

Resonant characters for rational arrangements

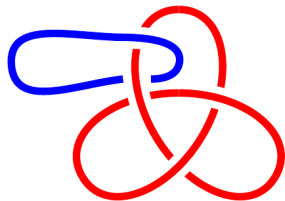
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Universidad de Zaragoza

Branched Coverings, Degenerations, and Related Topics 2016
Hiroshima, March 2016



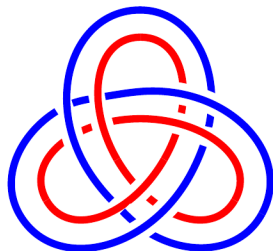
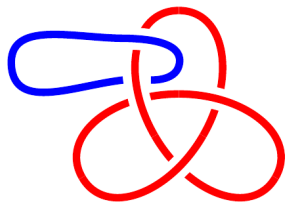
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- ▶ $\mathcal{L}(\vec{K}, \vec{L}) = n$ if $\vec{L} = n \cdot C \in H_1(\mathbb{S}^3 \setminus K; \mathbb{Z}) = \mathbb{Z}\langle C \rangle$.

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- ▶ Projection of $K \implies \tilde{K}$: $\mathcal{L}(K, \tilde{K})$ invariant of the projection.

Algebraic curves

Regular neighbourhood

- ▶ $\mathcal{C} = \bigcup_{i=1}^n \mathcal{C}_i \subset \mathbb{P}^2$ plane algebraic curve, $\deg \mathcal{C}_i = d_i$.

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weighted by $(g_j, e_j) = (g(A_j), A_j^2)$.



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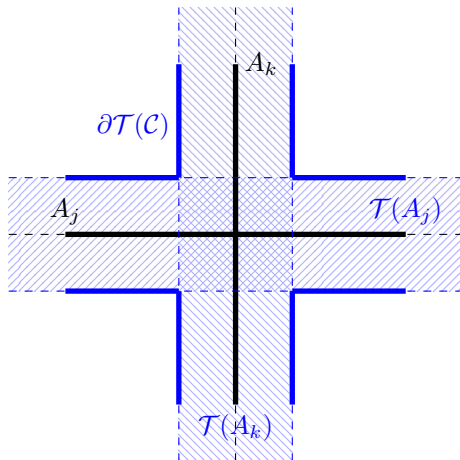
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▶ $\mathcal{T}(\mathcal{C})$ regular neighbourhood of $\sigma^{-1}(\mathcal{C}) \subset X$, plumbed union of tubular neighbourhoods $\pi_j : \mathcal{T}(A_j) \rightarrow A_j$ (Euler number A_j^2).



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▶ $\sigma(\mathcal{T}(\mathcal{C})) = \mathcal{R}(\mathcal{C}) \subset \mathbb{P}^2$, $\partial \mathcal{T}(\mathcal{C}) \equiv \partial \mathcal{R}(\mathcal{C})$



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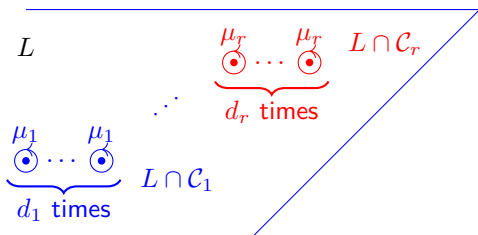
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- ▶ Goal: Define a linking number for cycles in Γ (see also Guerville-Meilhan).

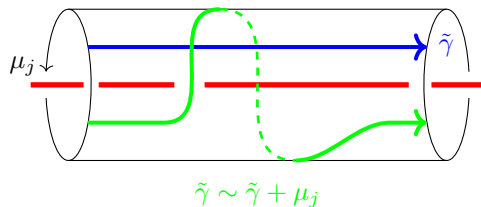


Liftings and characters

- ▶ γ simplicial cycle in $\Gamma \implies \tilde{\gamma}$ lifted 1-cycle in $\mathbb{P}^2 \setminus \mathcal{C}$

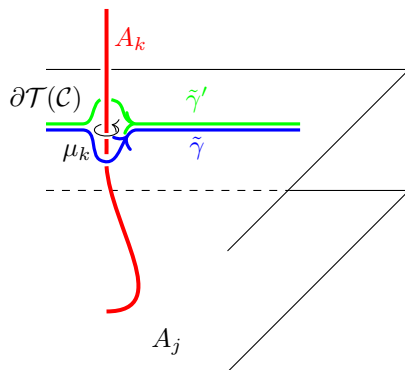
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Theorem (AFG)

The linking number of (ξ, γ) is a topological invariant of $(\mathbb{P}^2, \mathcal{C}, \xi)$ (in the case of line arrangements).



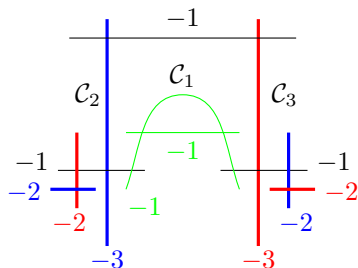
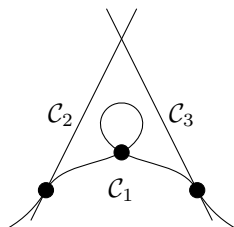
Nodal cubic and inflexion tangent lines

- $\mathcal{C} = \{\overbrace{(xyz + x^3 - y^3)}^{\mathcal{C}_1} \overbrace{(z^2 - 3(x - y)z + 9(x^2 - xy + y^2))}^{\mathcal{C}_2 \cup \mathcal{C}_3} = 0\}$:
nodal cubic and two tangent lines at inflexion points.



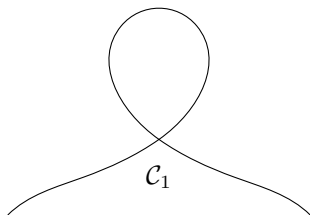
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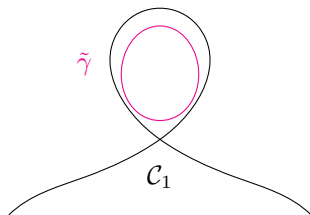
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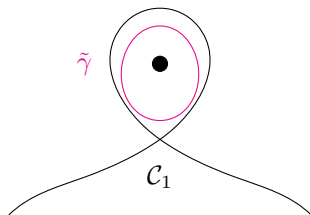
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- ▶ $\rho : Y \rightarrow \mathbb{P}^2$ branched covering associated to ξ

$$[x : y : z] \mapsto [3xyz - (\zeta - 1)(\bar{\zeta}x^3 - y^3) : -3xyz + (\bar{\zeta} - 1)(\zeta x^3 - y^3) : 9xyz]$$

The preimage of \mathcal{C}_1 : three lines.

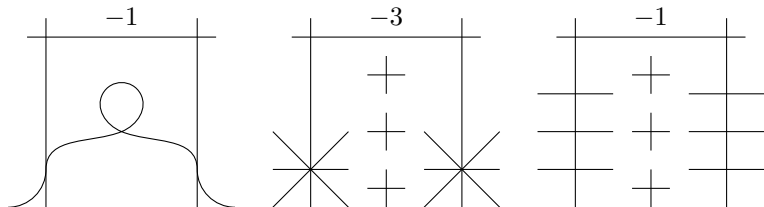


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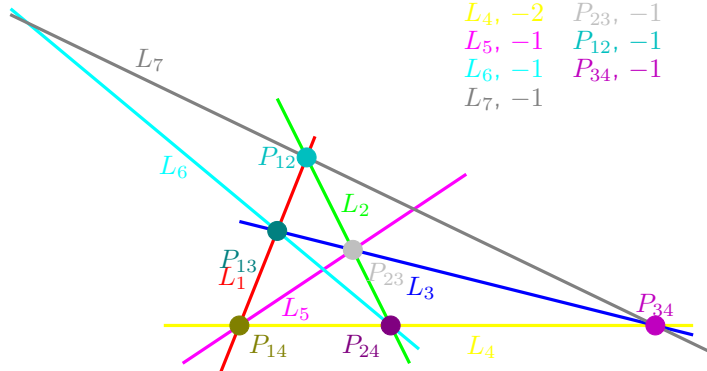
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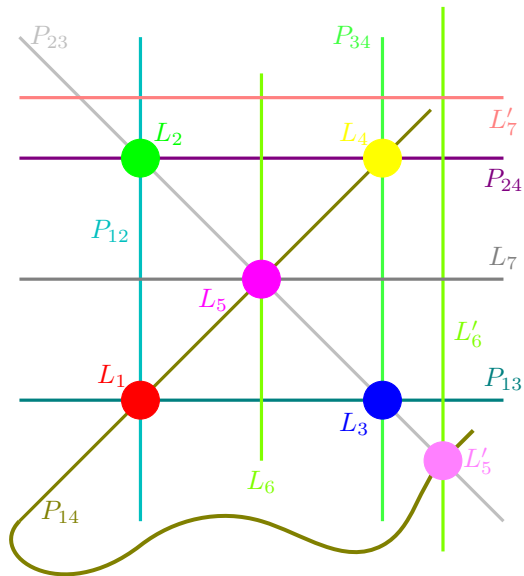


Extended Ceva arrangement

$L_1, -2$	$P_{14}, -1$
$L_2, -2$	$P_{24}, -1$
$L_3, -2$	$P_{13}, -1$
$L_4, -2$	$P_{23}, -1$
$L_5, -1$	$P_{12}, -1$
$L_6, -1$	$P_{34}, -1$
$L_7, -1$	



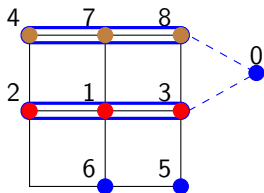
Extended Ceva arrangement



$$[xy^2 : x(y-z)^2 : y(y-z)(x-z)]$$

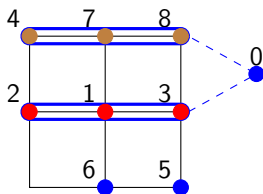
Zariski pairs of arrangements via linking numbers

Extended McLane arrangements



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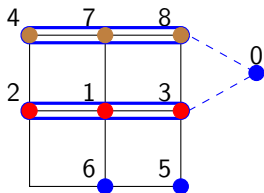


$$\xi : (\mu_0, \dots, \mu_8) \mapsto (1, \zeta, \zeta, \zeta, \bar{\zeta}, 1, 1, \bar{\zeta}, \bar{\zeta}), \text{ cycle } L_0, L_5, L_6.$$

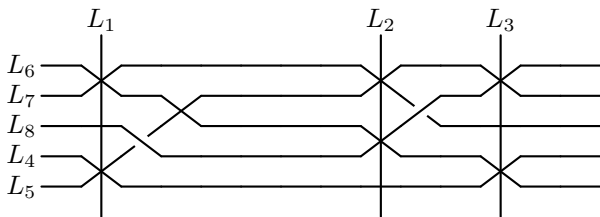


Zariski pairs of arrangements via linking numbers

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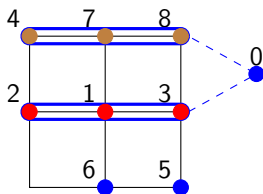


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Guerville's arrangement

Linking number oriented Zariski 4-tuple with equations in $\mathbb{Q}(\exp(\frac{2i\pi}{5}))$.



Characteristic Varieties from Alexander Invariant

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The characteristic variety $V'_k(G) \equiv V'_k(Z) \subset \mathbb{T}_Z$ is the zero locus of $J_k(G)$.



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Remark

These two definitions may differ only in $\mathbf{1} \in \mathbb{T}_Z$.



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Corollary

Torsion points are dense in $V_k(Z)$.



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Hodge Theory



Hypersurface complement. Projective part.

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Let $\sigma_\xi : H^0(\mathbb{P}^2; \mathcal{O}(\ell_\xi - 3)) \longrightarrow \bigoplus_{P \in \mathcal{P}} \left(\mathcal{O}_{\mathbb{P}^2, P} / \mathfrak{m}_P^{\left[\sum_{P \in L_j} q_j \right] - 1} \right)$. Then,

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Geometric interpretation

$\ker \sigma_\xi$: equations of curves of degree $\ell_\xi - 3$ passing through the points $P \in \mathcal{P}$ with multiplicity at least $\left\lfloor \sum_{P \in L_j} q_j \right\rfloor - 1$.



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There is a finite polytope stratification of the hypercube $[0, 1)^{r+1}$ such that the maps σ_ξ are constant on each strata.

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Problem

Compute $\dim_{\mathbb{C}} QP_\xi$ for resonant torsion characters.



Quasi-projective part

Theorem

*S smooth projective surface, $\mathcal{D} \subset S$ normal crossing divisor,
 $\mathcal{D} = D_1 \cup \cdots \cup D_s$. Then,*

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Quasi-projective part

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S smooth projective surface, $\mathcal{D} \subset S$ normal crossing divisor,
 $\mathcal{D} = D_1 \cup \cdots \cup D_s$. Then,

$$\dim_{\mathbb{C}} H_1(S \setminus \mathcal{D}; \mathbb{C}) - \dim_{\mathbb{C}} H_1(S; \mathbb{C}) = \dim_{\mathbb{C}} \ker \left(\bigoplus_{j=1}^s H_2(D_j; \mathbb{C}) \rightarrow H_2(S; \mathbb{C}) \right).$$

▶ $\mathbb{P}^2 \setminus \mathcal{C} \equiv X \setminus \left(\bigcup_{j=0}^r \hat{L}_j \cup \bigcup_{P \in \mathcal{P}} E_P \right) = X \setminus \left(\bigcup_{j=0}^t A_j \right)$

- ▶ Smooth model $\rho_{\xi} : \bar{X}^{\xi} \rightarrow X$, such that $\bar{X}^{\xi} \setminus X^{\xi} = \mathcal{D}^{\xi}$ normal crossing divisor.



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▶ Smooth model $\rho_{\xi} : \bar{X}^{\xi} \rightarrow X$, such that $\bar{X}^{\xi} \setminus X^{\xi} = \mathcal{D}^{\xi}$ normal crossing divisor.

▶ $\mathcal{D}^{\xi} = \bigcup_{j=0}^t \underbrace{\hat{\rho}_{\xi}^*(A_j)}_{\text{strict transform}} \cup \bigcup_{k=1}^{t'} \underbrace{B_k}_{\text{collapsed by } \rho_{\xi}}$.



Computing QP_ξ

$$\bigoplus_{j=0}^{t'} H_2(B_j; \mathbb{C}) \oplus \bigoplus_{j=0}^t H_2(\hat{\rho}_\xi^*(A_j); \mathbb{C}) \longrightarrow H_2(\bar{X}_\xi; \mathbb{C})$$



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- ▶ $H_2(\hat{\rho}_\xi^*(A_j); \mathbb{C})^\xi \neq 0 \iff \dim_{\mathbb{C}} H_2(\hat{\rho}_\xi^*(A_j); \mathbb{C}) = N \iff \rho_\xi$ is an unbranched covering over A_j



Computing QP_ξ

$$\bigoplus_{A_j \subset \mathcal{C}_\xi} H_2(\hat{\rho}_\xi^*(A_j); \mathbb{C})^\xi \longrightarrow H_2(\bar{X}_\xi; \mathbb{C})^\xi$$

- $A_j \subset \mathcal{C}_\xi$, \tilde{A}_j irreducible component of $\hat{\rho}_\xi^*(A_j)$, $\zeta = \exp \frac{2i\pi}{N}$,
 $H_2(\hat{\rho}_\xi^*(A_j); \mathbb{C})^\xi = \mathbb{C}\langle \hat{A}_j \rangle$,

$$\hat{A}_j = \frac{1}{\sqrt{N}} (\tilde{A}_j + \bar{\zeta} \cdot \xi(A_j) + \bar{\zeta}^2 \cdot \xi^2(A_j) + \cdots + \bar{\zeta}^{N-1} \xi^{N-1}(A_j))$$



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Theorem

$\dim_{\mathbb{C}} QP_\xi = \text{corank } \mathbf{I}_\xi$ (Hodge index theorem).



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- ▶ Fix spanning forest $T_\xi \subset \Gamma_\xi$.

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- ▶ $(\hat{A}_j^\xi)^2 = (\hat{A}_j)^2$
- ▶ If $A_j \cap A_k = \{P_1, \dots, P_m\} \rightarrow \{e_1, \dots, e_m\}$,

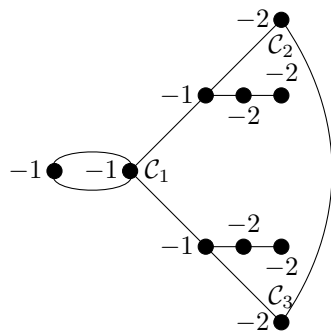
$$\hat{A}_j^\xi \cdot \hat{A}_k^\xi = \xi(\tilde{\gamma}_{e_1}) + \dots + \xi(\tilde{\gamma}_{e_m}).$$



Examples

Example

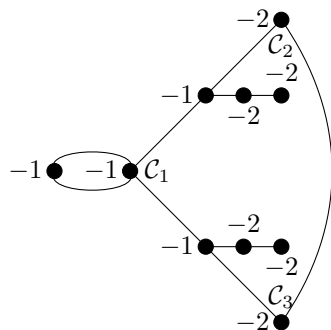
Nodal cubic and inflexion tangent lines



Examples

Example

Nodal cubic and inflexion tangent lines



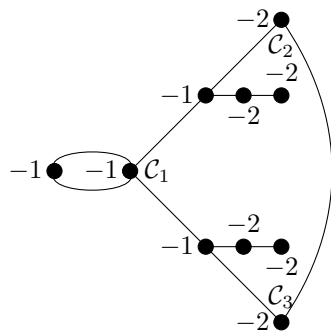
$$\begin{pmatrix} -1 & 1 + \zeta \\ 1 + \bar{\zeta} & -1 \end{pmatrix} = \begin{pmatrix} -1 & -\bar{\zeta} \\ -\zeta & -1 \end{pmatrix}$$



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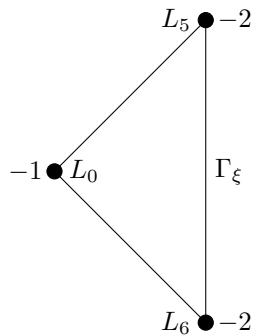
$$\dim_{\mathbb{C}} QP_{\xi} = 1$$



Examples

Example

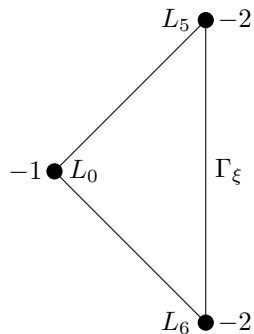
Extended McLane arrangement



Examples

Example

Extended McLane arrangement

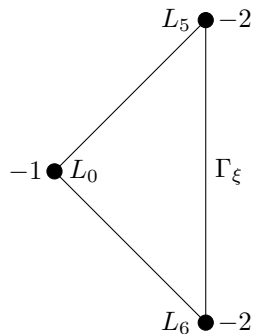


$$\begin{pmatrix} -1 & 1 & 1 \\ 1 & -2 & \zeta \\ 1 & \bar{\zeta} & -2 \end{pmatrix}$$

Examples

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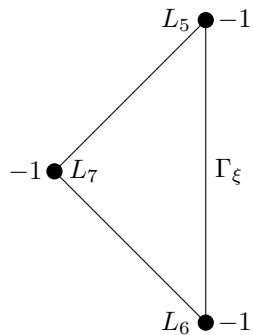
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Examples

Example

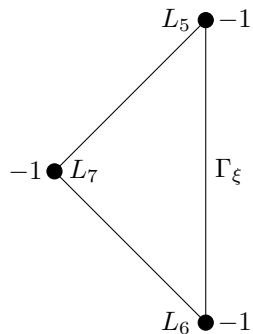
Extended Ceva arrangement



Examples

Example

Extended Ceva arrangement

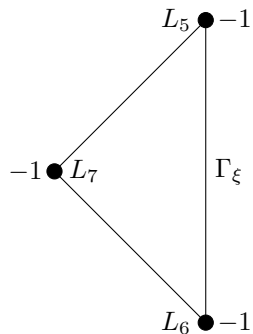


$$\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{pmatrix}$$

Examples

Example

Extended Ceva arrangement



$$\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{pmatrix}$$

$$\dim_{\mathbb{C}} QP_\xi = 2$$