Algebraic, geometric and topological properties of complex projective plane curves

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Topology & Geometry Seminar
Singapore, October 9th 2024

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- Complex algebraic curve: Riemann surface with singularities.
- Projective $\iff$ compact.
- $\mathcal{C} \subset \mathbb{P}^2$: irreducible components, degree, topological type of singularities.

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    - Link in $\mathbb{S}^3$
    - Smooth point, $y = 0$, unknot
    - Node: $y^2 - x^2 = 0$, Hopf link
    - Ordinary cusp: $y^2 - x^3 = 0$, trefoil knot
    - $y^p - x^q = 0$, (p, q) torus link

Local and global properties

Local

- Singularity defined by $\{f(x,y)=0\}$, $f \in \mathbb{C}\{x,y\}$ reduced:
  - Milnor number: $\mu = \dim_{\mathbb{C}} \mathbb{C}\{x,y\}/\operatorname{Jac} f$
  - Milnor fiber: $\Sigma_t : f(x,y)=t$ in a small ball, $\chi(\Sigma_t) = 1-\mu$

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- $\mathcal{O} := \mathbb{C}\{x,y\}/(f)$, $\overline{\mathcal{O}}$ its integral closure, $\delta = \dim_{\mathbb{C}} \overline{\mathcal{O}}/\mathcal{O}$.
- $2\delta = \mu + r - 1$, $r = \#$ of irreducible factors of f (only in characteristic 0).
- $f = y^2 - x^n$, $\delta = \left\lfloor \frac{n}{2} \right\rfloor$.

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Global

- $\mathcal{C}$ irreducible of degree d: $g=\dfrac{(d-1)(d-2)}{2}-\sum_{P\in\operatorname{Sing}\mathcal{C}}\delta_P$.
- $d=1,2$: rational curves.
- $d=3$: elliptic if smooth, rational if singular.
- $d=6$, 6 ordinary cusps $g=4$.

Embedded topology

- $\mathcal{C}$ irreducible of degree d: $H_1(\mathbb{P}^2 \setminus \mathcal{C}; \mathbb{Z}) \cong \mathbb{Z}/d$.
- $\mathcal{C} = \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_r$, $\mathcal{C}_i$ irreducible of degree d_i:

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- $\mathcal{C}$ degree 6 with six cusps NON on a conic: $\nexists \pi_1(\mathbb{P}^2 \smallsetminus \mathcal{C}) \twoheadrightarrow \mathfrak{S}_3$, actually $\cong \mathbb{Z}/2 \times \mathbb{Z}/3$

Zariski pairs

Definition

The combinatorics of $\mathcal{C} \subset \mathbb{P}^2$ is the topological type of $(\mathcal{T}, \mathcal{C})$, $\mathcal{T}$ closed regular neighbourhood of $\mathcal{C}$.

- $\mathcal{C}$ arrangement of lines: combinatorics $\equiv$ intersection pattern
- $\mathcal{C}$ irreducible: combinatorics $\equiv$ degree and topological type of singularities

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$\mathcal{C}_1, \mathcal{C}_2 \subset \mathbb{P}^2$ Zariski pair if same combinatorics $\nexists (\mathbb{P}^2, \mathcal{C}_1) \xrightarrow{\text{homeo}} (\mathbb{P}^2, \mathcal{C}_2)$

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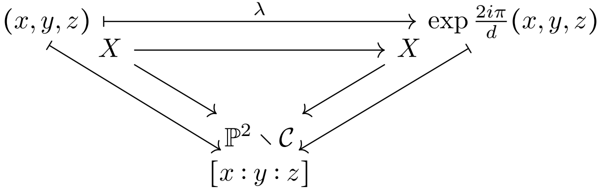
- Alexander-Zariski pairs: distinct Alexander polynomials.

$$\mathcal{C} = \{F_d(x,y,z) = 0\} \subset \mathbb{P}^3, \quad X := \{F_d(x,y,z) = 1\} \subset \mathbb{C}^3$$



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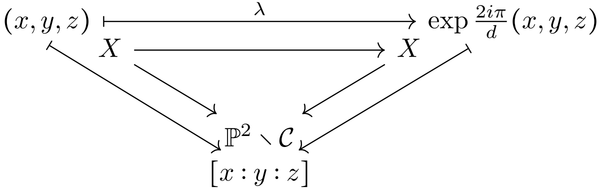

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- $\mathcal{C}$ Zariski sextic six points on a conic: $\Delta(t) = t^2 - t + 1$
- $\mathcal{C}$ Zariski sextic six points NOT on a conic: $\Delta(t) = 1$

Characteristics of Zariski pairs.

- $\exists$ Zariski pairs not Alexander Zariski (arrangements of cubics and conics):
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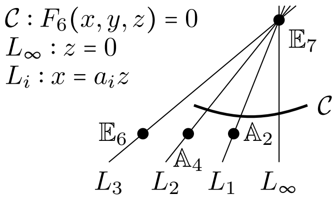
- ∃ Zariski pairs with homeomorphic complements
- ∃ arithmetic Zariski pairs

Definition

$\mathbb{K}$ number field, $\mathcal{C} \subset \mathbb{K}\mathbb{P}^2$ curve: $\{\mathcal{C} \otimes_\varphi \mathbb{C} \mid \varphi : \mathbb{K} \hookrightarrow \mathbb{C}\}$ arithmetic Zariski tuple if pairwise Zariski pairs

Complement-homeomorphic arithmetic Zariski pair



- $F_6 \in \mathbb{Q}[\sqrt{5}][x, y, z]$, $f(x, y) := F_6(x, y, 1)$ y-monic, $\deg_y f = 3$
- $\varphi \in \operatorname{Gal} \mathbb{Q}[\sqrt{5}]/\mathbb{Q}$, $\varphi(\sqrt{5}) = -\sqrt{5}$
- $\exists a, b, c, d, e \in \mathbb{Q}[\sqrt{5}]^*$ s.t. $f^\varphi(x, -y + ax + bx^2) = cf(x, dx + ey)$.
- $\mathbb{E}_7 : v(v^2 - u^3) = 0$
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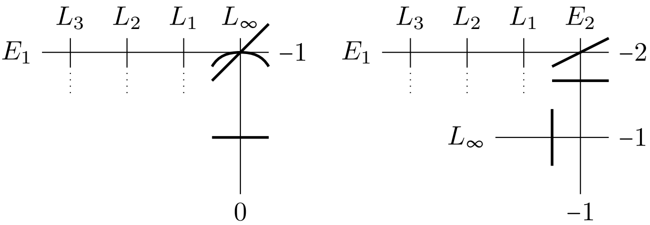

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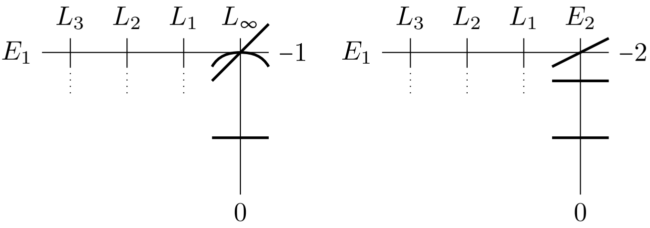

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Triangular combinatorics

Combinatorics $\mathscr{C}$ of degree d

- $\Sigma^{\mathscr{C}} := \{\text{curves satisfying } \mathscr{C}\}$ quasiprojective variety
- $\mathscr{M}^{\mathscr{C}} := \Sigma^{\mathscr{C}} / \mathrm{PGL}(3; \mathbb{C})$

Triangular combinatorics

$\mathscr{M}_{d,1} = \mathscr{M}_1^{d+3} := \mathscr{M}^{\mathscr{C}_1}$, $d > 1$

$\mathscr{C}_1$: irreducible components $\mathcal{C}_d, L_1^1, L_1^2, L_1^3$ (subscript is degree)

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- $\mathcal{D}_{2d}$ with three singular points, topological type $v^d - u^{d+1} = 0$
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- $\mathscr{C}_1 \leftrightarrow \mathscr{C}_2$ via standard Cremona transformation

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$\mathscr{C}_3$: the irreducible component $\mathcal{D}_{2d}$ of $\mathscr{M}_2^{2d+3}$

Moduli spaces

Theorem

$d > 2,\ \zeta = \exp\frac{2i\pi}{d},\ \mathscr{M}_{d,j} = \bigcup_{k=0}^{\lfloor\frac{d}{2}\rfloor}\mathscr{M}_{d,j}^{\zeta^k}$

- $\mathscr{M}_{d,j}^{\zeta^k} \neq \varnothing$ *and connected.*
- $\exists F_{d,j,a} \in \mathbb{Q}[a][x,y,z]$ *s.t* $\mathcal{C}_{d,j,k} : \{F_{d,j,\zeta^k}(x,y,z) = 0\} \in \mathscr{M}_{d,j}^{\zeta^k}$
- $\pi_1(\mathbb{P}^2 \setminus \mathcal{C}_{d,j,k})$ *is abelian except for* $d = 3,\ j = 1,2,3,\ k = 0.$
- $\left\{\mathcal{C}_{d,j,k} \mid 0 \leq k \leq \left\lfloor\frac{d}{2}\right\rfloor\right\}$ *is a Zariski tuple for:*
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Open problem

Is $\left\{\mathcal{C}_{d,3,k} \mid 0 \leq k \leq \left\lfloor \frac{d}{2} \right\rfloor\right\}$ a Zariski tuple for $d > 3$?

Construction of the curves

Proposition

For $\mathcal{C}_d + L_1^1 + L_1^2 + L_1^3 \in \Sigma^{\mathscr{C}_1}$*, there is an automorphism* $\Phi \in \mathrm{PGL}(3;\mathbb{C})$ *such that*

- $\Phi(L_1^1 + L_1^2 + L_1^3) = \{x = 0\} + \{y = 0\} + \{z = 0\}$
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Curves after Φ

$\mathcal{C}_d = \{G_d(x,y,z) = 0\}$ where

- $G_d(0,y,z) = (y+z)^d$, $G_d(x,0,z) = (x+z)^d$

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- For $\Sigma^{\mathscr{C}_2}$ apply $[x : y : z] \mapsto [yz : xz : xy]$
- For $\Sigma^{\mathscr{C}_3}$ forget the lines.

Zariski tuples

Sketch of the proof.

- $V = \{[x:y:z:t] \in \mathbb{P}^3 \mid t^d - G_d(x,y,z) = 0\}$

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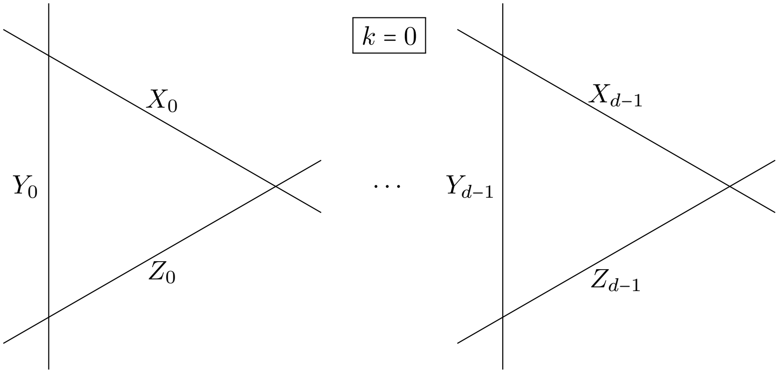
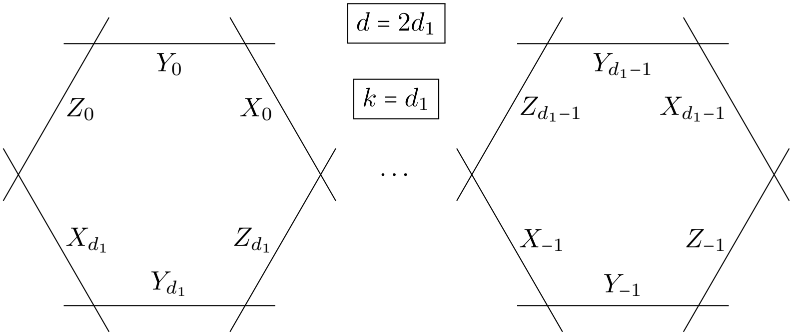
Apply covering arguments □

Preimages of the lines

$k = 0$



Preimages of the lines



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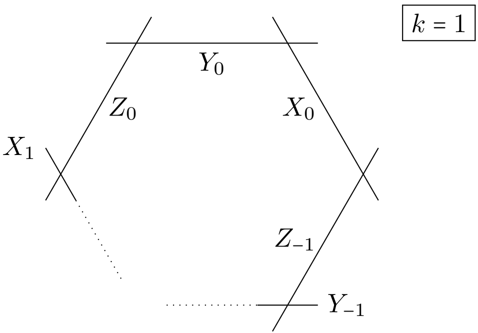

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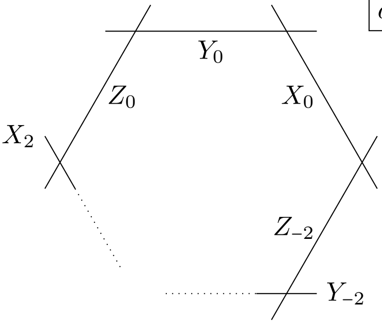

$d = 2d_1 + 1$

$k = 2$

Thank you for your attention!