Braid monodromy and topology of algebraic curves

Enrique Artal (Universidad de Zaragoza)

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Joint work [ACC02, ACC02a] with:

Jorge Carmona (Universidad Complutense)
José I. Cogolludo (Universidad de Zaragoza)

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- $\mathscr{M} := \mathscr{M}(k_1 T_1, \ldots, k_r T_r; d) :=$

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- ▷ Σ^{irr}: irreducible curves

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$\tilde{\Sigma} \subset \Sigma$ connected component $\mathcal{C}_1, \mathcal{C}_2 \in \tilde{\Sigma} \Rightarrow \exists$ *oriented* isotopy H such that $h_0 = 1_{\mathbb{P}^2}$, $h_1(\mathcal{C}_1) = \mathcal{C}_2$.

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What about the converse?

Does there exist an *oriented* homeomorphism

$$\Phi : \mathbb{P}^2 \longrightarrow \mathbb{P}^2$$

such that $\Phi(\mathcal{C}_1) = \mathcal{C}_2$?

2. Previous results

- ▷ Works of Greuel [GLS98, GLS98a, GL99, GLS99, GLS02], Shustin [SHU97, SHU97a], Lossen about irreducibility, smoothness, existence,. . .

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- ▷ Works of Greuel [GLS98, GLS98a, GL99, GLS99, GLS02], Shustin [SHU97, SHU97a], Lossen about irreducibility, smoothness, existence,. . .
- ▷ Existence and connectedness have been solved for $d \leq 5$ by Namba [NMB86] and Degtyarev [DEG90], see here.

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  - $\Sigma'(6\mathbb{A}_2;6)$, $\Sigma''(6\mathbb{A}_2;6)$,... other ones (at least one)
- ▷ Study the case $d = 6$, $T_i = \mathbb{A}_k\,, \mathbb{D}_l\,, \mathbb{E}_r$

3. Sextics with simple points

- $\mathcal{C} \in \Sigma$, $\pi : \widehat{Y} \longrightarrow \mathbb{P}^2$ double covering ramified along $\mathcal{C}$, $\tau : Y \longrightarrow \widehat{Y}$ minimal resolution, Y $K3$ surface (see Barth-Peters-Van de Ven [BPV84])

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- $\mu(\mathcal{C})$ sum of Milnor numbers of $\mathrm{Sing}(\mathcal{C})$, Y $K3 \Rightarrow \mu(\mathcal{C}) \leq 19$
- Characterization of $\Sigma \neq \emptyset$ by Urabe, Yang [YA96] using Nikulin's results (intersection form lattice of a $K3$ surface)
  - Complete list when $\mu(\mathcal{C}) = 19$ and supplementary list for $\mu(\mathcal{C}) = 18$
  - $\Sigma \neq \emptyset$ if and only if the graph of singular points is a subgraph of a graph in one on the list
  - Yang also studies $\Sigma(\Gamma)$

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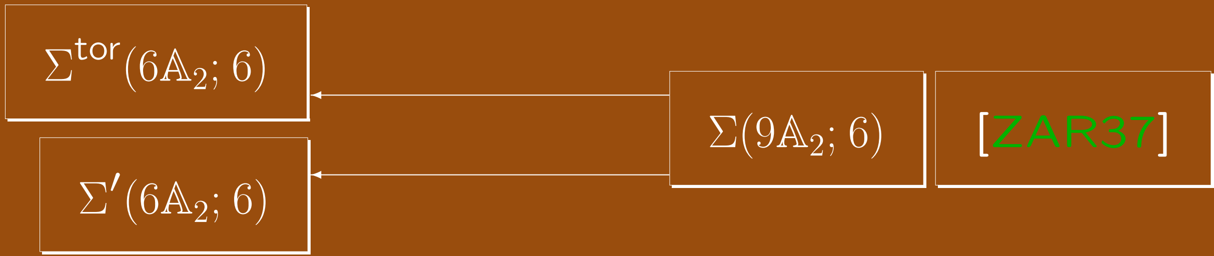

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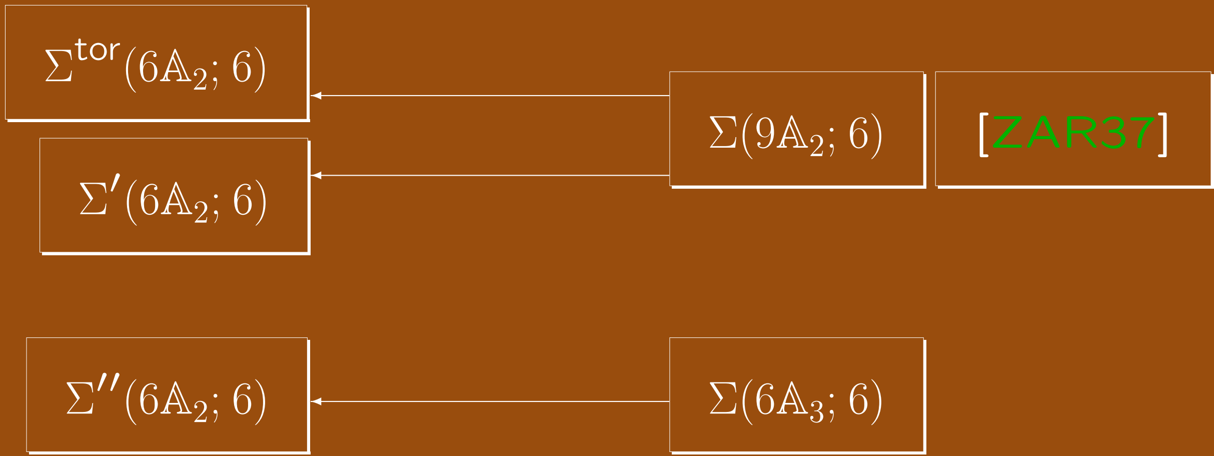

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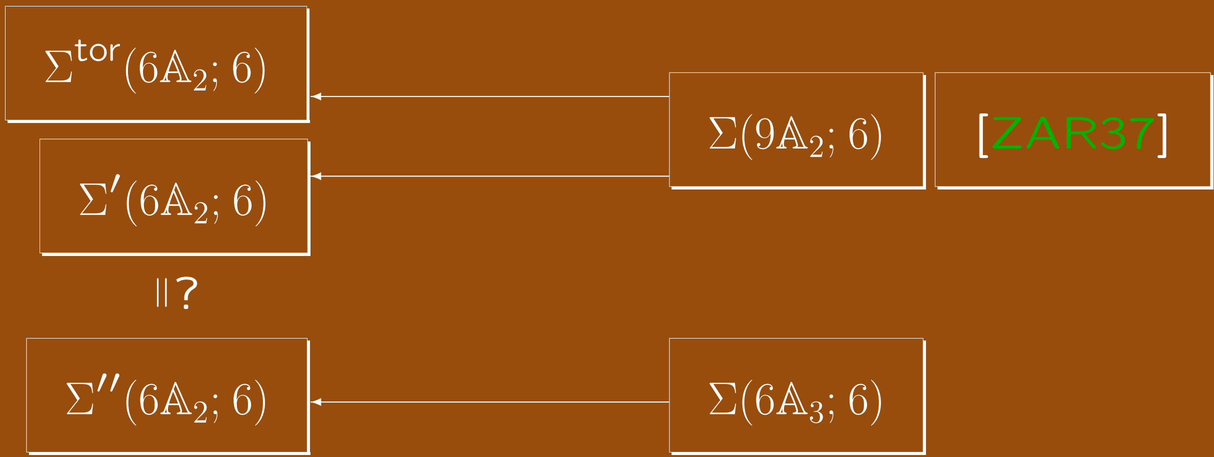

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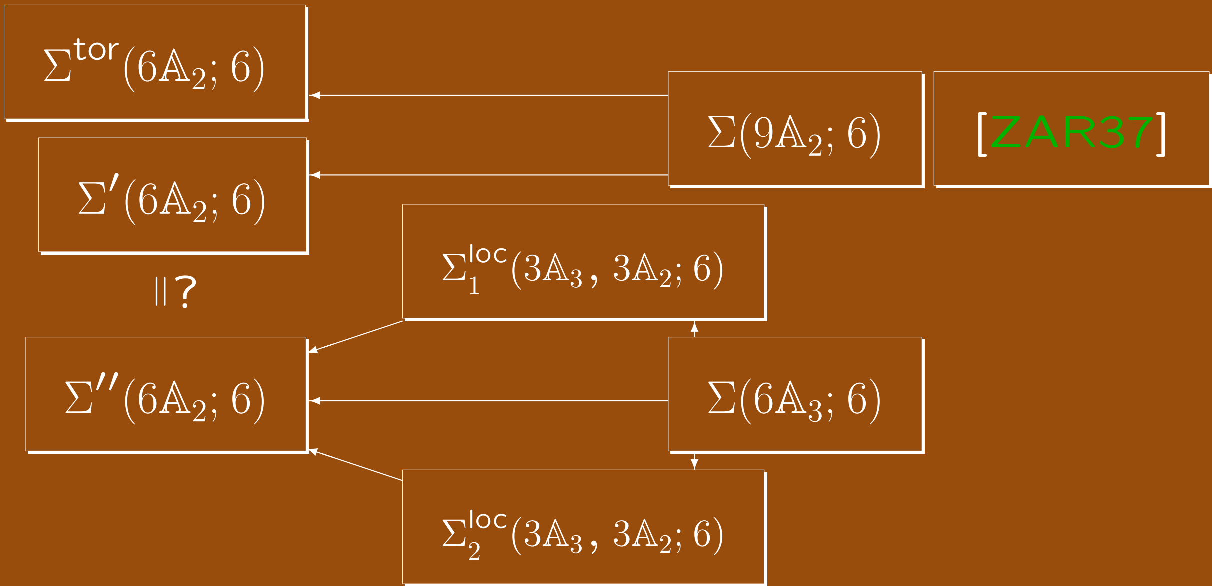

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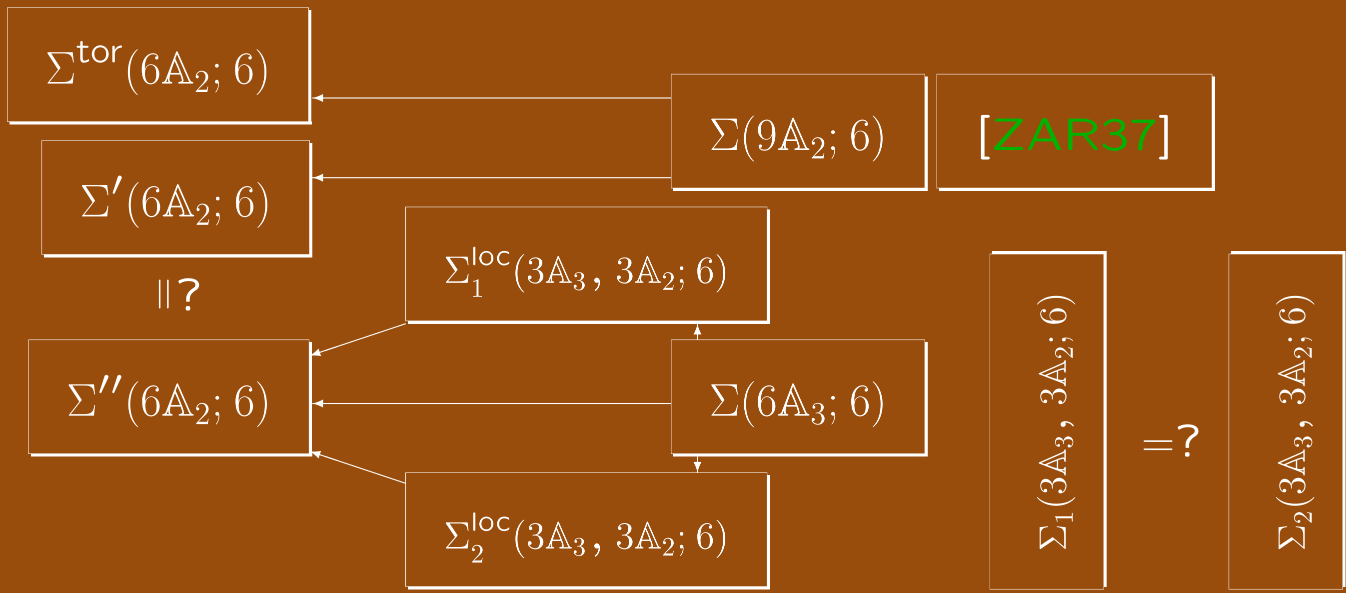

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- Σ_1: tangent line at $\mathbb{A}_{15}$ pass through $\mathbb{A}_3$
- Σ_2: *generic*
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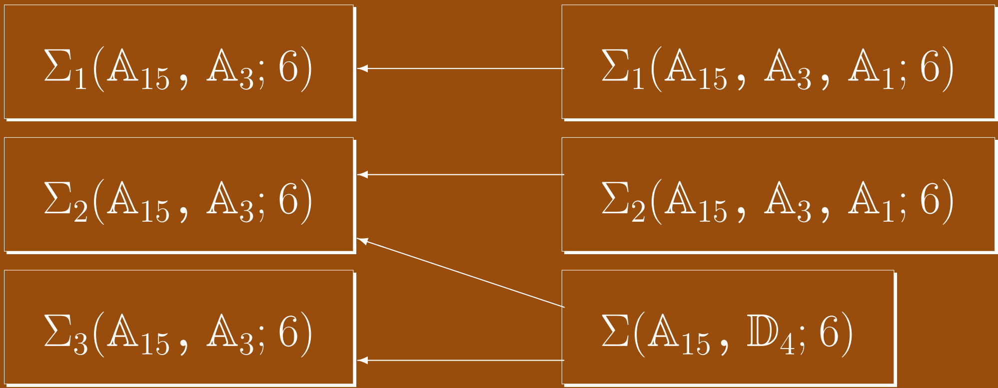

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$$\Sigma^{\mathsf{tor,irr}}(\mathbb{A}_{17}, \mathbb{A}_1; 6)$$

$$\Sigma_1^{\mathsf{irr}}(\mathbb{A}_{17}, \mathbb{A}_1; 6)$$

$$\Sigma^{\mathsf{tor,irr}}(\mathbb{A}_{17}, \mathbb{A}_1; 6) \longleftarrow \Sigma(\mathbb{A}_{17}, \mathbb{A}_2; 6)$$

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- $\Sigma_i(\mathbb{A}_{18}, \mathbb{A}_1; 6)$: $\exists$ conjugate representatives with coefficents in $\mathbb{Q}(19s^3 + 50s^2 + 36s + 8)$
- $\Sigma_j(\mathbb{A}_{19}; 6)$: $\exists$ conjugate representatives in $\mathbb{Q}(\sqrt{5})$ (see [YOS79] for a more complicated extension)

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- Many topological invariants come from algebraic properties
- Look for other invariants

5. Braid monodromy for affine curves

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$$f(x, y) = y^d + f_1(x)y^{d-1} + \cdots + f_{d-1}(x)y + f_d(x),$$

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- $\mathscr{D} := \{x \in \mathbb{C} \mid D(x) = 0\} = \{x_1, \ldots, x_r\}$
- $V := \{p(t) \in \mathbb{C}[t] \mid p$ monic of degree $d\}$, D discriminant hypersurface
- $V \setminus D \equiv \{A \subset \mathbb{C} \mid \#A = d\}$

$$\begin{aligned}\tilde{f} : \mathbb{C} \setminus \mathscr{D} &\longrightarrow V \setminus D \\ x &\longmapsto f(x, t)\end{aligned}$$

$$* := R \text{ s. t. } \mathscr{D} \subset \{z \in \mathbb{C} \mid |z| < R\}, \ \mathbf{y}^* := \tilde{f}(*)$$

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Braid monodromy of $\mathcal{C}^{\mathsf{aff}}$:

$$\nabla := \tilde{f}_* : \pi_1(\mathbb{C} \setminus \mathscr{D}; *) \longrightarrow \pi_1(V \setminus D; \mathbf{y}^*) \stackrel{..}{=} B_{\mathbf{y}^*}$$

Geometric bases of the free group $\pi_1(\mathbb{C} \setminus \mathscr{D}; *)$

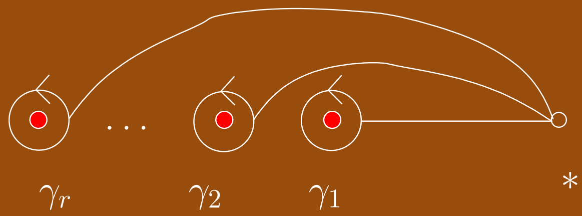

Figure 1: Geometric basis

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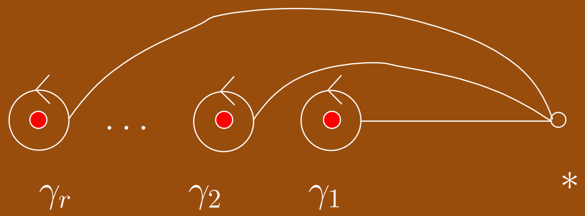

Figure 1: Geometric basis

- Each loop is meridian of a point of $\mathscr{D}$
- $c_\gamma := \gamma_r \cdot \ldots \cdot \gamma_1$ is the boundary of a big geometric disk; c_γ^{-1} is meridian of ∞
- $(\nabla(\gamma_1), \ldots, \nabla(\gamma_r)) \in (B_{\mathbf{y}^*})^r$

$$\mathbf{y}^0 := \{-1, \ldots, -d\}$$

$$B_{\mathbf{y}^0} \equiv B_d := \langle \sigma_1, \ldots, \sigma_{d-1} : [\sigma_i, \sigma_j] = 1, \; |i - j| \geq 2, \; \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \; i = 1, \ldots, d-2 \rangle$$

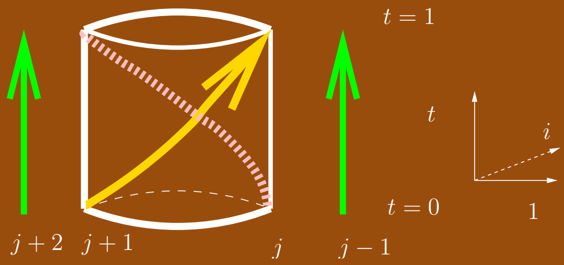

Figure 2: σ_j

- $\tau \in B(\mathbf{y}^*, \mathbf{y}^0)$ braid starting at $\mathbf{y}^*$ and ending at $\mathbf{y}^0$
- $\Phi_\tau : B_{\mathbf{y}^*} \longrightarrow B_d$, $\Phi_\tau(\sigma) := \tau \cdot \sigma \cdot \tau^{-1}$

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Braid monodromy $+ \cdots$

$\Downarrow$

An element of $(B_d)^r$

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  - Right action of B_r on $\mathscr{G}$:

$$(\gamma_1, \ldots, \gamma_r)^{\sigma_i} :=$$
$$(\gamma_1, \ldots, \gamma_{i-1}, \gamma_{i+1}, \gamma_{i+1}\gamma_i\gamma_{i+1}^{-1}, \gamma_{i+2}, \ldots, \gamma_r)$$

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  - Right action of B_d on B_d^r by simultaneous conjugation.

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  - Right action of B_d on B_d^r by simultaneous conjugation.
  - Pseudogeometric basis of $\pi_1(\mathbb{C} \setminus \mathscr{D}; *)$: c_γ^{-1} is a meridian of the line at infinity

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$$\cong$$

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Braid monodromy does not depend on Jung automorphisms as:

$$(x, y) \mapsto (ax + b, cy + p(x))$$

$a, c \in \mathbb{C}^*$, $b \in \mathbb{C}$, $p(x) \in \mathbb{C}[x]$

6. An example

$$\#\mathscr{M}(\mathbb{E}_6, \mathbb{A}_7, \mathbb{A}_3, \mathbb{A}_2, \mathbb{A}_1; 6) = 2$$

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Representantatives $\mathcal{C}_\beta$, $\beta^2 = 2$, with equations

$$f_\beta(x, y, z) g_\beta(x, y, z) = 0$$

having coefficients in $\mathbb{Q}(\sqrt{2})$

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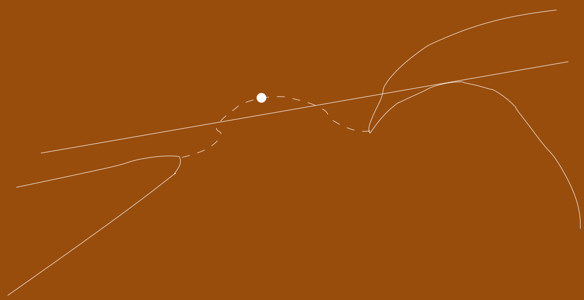
$$\begin{aligned} f_\beta(x, y, z) :=& y^2 z^3 + (303 - 216\,\beta)\, y z^2 x^2 + \\ &+ (-636 + 450\,\beta)\, y z x^3 + \\ &+ (-234\,\beta + 331)\, y x^4 + (-18\,\beta + 27)\, z x^4 + \\ &+ (18\,\beta - 26)\, x^5, \\ g_\beta(x, y, z) :=& y + \left(\frac{10449}{196} - \frac{3645}{98}\beta\right) z + \\ &+ \left(-\frac{432}{7} + \frac{297}{7}\beta\right) x. \end{aligned} \tag{1}$$

Curve $C_{\sqrt{2}}$



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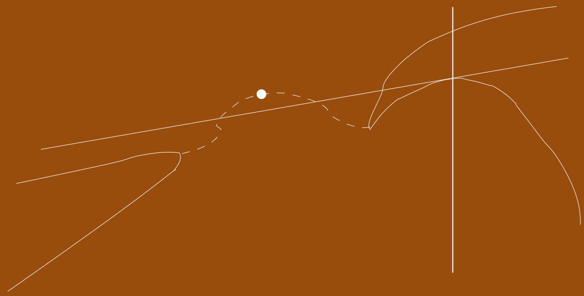

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$\gamma_2^{\sqrt{2}} \longmapsto$

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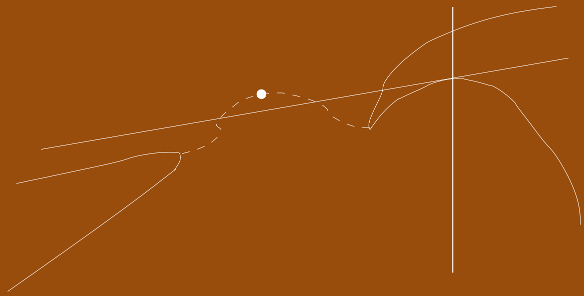

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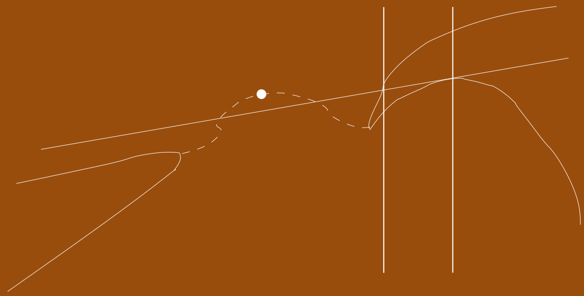

$\gamma_1^{\sqrt{2}} \mapsto \sigma_2^8$

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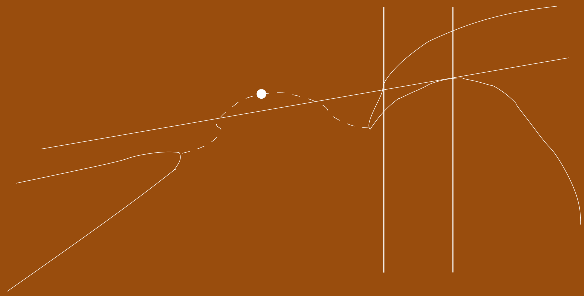

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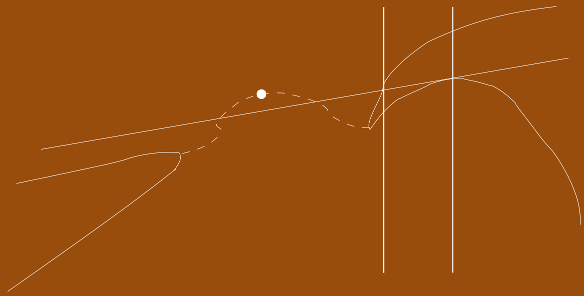

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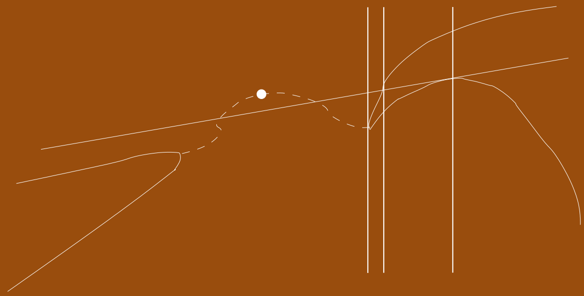

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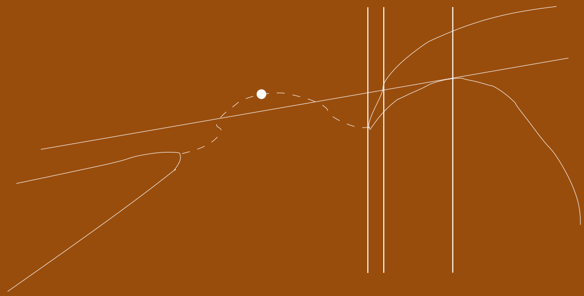

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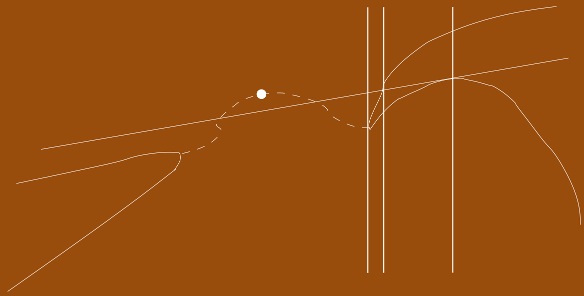

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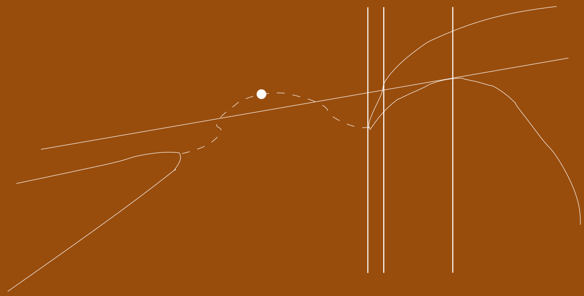

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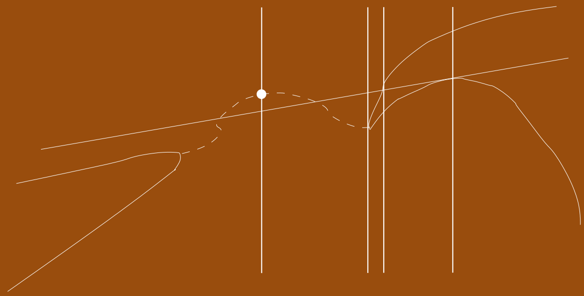

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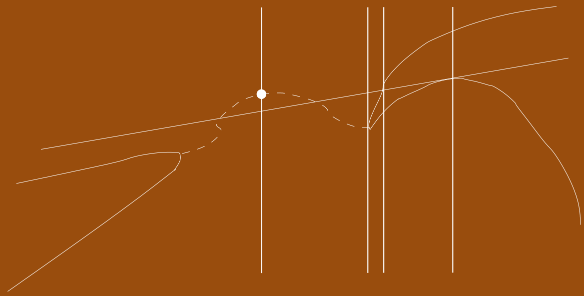

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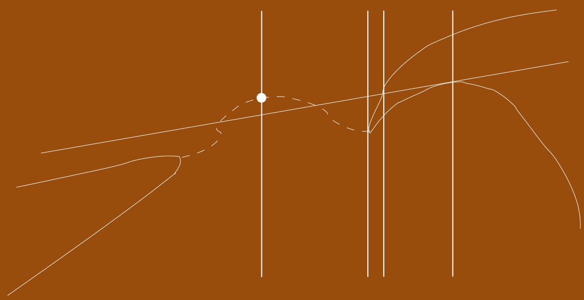

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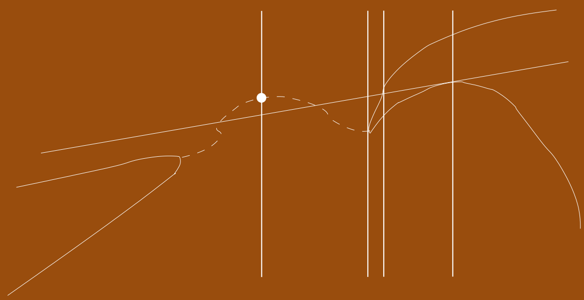

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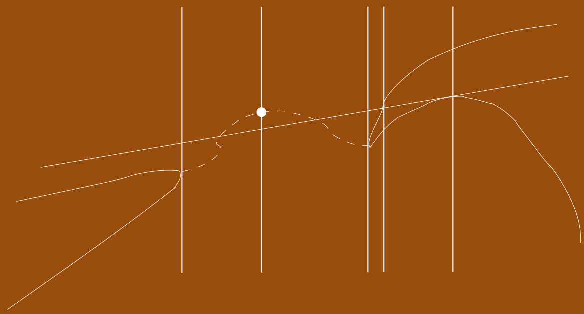

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Curve $C_{-\sqrt{2}}$

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$$\gamma_1^{-\sqrt{2}} \mapsto \sigma_2^3$$
$$\gamma_2^{-\sqrt{2}} \mapsto \left(\sigma_2\sigma_1^{-1}\sigma_2\right) * \sigma_1$$
$$\gamma_3^{-\sqrt{2}} \mapsto \sigma_2 * \sigma_1^8$$
$$\gamma_4^{-\sqrt{2}} \mapsto \sigma_1^{-2} * \sigma_2^4$$
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Look for topological consequences

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- J. Carmona (2002) [CAR02]: Same result without the restrictions

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Then, braid monodromies of the triples are equal.

Corollary 3. *$\mathcal{C}^{\varphi}_{\sqrt{2}} \cup L$ and $\mathcal{C}^{\varphi}_{-\sqrt{2}} \cup L$ are non-homeomorphic curves, conjugated in $\mathbb{Q}(\sqrt{2})$*

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Long exact sequence of homotopy

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Look for a presentation

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$$\mu_i^{\sigma_i} = \mu_{i+1} \qquad \mu_{i+1}^{\sigma_i} = \mu_{i+1} * \mu_i \qquad a * b := aba^{-1}$$

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$$\begin{array}{ccc} \pi_1(\mathbb{C} \setminus \mathbf{y}^*; M) & \stackrel{\sigma}{\longrightarrow} & \pi_1(\mathbb{C} \setminus \mathbf{y}^*; M) \\ \Psi_\tau \downarrow & & \downarrow \Psi_\tau \\ \pi_1(\mathbb{C} \setminus \mathbf{y}^0; M) & \stackrel{\Phi_\tau(\sigma)}{\longrightarrow} & \pi_1(\mathbb{C} \setminus \mathbf{y}^0; M) \end{array}$$

- Automorphism $\Psi_\tau : \pi_1(\mathbb{C}\setminus\mathbf{y}^*; M) \longrightarrow \pi_1(\mathbb{C}\setminus\mathbf{y}^0; M)$ induced by $\tau \in B(\mathbf{y}^*, \mathbf{y}^0)$
- $\Psi_\tau(\mu_j) = \mu_j^0$
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$$\begin{array}{ccc} \pi_1(\mathbb{C} \setminus \mathbf{y}^*; M) & \xrightarrow{\sigma} & \pi_1(\mathbb{C} \setminus \mathbf{y}^*; M) \\ \Psi_\tau \downarrow & & \downarrow \Psi_\tau \\ \pi_1(\mathbb{C} \setminus \mathbf{y}^0; M) & \xrightarrow{\Phi_\tau(\sigma)} & \pi_1(\mathbb{C} \setminus \mathbf{y}^0; M) \end{array}$$

- Lift a pseudo-geometric basis $\gamma_1, \ldots, \gamma_r$ of $\pi_1(\mathbb{C}\setminus \mathscr{D}; *)$ to $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_r$ in $\mathbb{C} \times \{M\}$, see Figure 8
- $\mu_i^{\tilde{\gamma}_j} =?$

$$\pi_1(\mathbb{C}^2 \backslash \mathcal{C}^{\varphi}; (*, M)) = \Big\langle \mu_1, \ldots, \mu_d, \tilde{\gamma}_1, \ldots, \tilde{\gamma}_r \ : \ \mu_i^{\tilde{\gamma}_j} = \mu_i^{\nabla(\gamma_j)}, i = 1, \ldots, d, \ j = 1, \ldots, r \Big\rangle \cong \Big\langle \mu_1^0, \ldots, \mu_d^0, \tilde{\gamma}_1, \ldots, \tilde{\gamma}_r \ : \ (\mu_i^0)^{\tilde{\gamma}_j} = (\mu_i^0)^{\nabla_\tau(\gamma_j)}, i = 1, \ldots, d, \ j = 1, \ldots, r \Big\rangle \tag{3}$$

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$$\pi_1(\mathbb{C}^2 \setminus \mathcal{C}^\varphi; (*, M)) = \Big\langle \mu_1, \ldots, \mu_d, \tilde{\gamma}_1, \ldots, \tilde{\gamma}_r \ : \ \mu_i^{\tilde{\gamma}_j} = \mu_i^{\nabla(\gamma_j)}, i = 1, \ldots, d, \ \ j = 1, \ldots, r \Big\rangle \cong$$

$$\Big\langle \mu_1^0, \ldots, \mu_d^0, \tilde{\gamma}_1, \ldots, \tilde{\gamma}_r \ : \ (\mu_i^0)^{\tilde{\gamma}_j} = (\mu_i^0)^{\nabla_\tau(\gamma_j)}, i = 1, \ldots, d, \ \ j = 1, \ldots, r \Big\rangle \tag{3}$$

- $\nabla_\tau(\gamma_j) \in B_d$ is determined by the presentation
- *A priori* these data are not topological invariants
- The goal is to prove that the *oriented topology* of $(\mathcal{C}^\varphi, L, P)$ does determine these data.

Step 1. *Meridians of $\mathcal{C}$ are determined by the* oriented topology *of* $(\mathcal{C}^{\varphi}, L, P)$

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Step 3. *Let us choose $*$ near one x_i; the element* $c := \mu_d \cdot \ldots \cdot \mu_1$ *is well-defined by the oriented topology of* $(\mathcal{C}^{\varphi}, L, P)$

Step 4. *An ordered family* $\hat{\mu}_1, \ldots, \hat{\mu}_d$ *of meridians of* $\mathcal{C}$ *such that* $c = \hat{\mu}_d \cdot \ldots \cdot \hat{\mu}_1$ *is a geometric basis of* K

Step 4. *An ordered family* $\hat{\mu}_1, \ldots, \hat{\mu}_d$ *of meridians of* $\mathcal{C}$ *such that* $c = \hat{\mu}_d \cdot \ldots \cdot \hat{\mu}_1$ *is a geometric basis of* K

Step 5. *The element* $\tilde{\gamma}_j$ *is the unique lift of* $\gamma_j \in H$, *which is a meridian of the line* $x = x_j z$ *an such that the conjugation by* $\tilde{\gamma}_j$ *induces on* K *a braid-like automorphism with respect to the family of geometric bases of* K

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Step 6. *The product $(\tilde{\gamma}_r \cdot \ldots \cdot \tilde{\gamma}_1)^{-1}$ is a meridian of the line L in $\pi_1(\mathbb{P}^2 \setminus (L_1 \cup \cdots \cup L_r \cup L); (*, M))$*

Sketch of the proof of Corollary 3

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- Let us suppose there exists a homeomorphism $\Phi : \mathbb{P}^2 \longrightarrow \mathbb{P}^2$ such that $\Phi(\mathcal{C}^{\varphi}_{\sqrt{2}} \cup L) = \mathcal{C}^{\varphi}_{-\sqrt{2}} \cup L$

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- It is easily seen that $\Phi(P) = P$, $\Phi(L) = L$ and $\Phi(\mathcal{C}^{\varphi}_{\sqrt{2}}) = \mathcal{C}^{\varphi}_{-\sqrt{2}}$
- By orientation properties of algebraic knots, the homeomorphism Φ preserves the orientation of $\mathbb{P}^2$
- Since curves have real equations, eventually applying complex conjugation, we may suppose that Φ preserves the orientations of the quintics in $\mathcal{C}_{\sqrt{2}}$ and $\mathcal{C}_{-\sqrt{2}}$

- From the relationship of intersection and linking numbers, we deduce that Φ preserves the orientations of L, $\mathcal{C}^{\varphi}_{\sqrt{2}}$ and $\mathcal{C}^{\varphi}_{-\sqrt{2}}$

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- Φ verifies the conditions stated in Theorem 2
- Contradiction with Theorem 1

References

[ACC02] E. Artal, J. Carmona and J.I. Cogolludo, *Braid monodromy and topology of plane curves*, accepted in Duke Math. J., 2002.

[ACC02a] E. Artal, J. Carmona and J.I. Cogolludo, *Effective invariants of braid monodromy*, Preprint, 2002.

[ACC02b] E. Artal, J. Carmona, and J. I. Cogolludo, *On sextic curves with big Milnor number*, Trends in Singularities (A. Libgober and M. Tibăr, eds.), Trends in Mathematics, Birkhäuser Verlag Basel/Switzerland, 2002, pp. 1–29.

[ARTIN47] ________ , *Theory of braids*, Ann. of Math. (2) **48** (1947), 101–126.

[BPV84] W. Barth, C. Peters, and A. Van de Ven, *Compact complex surfaces*, Erg. der Math. und ihrer Grenz., A Series of Modern Surveys in Math., 3, vol. 4, Springer-Verlag, Berlin, 1984.

[CAR02] J. Carmona, *Ph.D. thesis*.

[CHI37] O. Chisini, *Una suggestiva rappresentazione reale per le curve algebriche piane*, Ist. Lombardo, Rend., II. Ser. **66** (1933), 1141–1155.

[DEG90] A. I. Degtyarëv, *Isotopic classification of complex plane projective curves of degree* 5, Leningrad Math. J. **1** (1990), no. 4, 881–904.

[GLS98] Gert-Martin Greuel, Christoph Lossen and Eugenii Shustin, *New asymptotics in the geometry of equisingular families of curves*, Internat. Math. Res. Notices (1997), no. 13, 595–611. MR 98g:14039

[GLS98a] ______, *Geometry of families of nodal curves on the blown-up projective plane*, Trans. Amer. Math. Soc. **350** (1998), no. 1, 251–274. MR 98j:14034

[GLS99] ______, *Plane curves of minimal degree with prescribed singularities*, Invent. Math. **133** (1998), no. 3, 539–580. MR 99g:14035

[GLS02] ______, *The variety of plane curves with ordinary singularities is not irreducible*, Internat. Math. Res. Notices (2001), no. 11, 543–550. MR 2002e:14042

[GL99] Gert-Martin Greuel and Eugenii Shustin, *Geometry of equisingular families of curves*, Singularity theory (Liverpool, 1996), Cambridge Univ. Press, Cambridge, 1999, pp. xvi, 79–108. MR 2000e:14036

[VK33] E.R. van Kampen, *On the fundamental group of an algebraic curve*, Amer. J. Math. **55** (1933), 255–260.

[KT00] Vik. S. Kulikov and M. Teicher, *Braid monodromy factorizations and diffeomorphism types*, Izv. Ross. Akad. Nauk Ser. Mat. **64** (2000), no. 2, 89–120.

[LIB86] A. Libgober, *On the homotopy type of the complement to plane algebraic curves*, J. Reine Angew. Math. **367** (1986), 103–114.

[MOI81] B. G. Moishezon, *Stable branch curves and braid monodromies*, L.N.M. 862, Algebraic geometry (Chicago, Ill., 1980), Springer, Berlin, 1981, pp. 107–192.

[NMB86] M. Namba, *Geometry of projective algebraic curves*, Marcel Dekker Inc., New York, 1984. MR 86d:14021

[SHU97] Eugenii Shustin, *Geometry of equisingular families of plane algebraic curves*, J. Algebraic Geom. **5** (1996), no. 2, 209–234. MR 97g:14025

[SHU97a] ________, *Smoothness of equisingular families of plane algebraic curves*, Internat. Math. Res. Notices (1997), no. 2, 67–82. MR 97j:14031

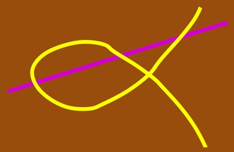
[YA96] J.-G. Yang, *Sextic curves with simple singularities*, Tohoku Math. J. (2) **48** (1996), no. 2, 203–227.

[YOS79] H. Yoshihara, *On plane rational curves*, Proc. Japan Acad. Ser. A Math. Sci. **55** (1979), no. 4, 152–155.

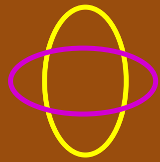
[ZAR29] O. Zariski, *On the problem of existence of algebraic functions of two variables possessing a given branch curve*, Amer. J. Math. **51** (1929), 305–328.

[ZAR31] ________, *On the irregularity of cyclic multiple planes*, Ann. Math. **32** (1931), 445–489.

[ZAR37] ________, *The topological discriminant group of a riemann surface of genus p*, Amer. J. Math. **59** (1937), 335–358.



(a) Nodal cubic and line

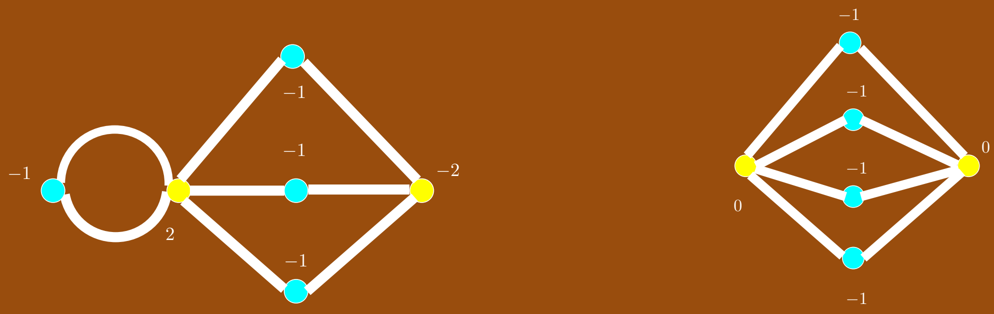


(b) Two conics

Figure 3: $\Sigma(4\mathbb{A}_1; 4)$

Define $\Sigma(\Gamma)$ and $\mathscr{M}(\Gamma)$ where Γ is:

- A weighted bi-coloured graph, which is dual to $\sigma^{-1}(\mathcal{C})$, $\sigma : Y \longrightarrow \mathbb{P}^2$, minimal embedded resolution of $\operatorname{Sing}(\mathcal{C})$.
- Weight $\equiv$ self-intersection number
- Vertices $\alpha \equiv$ exceptional divisor of σ
- Vertices $\beta \equiv$ strict transform of $\mathcal{C}$



(a) Nodal cubic and line

(b) Two conics

Figure 4: Graphs

If $d \leq 5$ and $\Sigma(\Gamma) \neq \emptyset$, $\Sigma(\Gamma)$ is irreducible

Go back

Definition of meridian

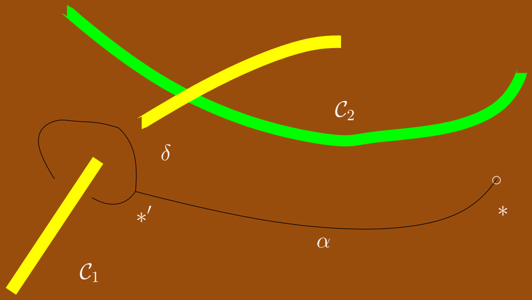


Figure 5: Meridian

- X surface, $\mathcal{C} \subset X$ curve, $\mathcal{C}_1 \subset \mathcal{C}$ irreducible component, $* \in X \setminus \mathcal{C}$, $G := \pi_1(X \setminus \mathcal{C}; *)$

Definition of meridian

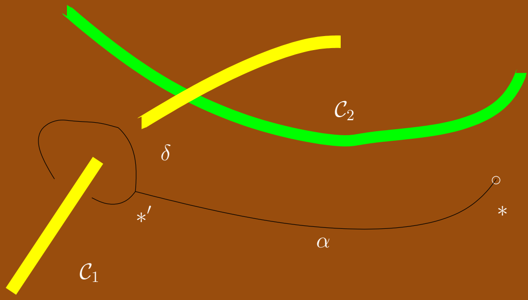


Figure 5: Meridian

- X surface, $\mathcal{C} \subset X$ curve, $\mathcal{C}_1 \subset \mathcal{C}$ irreducible component, $* \in X \setminus \mathcal{C}$, $G := \pi_1(X \setminus \mathcal{C}; *)$
- Δ small analytic disk $\pitchfork \mathcal{C}_1$, $*' \in \partial\Delta$, α path from $*$ to $*'$, δ loop en $*'$ running once and counterclockwise $\partial\Delta$

Definition of meridian

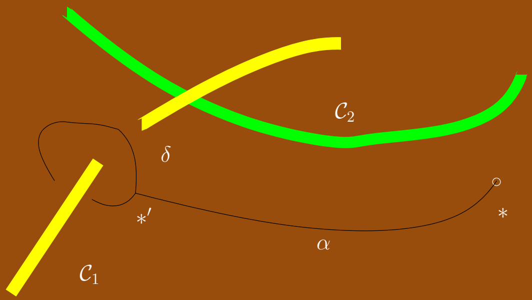


Figure 5: Meridian

- X surface, $\mathcal{C} \subset X$ curve, $\mathcal{C}_1 \subset \mathcal{C}$ irreducible component, $* \in X \setminus \mathcal{C}$, $G := \pi_1(X \setminus \mathcal{C}; *)$
- Δ small analytic disk $\pitchfork \mathcal{C}_1$, $*' \in \partial\Delta$, α path from $*$ to $*'$, δ loop en $*'$ running once and counterclockwise $\partial\Delta$
- $\alpha \cdot \delta \cdot \alpha^{-1}$ is a *meridian* of $\mathcal{C}_1$ in G. The set of meridians of $\mathcal{C}_1$ is a conjugation class. Go back

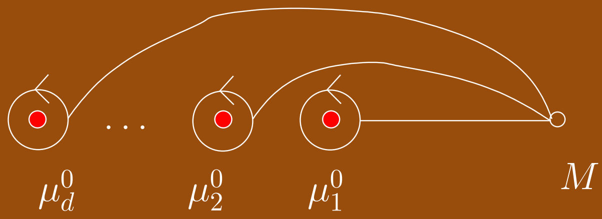


Figure 6: Geometric basis in the fiber

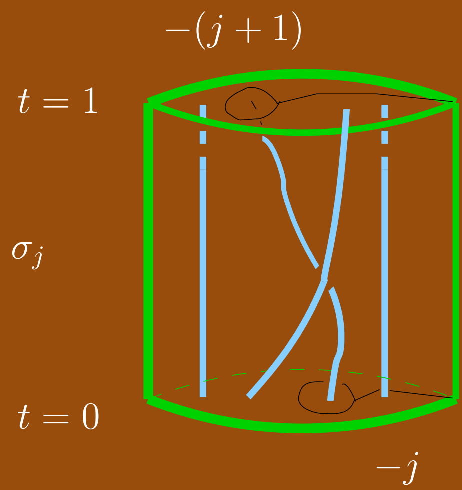


Figure 7: Action of σ_j

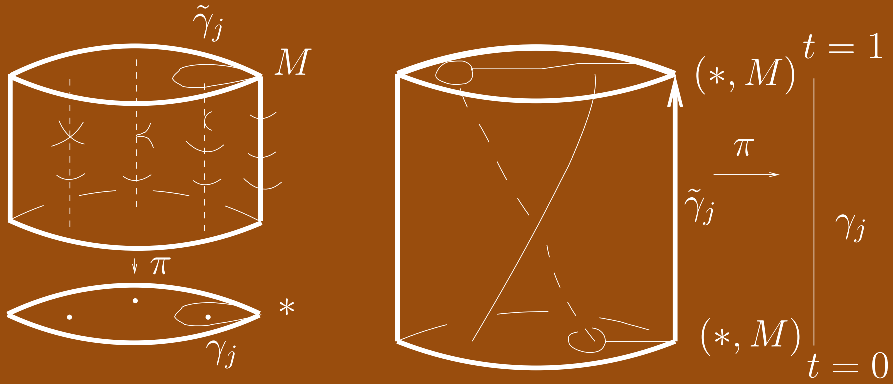


Figure 8: Adapted polydisks and conjugation