$\begin{array}{rcccc}
 & & & \sigma \in C_k & \\
 & & & \downarrow & \\
 & & \cdots & \vdots & \cdots \\
\tau_1 \in C_{k-1} & \to & \cdots & \pm 1 & \cdots \\
\tau_1 \subset \partial\sigma & & \cdots & \vdots & \cdots \\
\tau_2 \in C_{k-1} & \to & \cdots & 0 & \cdots \\
\tau_2 \not\subset \partial\sigma & & \cdots & \vdots & \cdots
\end{array}
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$\sigma = \tau_1 \cup \{v\}$ $\qquad$ $X_v = (t-1)^{n_v}$

Differentials

$C_{*+1}(\mathbb{T}^{\chi}_{\mathbb{R}}(\mathbb{B}_{\Gamma})) \otimes_{\mathbb{C}[t^{\pm 1}]} \mathbb{C}(t)$

$$
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where $\Phi_n(t)$ *is the* n^{th}*-cyclotomic polynomial and* $m_j \leq \ell + 1$.

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1. In that case $H_*(A_\Gamma^\xi;\mathbb{C}) \cong H_*(F_\chi;\mathbb{C})$ (equivariantly with respect to the $\mathbb{C}[t^{\pm 1}]$ and the monodromy action)
2. As a consequence, Monodromy Theorem holds for H_ℓ, $\ell \leq k$: eigenvalues are roots of unity and Jordan blocks are of size at most $\ell + 1$.

Idea of the proof

- $M = \mathbb{B}_\Gamma$ at least $(n-1)$-connected
- Compare Fitting ideals $\partial : C_n(M) \to C_{n-1}(M)$ over $\mathbb{Z}$
- $\partial : C_{n+1}(\mathbb{T}^\chi_\mathbb{R}(M)) \to C_n(\mathbb{T}^\chi_\mathbb{R}(M))$ over $\mathbb{C}[t^{\pm 1}]$
- $\{\text{Minors of } \partial\frac{(\sigma_1,\ldots,\sigma_r)}{(\tau_1,\ldots,\tau_r)}\} \leftrightarrow \{(K, L) \text{ admissible } n\text{-pairs}\}$
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Lemma

A minor does not vanish if and only if (K,L) is acyclic. In that case its value is

$$\frac{\left(\prod_{\sigma\in K\setminus M_{n-1}} X_\sigma\right)\left(\prod_{\tau\in L\setminus M_{n-2}} X_\tau\right)}{\prod_{\tau\in M_{n-1}\setminus M_{n-2}} X_\tau}.$$

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Remark

The statement about roots of unity is proved.

Jordan blocks I

To study the behavior for one cyclotomic, say Φ_d, fix a primitive d^{th}-root of unity ζ and consider the complex

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A minor associated to (K, L)

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1. Mark in red (resp. blue) the vertices v such that $d | n_v$ (resp. $d \not| n_v$)

Jordan blocks I

To study the behavior for one cyclotomic, say Φ_d, fix a primitive d^{th}-root of unity ζ and consider the complex

$$C_*(\mathbb{T}^{\chi}_{\mathbb{R}}(K)) \otimes_{\mathbb{C}[t^{\pm 1}]} \mathbb{C}[[T]], \quad t = \zeta + T.$$

Fitting ideals

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$\mathbb{C}[t^{\pm 1}] \supset (T^N) \supset (T^{N-n_1}) \supset \cdots \supset (T^{N-(n_1+\cdots+n_k})) \supset (0)$,
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Strategy

1. Mark in red (resp. blue) the vertices v such that $d | n_v$ (resp. $d \not| n_v$)
2. The multiplicity of a simplex is the number of red vertices.

Jordan Blocks II

Strategy

Jordan Blocks II

Strategy

3. The multiplicity of $M_n \supset K \supset M_{n-1}$ (resp. $M_{n-1} \supset L \supset M_{n-2}$) is the sum of the multiplicities of the n-simplices (resp. $(n-1)$-simplices).

Jordan Blocks II

Strategy

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4. Find an acyclic pair (K, L) (with fixed number j of n-simplices) reaching the minimum m_j of the multiplicities. The corresponding ideal is $T^{m_j - m_0}$ where m_0 is the sum of multiplicities of all $(n-1)$-simplices.

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Jordan Blocks II

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5. Start with K, L acyclic
6. Consider filtrations

$$M_n = M_n^{(n+1)} \supset M_n^{(n)} \supset \cdots \supset M_n^{(1)} \supset M_n^{(0)} \supset M_{n-1} \supset M_{n-1} = M_{n-1}^{(n)}$$

$$M_{n-1} = M_{n-1}^{(n)} \supset M_{n-1}^{(n-1)} \supset \cdots \supset M_{n-1}^{(1)} \supset M_{n-1}^{(0)} \supset M_{n-2}$$

Jordan Blocks III

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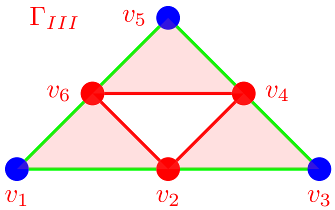
   - Add eventually multiplicity-2 edges to obtain a spanning tree K.
   - The multiplicity of K is independent of the choices.
8. For the next Fitting ideal, we may consider an acyclic (K_1, L_1) obtained from (K, L) by taking out an n-simplex of K (of multiplicity $\leq n + 1$) and adding an $(n - 1)$-simplex to L (of multiplicity ≥ 0). Hence the multiplicity decreases at most by $n + 1$ and the second statement of Monodromy Theorem holds.

Computations

Test Example III



Computations

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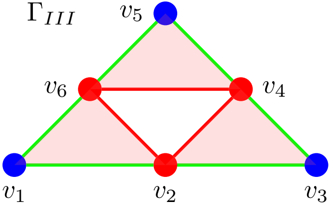

Test Example IV