Orbifolds for understanding symmetric curves and line arrangements

Enrique ARTAL BARTOLO

Departamento de Matemáticas
Facultad de Ciencias
Instituto Universitario de Matemáticas y sus Aplicaciones
Universidad de Zaragoza

Geometry, Topology and Algebra of Singular Spaces
70th anniversary of Prof. Alex Suciu and Prof. Laurențiu Păunescu
IMAR, Bucharest, Romania, July 11th 2023

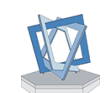
Funded by MCIN/AEI/10.13039/501100011033 (PID2020-114750GB-C31) and Gobierno de Aragón (E22_20R: "Álgebra y Geometría").



Questions on McLane arrangements

McLane arrangements $\mathscr{ML}$

- $\mathscr{ML}_\zeta := \{xyz(x-y)(x-z)(\zeta y+z)(\bar{\zeta}x+\zeta y+z)(x+\bar{\zeta}y-z)=0\}$, $\zeta = \exp\frac{\pm 2i\pi}{3}$



Questions on McLane arrangements

McLane arrangements $\mathscr{ML}$

- $\mathscr{ML}_\zeta := \{xyz(x-y)(x-z)(\zeta y+z)(\bar{\zeta}x+\zeta y+z)(x+\bar{\zeta}y-z)=0\}$, $\zeta = \exp\frac{\pm 2i\pi}{3}$
- $\mathscr{ML}$ smallest combinatorics with no real equation



Questions on McLane arrangements

McLane arrangements $\mathscr{ML}$

- $\mathscr{ML}_\zeta := \{xyz(x-y)(x-z)(\zeta y+z)(\bar{\zeta}x+\zeta y+z)(x+\bar{\zeta}y-z)=0\}$, $\zeta = \exp\frac{\pm 2i\pi}{3}$
- $\mathscr{ML}$ smallest combinatorics with no real equations for every line
- $\mathscr{ML} := \{(x^2+xy+y^2)(x^3-z^3)(y^3-z^3)=0\}$



Questions on McLane arrangements

McLane arrangements $\mathscr{ML}$

- $\mathscr{ML}_\zeta := \{xyz(x-y)(x-z)(\zeta y+z)(\bar{\zeta}x+\zeta y+z)(x+\bar{\zeta}y-z)=0\}$, $\zeta = \exp\frac{\pm 2i\pi}{3}$
- $\mathscr{ML}$ smallest combinatorics with no real equations for every line
- $\mathscr{ML} := \{(x^2+xy+y^2)(x^3-z^3)(y^3-z^3)=0\}$

Question

Can $\mathscr{ML}$ be defined by an irreducible polynomial in $\mathbb{Q}[x,y,z]$?



Questions on McLane arrangements

McLane arrangements $\mathscr{ML}$

- $\mathscr{ML}_\zeta := \{xyz(x-y)(x-z)(\zeta y+z)(\bar{\zeta}x+\zeta y+z)(x+\bar{\zeta}y-z)=0\}$, $\zeta = \exp\frac{\pm 2i\pi}{3}$
- $\mathscr{ML}$ smallest combinatorics with no real equations for every line
- $\mathscr{ML} := \{(x^2+xy+y^2)(x^3-z^3)(y^3-z^3)=0\}$

Question

Can symmetry provide more understanding to the topology of $(\mathbb{P}^2, \mathscr{ML})$?



Questions on McLane arrangements

McLane arrangements $\mathscr{ML}$

- $\mathscr{ML}_\zeta := \{xyz(x-y)(x-z)(\zeta y+z)(\bar{\zeta}x+\zeta y+z)(x+\bar{\zeta}y-z)=0\}$, $\zeta = \exp\frac{\pm 2i\pi}{3}$
- $\mathscr{ML}$ smallest combinatorics with no real equations for every line
- $\mathscr{ML} := \{(x^2+xy+y^2)(x^3-z^3)(y^3-z^3)=0\}$

Question

Can $\mathscr{ML}$ be defined by an irreducible polynomial in $\mathbb{Q}[x,y,z]$?

Strategy

Work with $\operatorname{Aut}\mathscr{ML}$ and study $\mathbb{P}^2/\operatorname{Aut}\mathscr{ML}$.



Questions on McLane arrangements

McLane arrangements $\mathscr{ML}$

- $\mathscr{ML}_\zeta := \{xyz(x-y)(x-z)(\zeta y+z)(\bar{\zeta}x+\zeta y+z)(x+\bar{\zeta}y-z)=0\}$, $\zeta = \exp\frac{\pm 2i\pi}{3}$
- $\mathscr{ML}$ smallest combinatorics with no real equations for every line
- $\mathscr{ML} := \{(x^2+xy+y^2)(x^3-z^3)(y^3-z^3)=0\}$

Question

Can $\mathscr{ML}$ be defined by an irreducible polynomial in $\mathbb{Q}[x,y,z]$?

Notation

- $\mu_n := \{\zeta \in \mathbb{C} \mid \zeta^n = 1\}$
- $\mu_n \times \mathbb{C} \to \mathbb{C}$, $\zeta \cdot t := \zeta^a t$: $\boxed{\frac{1}{n}(a)}$



Questions on McLane arrangements

McLane arrangements $\mathscr{ML}$

- $\mathscr{ML}_\zeta := \{xyz(x-y)(x-z)(\zeta y+z)(\bar{\zeta}x+\zeta y+z)(x+\bar{\zeta}y-z)=0\}$, $\zeta = \exp\frac{\pm 2i\pi}{3}$
- $\mathscr{ML}$ smallest combinatorics with no real equations for every line
- $\mathscr{ML} := \{(x^2+xy+y^2)(x^3-z^3)(y^3-z^3)=0\}$

Question

Can $\mathscr{ML}$ be defined by an irreducible polynomial in $\mathbb{Q}[x,y,z]$?

Notation

- $\mu_n := \{\zeta \in \mathbb{C} \mid \zeta^n = 1\}$
- $\mu_n \times \mathbb{C} \to \mathbb{C}$, $\zeta \cdot t := \zeta^a t$: $\boxed{\frac{1}{n}(a)}$
- $\prod_{j=1}^{r} \mu_{m_j} \times \mathbb{C}^n$, $\zeta_j \cdot (z_1,\ldots,z_n) := (\zeta_j^{a_{j1}} \cdot z_1, \ldots, \zeta_j^{a_{jn}} \cdot z_n)$.



Questions on McLane arrangements

McLane arrangements $\mathscr{ML}$

- $\mathscr{ML}_\zeta := \{xyz(x-y)(x-z)(\zeta y+z)(\bar{\zeta}x+\zeta y+z)(x+\bar{\zeta}y-z)=0\}$, $\zeta = \exp\frac{\pm 2i\pi}{3}$
- $\mathscr{ML}$ smallest combinatorics with no real equations for every line
- $\mathscr{ML} := \{(x^2+xy+y^2)(x^3-z^3)(y^3-z^3)=0\}$

Question

Can $\mathscr{ML}$ be defined by an irreducible polynomial in $\mathbb{Q}[x,y,z]$?

Notation

- $\mu_n := \{\zeta \in \mathbb{C} \mid \zeta^n = 1\}$
- $\mu_n \times \mathbb{C} \to \mathbb{C}$, $\zeta \cdot t := \zeta^a t$: $\boxed{\frac{1}{n}(a)}$
- $\begin{pmatrix} \frac{1}{m_1} \\ \vdots \\ \frac{1}{m_r} \end{pmatrix} \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{r1} & \cdots & a_{rn} \end{pmatrix}$



Questions on McLane arrangements

McLane arrangements $\mathscr{ML}$

- $\mathscr{ML}_\zeta := \{xyz(x-y)(x-z)(\zeta y+z)(\bar{\zeta}x+\zeta y+z)(x+\bar{\zeta}y-z)=0\}$, $\zeta = \exp\frac{\pm 2i\pi}{3}$
- $\mathscr{ML}$ smallest combinatorics with no real equations for every line
- $\mathscr{ML} := \{(x^2+xy+y^2)(x^3-z^3)(y^3-z^3)=0\}$

Question

Can $\mathscr{ML}$ be defined by an irreducible polynomial in $\mathbb{Q}[x,y,z]$?

Notation

- $\mu_n := \{\zeta \in \mathbb{C} \mid \zeta^n = 1\}$
- $\mu_n \times \mathbb{C} \to \mathbb{C}$, $\zeta \cdot t := \zeta^a t$: $\boxed{\frac{1}{n}(a)}$
- $\frac{1}{m}\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{r1} & \cdots & a_{rn} \end{pmatrix}$ if $m = m_1 = \cdots = m_r$



A symmetric curve

Tricuspidal quartic

- $\mathcal{C} : x^2y^2 + x^2z^2 + y^2z^2 - 2xyz(x + y + z) = 0$
- Symmetric by the natural action of $\mathfrak{S}_3$.



A symmetric curve

Tricuspidal quartic

- $\mathcal{C} : x^2y^2 + x^2z^2 + y^2z^2 - 2xyz(x + y + z) = 0$
- Symmetric by the natural action of $\mathfrak{S}_3$.
- $\mathcal{C} : z^4 - 24xyz^2 + 32(x^3 + y^3)z - 48x^2y^2 = 0.$
- Symmetric by the action of $\mathbb{D}_6$:
  - $[x : y : z] \mapsto [\zeta x : \overline{\zeta} y : z]$.
  - $[x : y : z] \mapsto [y : x : z]$.



A symmetric curve

Tricuspidal quartic

- $\mathcal{C} : x^2y^2 + x^2z^2 + y^2z^2 - 2xyz(x + y + z) = 0$
- Symmetric by the natural action of $\mathfrak{S}_3$.
- $\mathcal{C} : z^4 - 24xyz^2 + 32(x^3 + y^3)z - 48x^2y^2 = 0.$
- Symmetric by the action of $\mathbb{D}_6$:
  - $[x : y : z] \mapsto [\zeta x : \overline{\zeta} y : z]$.
  - $[x : y : z] \mapsto [y : x : z]$.

Strategy

Study $\mathcal{C} \subset \mathbb{P}^2$ via $\mathbb{P}^2/\mathbb{D}_6$



A symmetric curve

Tricuspidal quartic

- $\mathcal{C} : x^2y^2 + x^2z^2 + y^2z^2 - 2xyz(x + y + z) = 0$
- Symmetric by the natural action of $\mathfrak{S}_3$.
- $\mathcal{C} : z^4 - 24xyz^2 + 32(x^3 + y^3)z - 48x^2y^2 = 0.$
- Symmetric by the action of $\mathbb{D}_6$:
  - $[x : y : z] \mapsto [\zeta x : \overline{\zeta} y : z]$.
  - $[x : y : z] \mapsto [y : x : z]$.

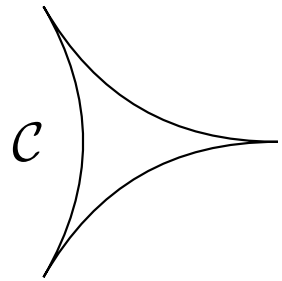
Strategy

Study $\mathcal{C} \subset \mathbb{P}^2$ via $\mathbb{P}^2/\mathbb{D}_6$

- Birational model of $\mathbb{P}^2 \to \mathbb{P}^2/\mathbb{D}_6$:
  $[x : y : z] \mapsto [4x^3y^3 : (x^3 + y^3)^2 : xyz(x^3 + y^3)]$
- $\mathcal{C}$ comes from $\mathcal{C}_1 : z^4 - 6xyz^2 + 8xy^2z - 3x^2y^2 = 0.$

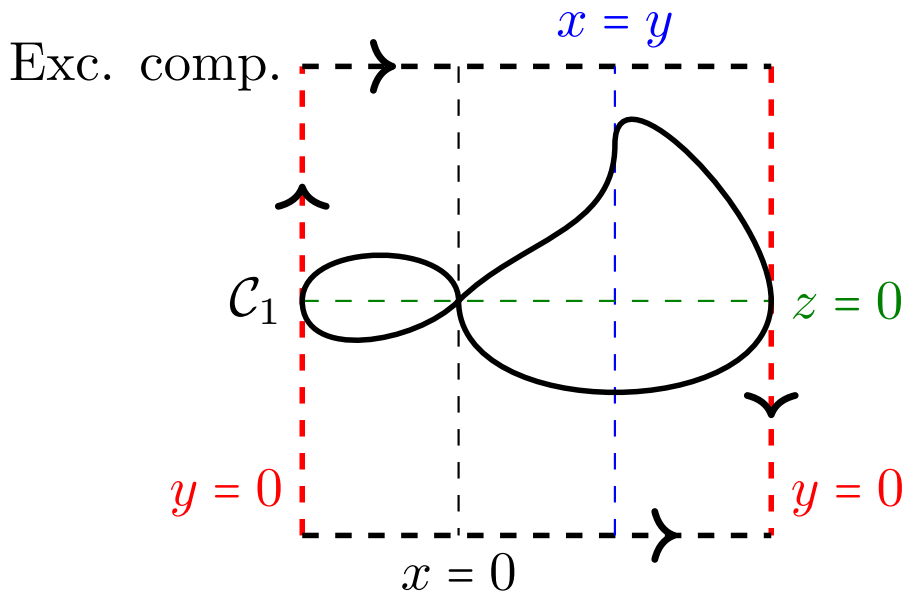


Geometry





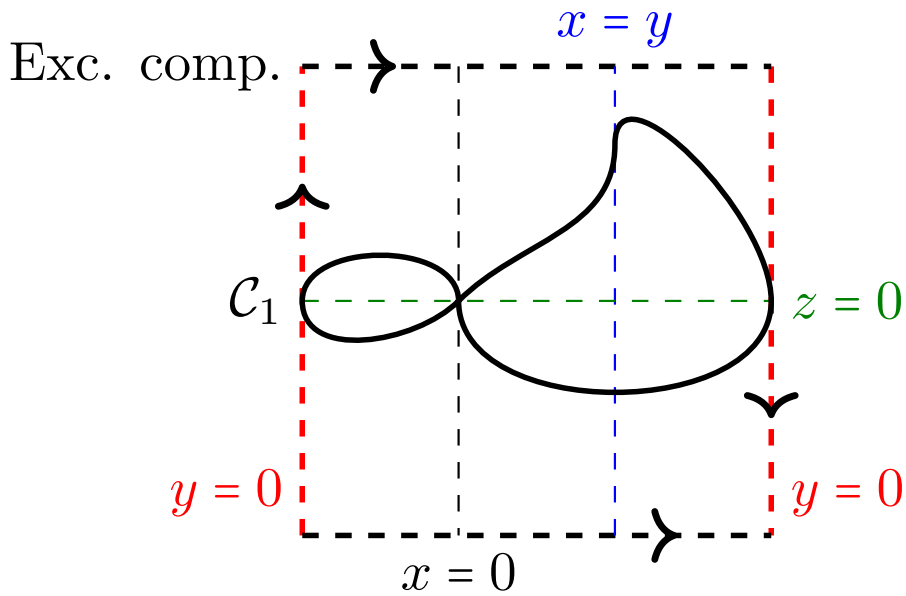
Geometry



- $y = 0$: $\mathcal{C}_1$ is like $u^2 = v^4$.
- $x = 0$: ordinary node with flex vertical branch.
- $x = y$: vertical tangent at flex.



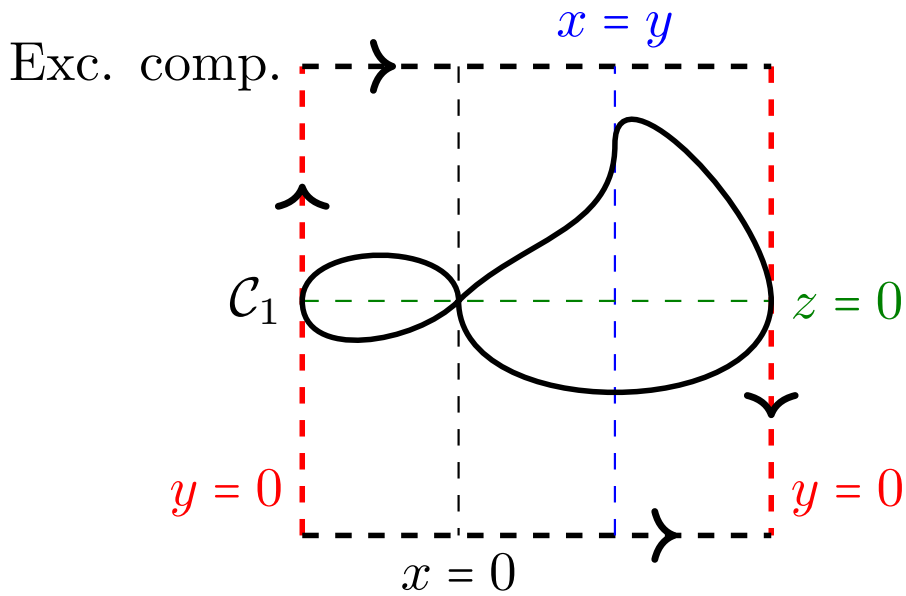
Geometry



- $y = 0$: $\mathcal{C}_1$ is like $u^2 = v^4$.
- $x = 0$: ordinary node with flex vertical branch.
- $x = y$: vertical tangent at flex.
- All vertical non-transversal lines are real (3).



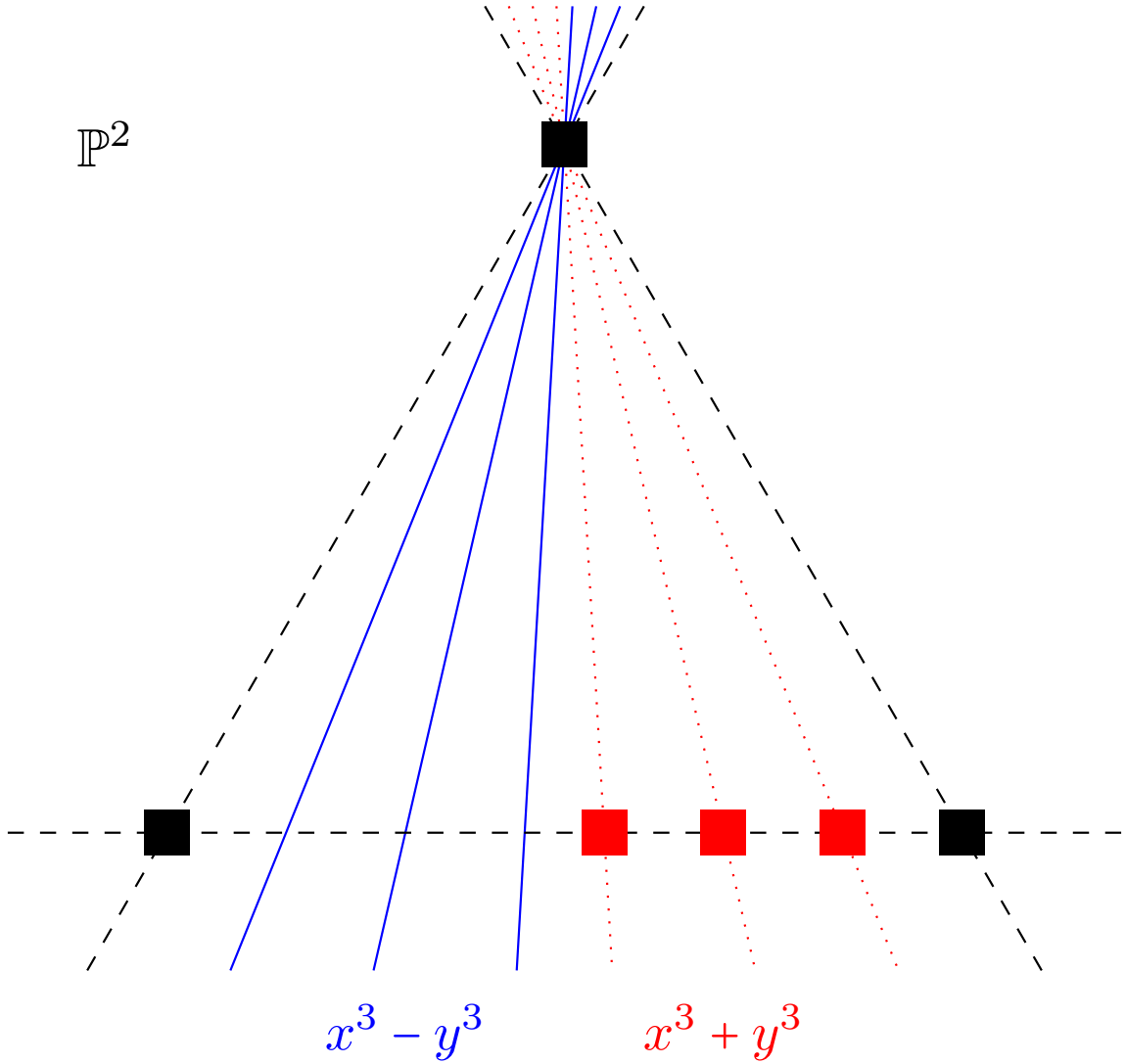
Geometry



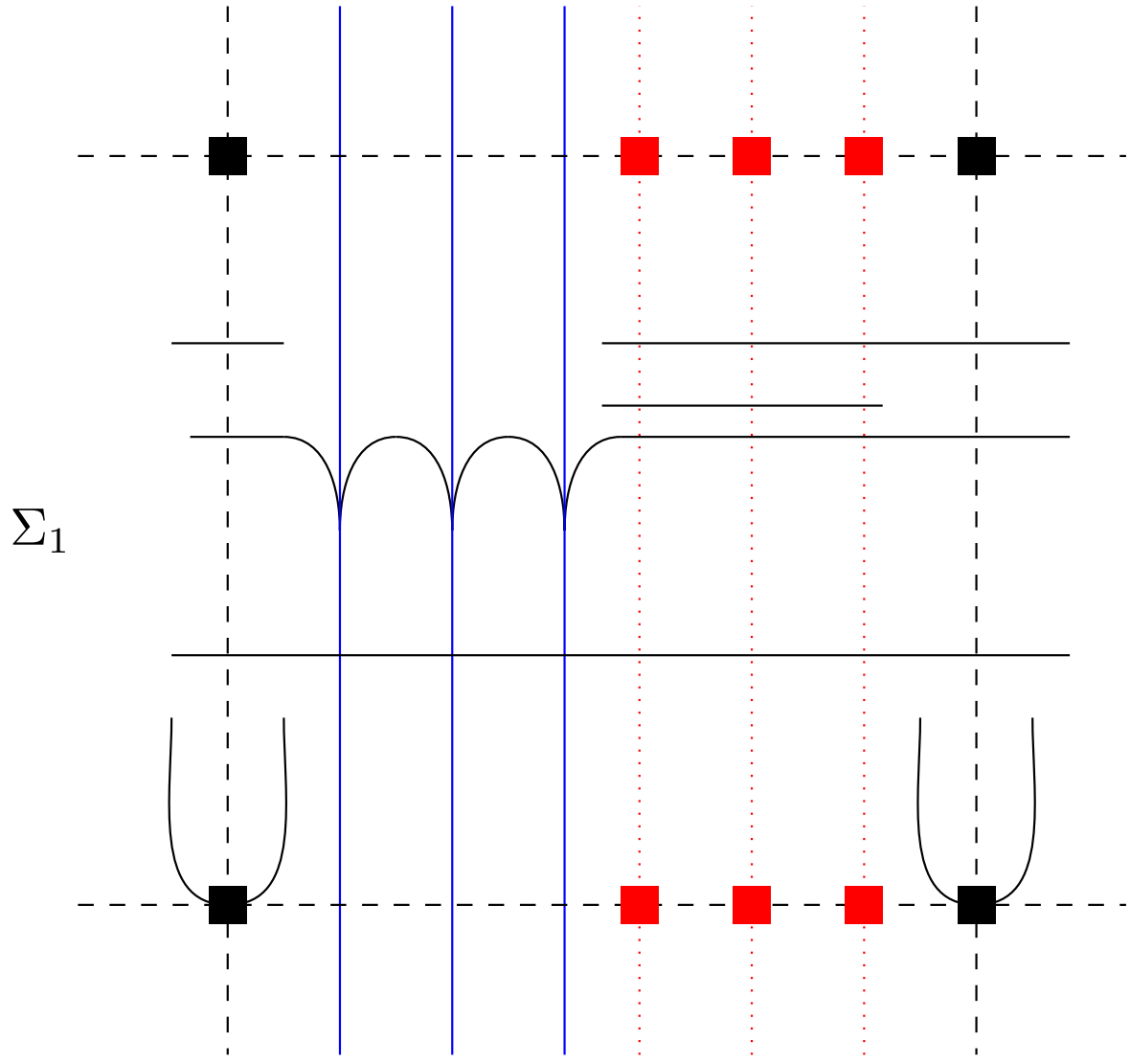
- $y = 0$: $\mathcal{C}_1$ is like $u^2 = v^4$.
- $x = 0$: ordinary node with flex vertical branch.
- $x = y$: vertical tangent at flex.
- All vertical non-transversal lines are real (3).
- Klein bottle from the real blow-up of $\mathbb{P}^2$ at $[0 : 0 : 1]$.



The movie of the action

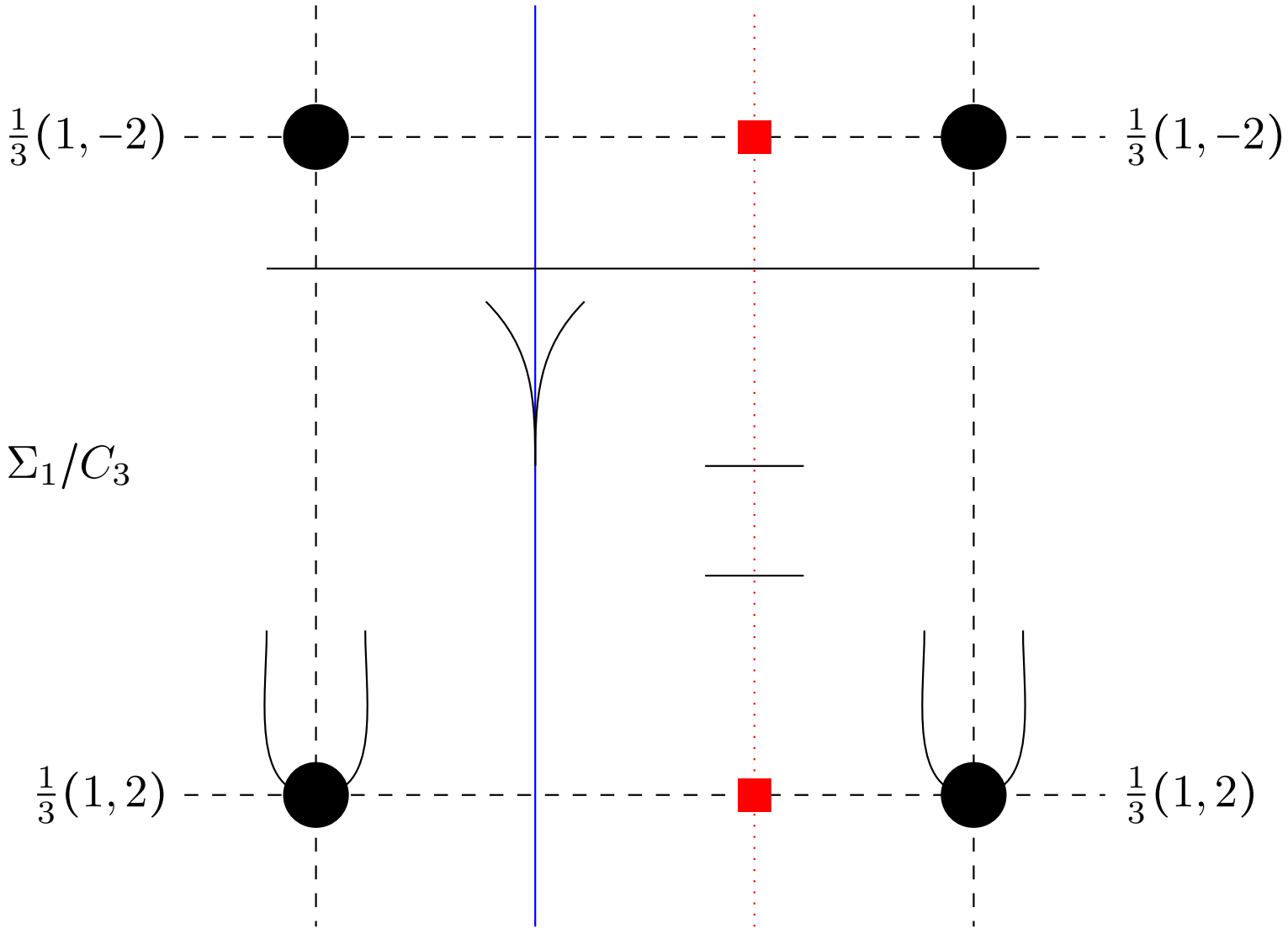


The movie of the action

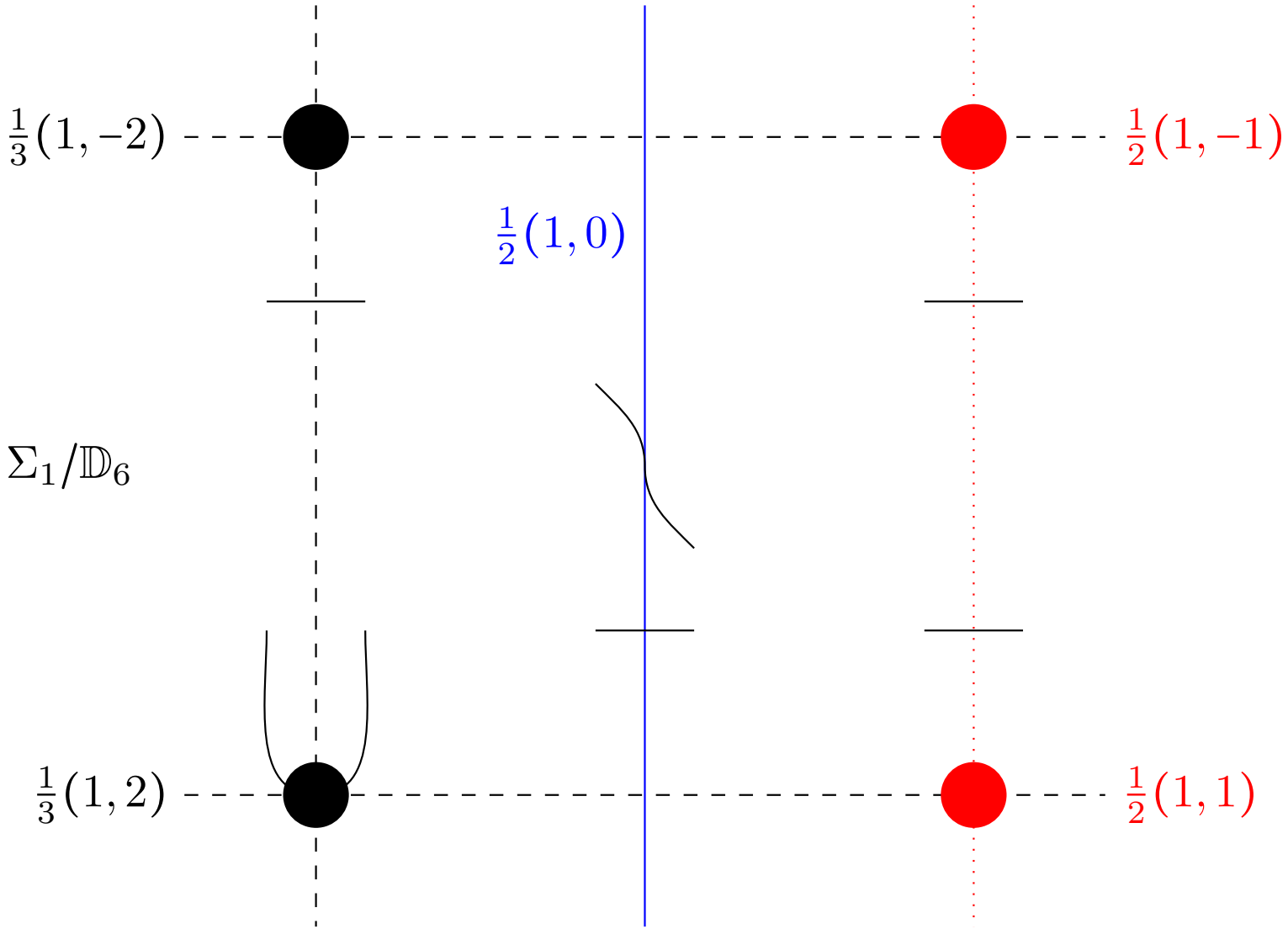




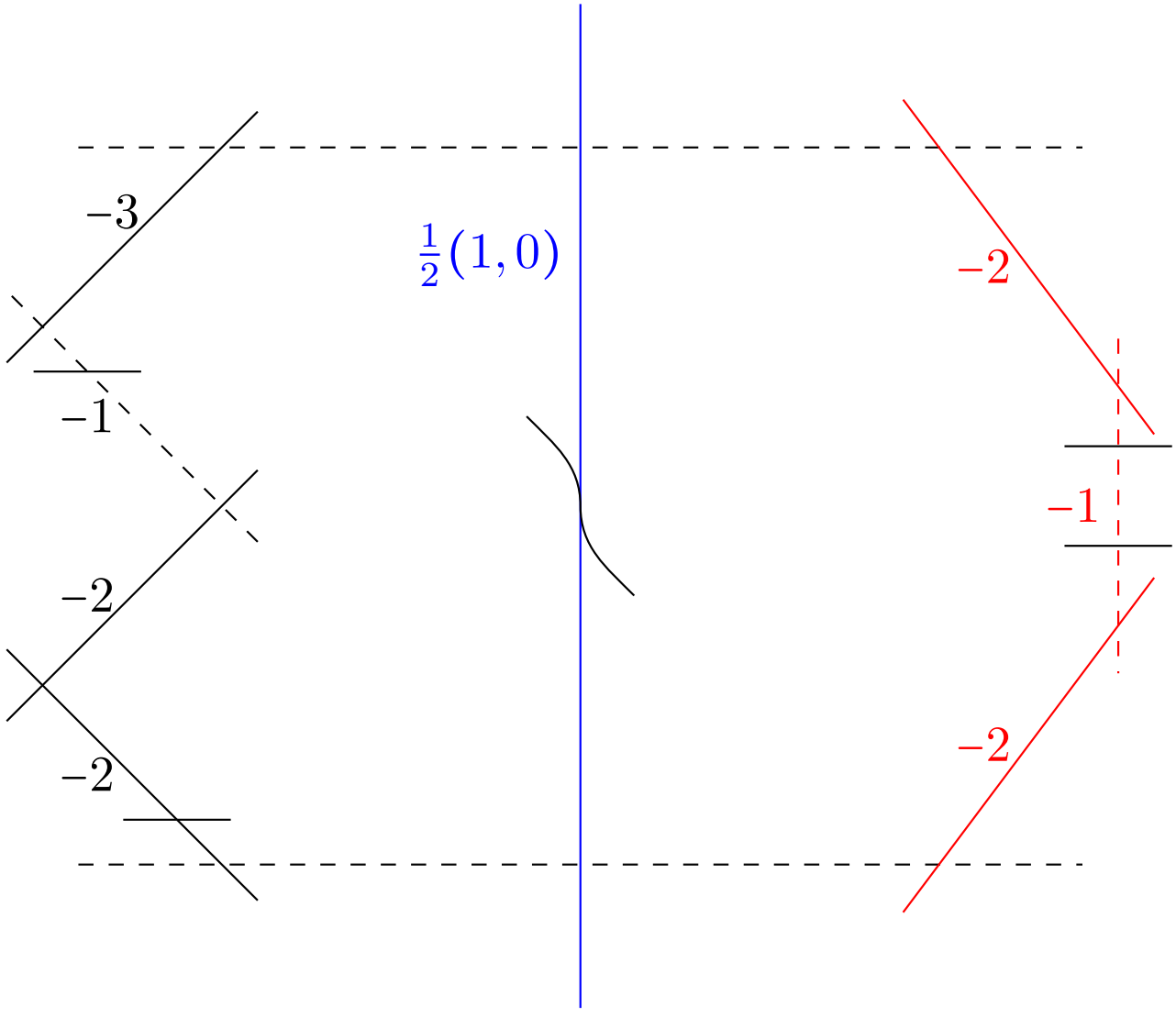
The movie of the action



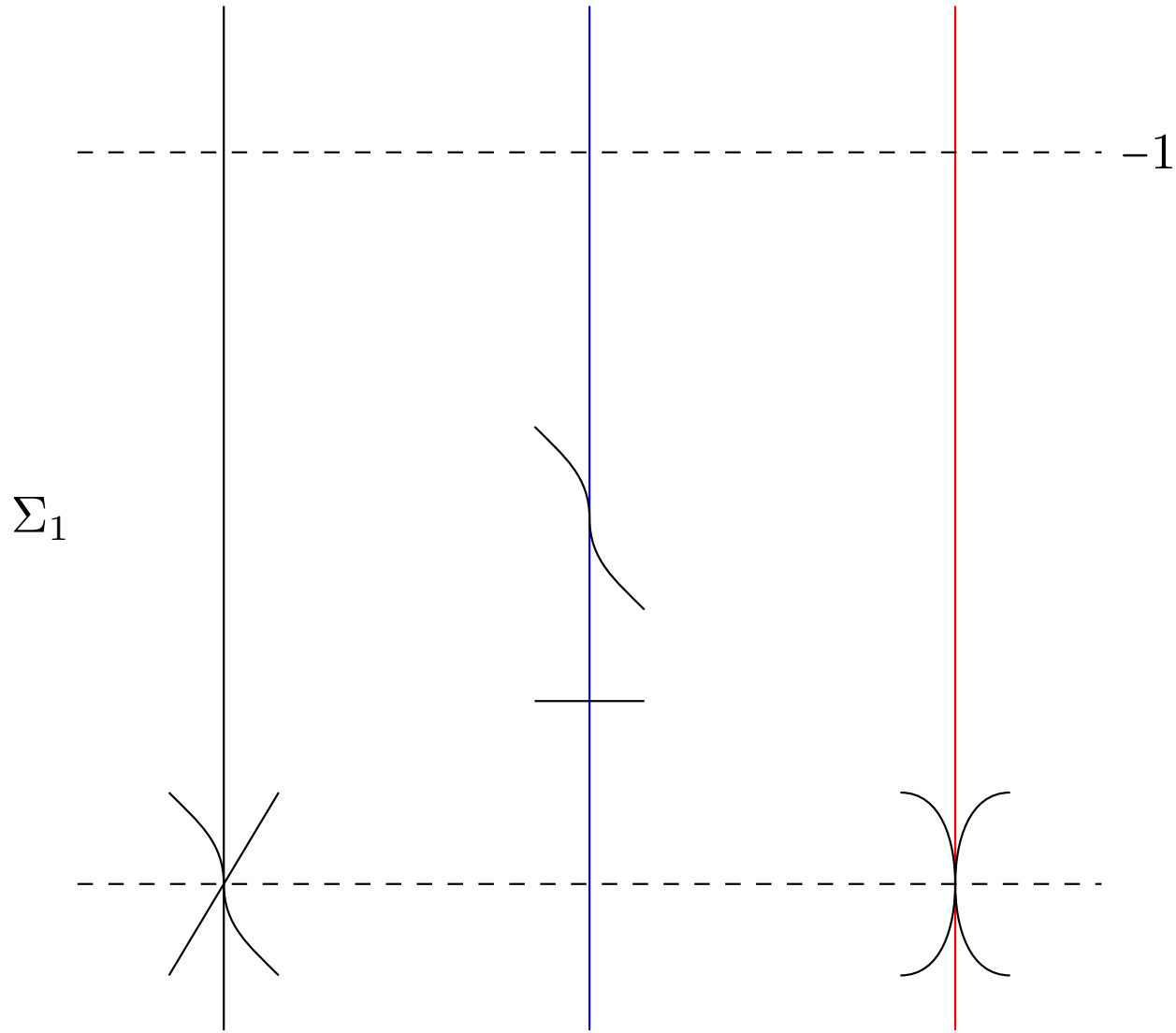
The movie of the action



The movie of the action



The movie of the action





Orbifolds

Definition

An orbifold is a pair $(X, \mathcal{O})$, X topological space and $\mathcal{O}$ is the following data: $\forall p \in X$ there exist

- G finite group
- $V \subset \mathbb{C}^n$ open set (or $\mathbb{R}^n$)
- $U \subset X$ open neighbourhood of p
- $\tilde{\mathbf{x}} : V/G \to U$ homeomorphism $(\mathbf{x} : V \to U)$.
- Analytic (or continuous, differentiable) compatibility of the change of charts.

$p \in X \rightsquigarrow$ *isotropy group* G_p, p regular if $G_p = \{1\}$.

Examples

$\mathbb{C}^n/G$ (G finite), weighted projective spaces, ...



Maps and orbifold fundamental groups

Definition

$(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$, $(f, f_{\mathcal{O}}) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ orbifold map composed by an underlying holomorphic (continuous, ...) map $f : X \to Y$ and a law of local liftings.



Maps and orbifold fundamental groups

Definition

$(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$, $(f, f_{\mathcal{O}}) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ orbifold map composed by an underlying holomorphic (continuous, ...) map $f : X \to Y$ and a law of local liftings.

Definition

$\mathcal{X} = (X, \mathcal{O}_X)$ connected orbifold, p regular point; $\pi_1^{\mathrm{orb}}(\mathcal{X}; p)$ boundary homotopy classes of orbifold loops.



Maps and orbifold fundamental groups

Definition

$(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$, $(f, f_{\mathcal{O}}) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ orbifold map composed by an underlying holomorphic (continuous, ...) map $f : X \to Y$ and a law of local liftings.

Definition

$\mathcal{X} = (X, \mathcal{O}_X)$ connected orbifold, p regular point; $\pi_1^{\mathrm{orb}}(\mathcal{X}; p)$ boundary homotopy classes of orbifold loops.

Examples



Maps and orbifold fundamental groups

Definition

$(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$, $(f, f_{\mathcal{O}}) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ orbifold map composed by an underlying holomorphic (continuous, ...) map $f : X \to Y$ and a law of local liftings.

Definition

$\mathcal{X} = (X, \mathcal{O}_X)$ connected orbifold, p regular point; $\pi_1^{\mathrm{orb}}(\mathcal{X}; p)$ boundary homotopy classes of orbifold loops.

Examples

- $\pi_1(\mathbb{C}/\mu_n)$ generated by the class of γ.

$$\begin{array}{ccc} & & \mathbb{C} \\ & \overset{\tilde{\gamma}}{\nearrow} & \downarrow \pi \\ [0,1] & \xrightarrow{\gamma} & \mathbb{C}/\mu_n \\ t & \longmapsto & [\exp \frac{2i\pi t}{n}] \end{array}$$



Maps and orbifold fundamental groups

Definition

$(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$, $(f, f_{\mathcal{O}}) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ orbifold map composed by an underlying holomorphic (continuous, ...) map $f : X \to Y$ and a law of local liftings.

Definition

$\mathcal{X} = (X, \mathcal{O}_X)$ connected orbifold, p regular point; $\pi_1^{\mathrm{orb}}(\mathcal{X}; p)$ boundary homotopy classes of orbifold loops.

Examples

- $\pi_1(\mathbb{C}/\mu_n)$ generated by the class of γ.

$$\begin{array}{ccc} & & \mathbb{C} \\ & \overset{\tilde{\gamma}}{\nearrow} & \downarrow \pi \\ [0,1] & \xrightarrow{\gamma} & \mathbb{C}/\mu_n \\ t & \longmapsto & [\exp \frac{2i\pi t}{n}] \end{array}$$

- Orbifold covers and orbifold fundamental groups mimic covers and fundamental groups.



$\pi_1^{\mathrm{orb}}(\Sigma_1/\mathbb{D}_6)$

Short answer

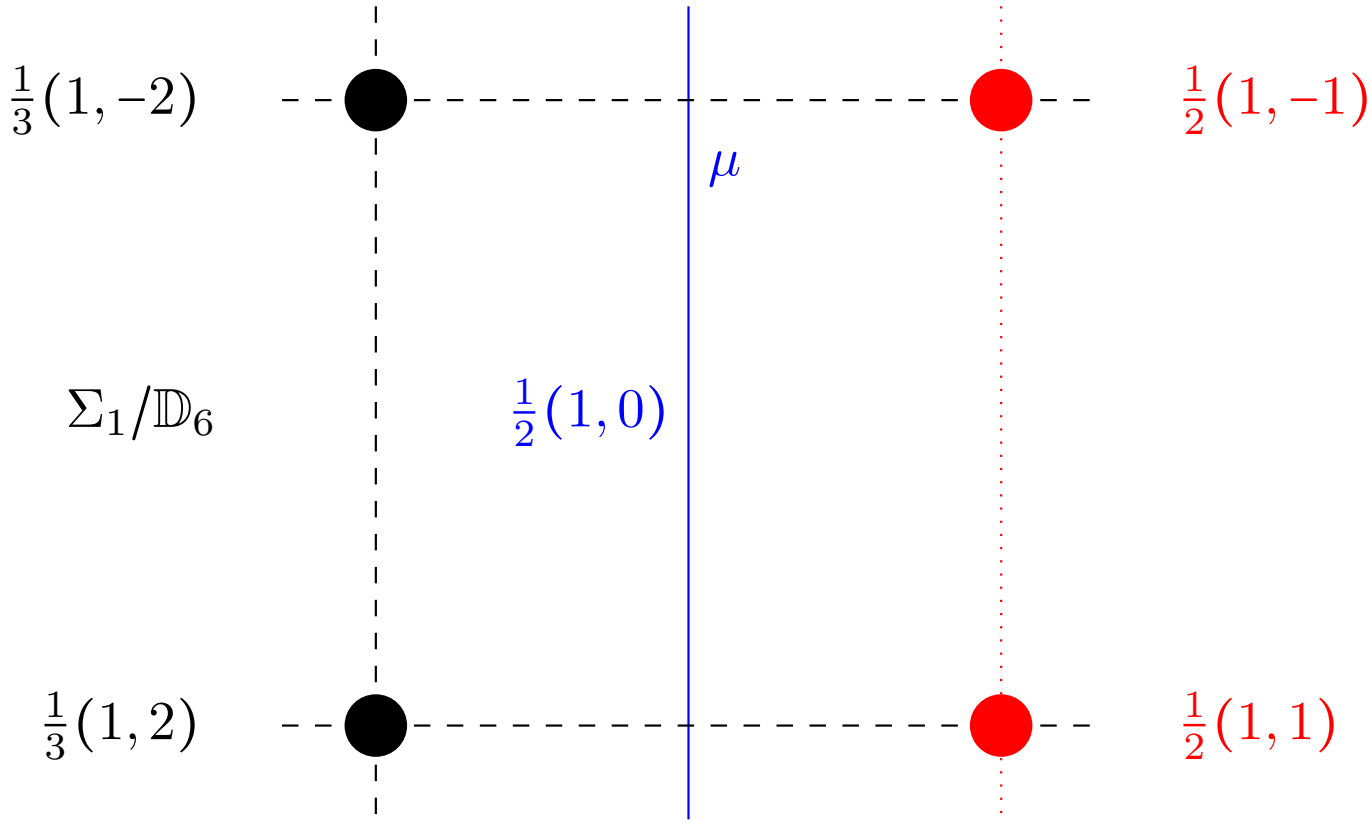
$\mathbb{D}_6$



$\pi_1^{\mathrm{orb}}(\Sigma_1/\mathbb{D}_6)$

Long answer

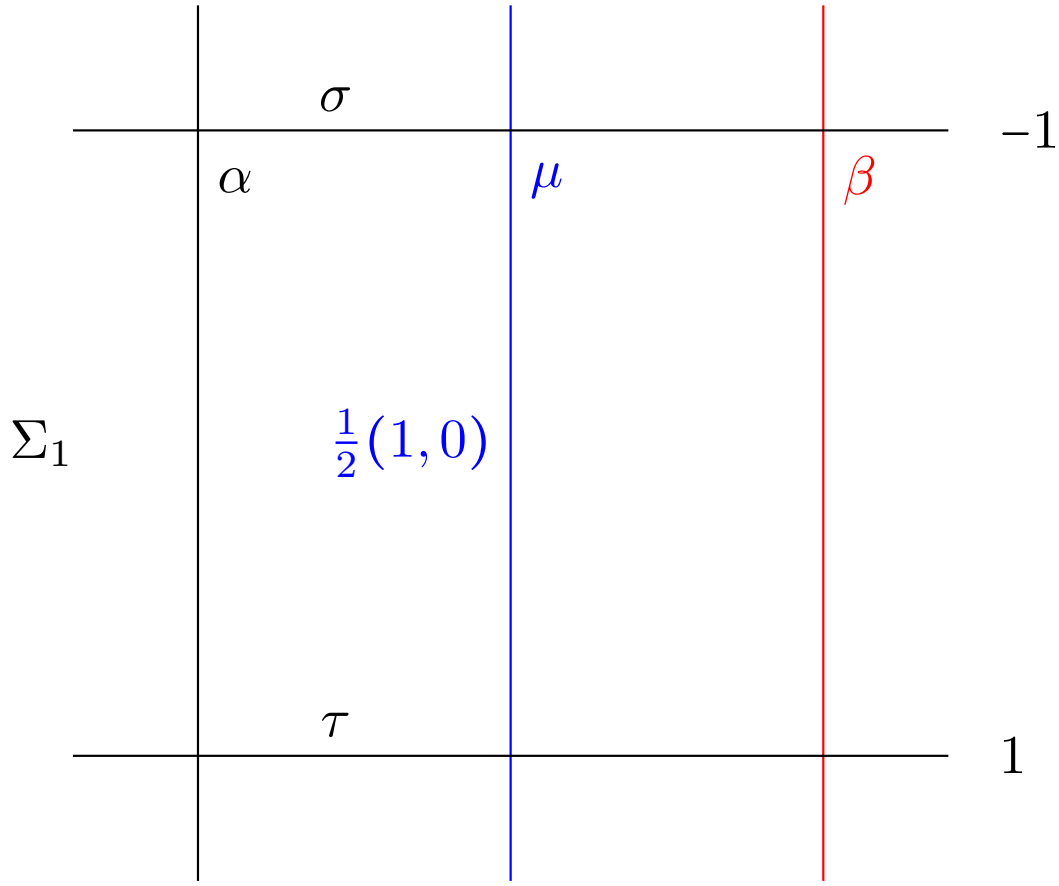
$\pi_1^{\mathrm{orb}}(\Sigma_1/\mathbb{D}_6) \cong \pi_1(\Sigma_1/\mathbb{D}_6 \setminus (\{\text{non regular points}\} \cup \text{blue line}))/\langle \mu^2 \rangle$



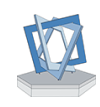
$\pi_1^{\mathrm{orb}}(\Sigma_1/\mathbb{D}_6)$

Long answer

$\pi_1^{\mathrm{orb}}(\Sigma_1/\mathbb{D}_6) \cong \pi_1(\Sigma_1/\mathbb{D}_6 \setminus (\{\text{non regular points}\} \cup \text{blue line}))/\langle \mu^2 \rangle$



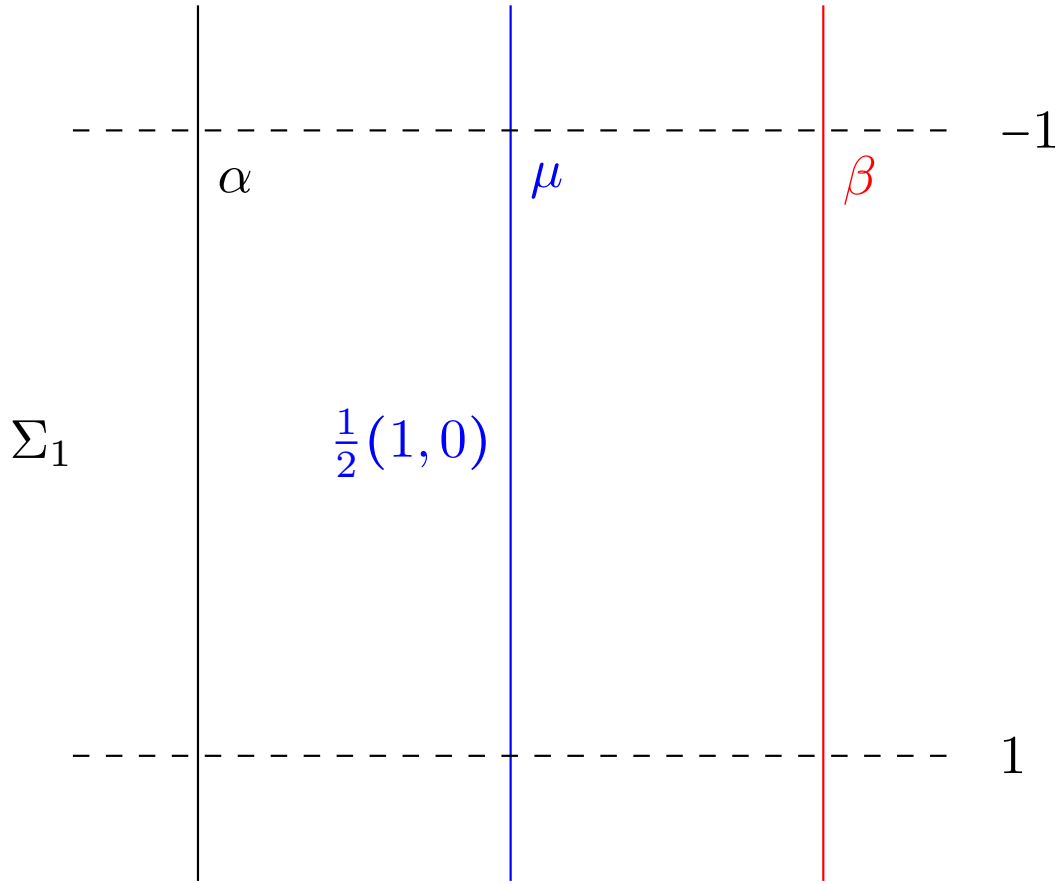
$$\langle \mu, \alpha, \beta, \sigma, \tau \mid \mu^2 = 1, \mu\alpha\beta = \sigma, \sigma\tau = 1, [\sigma, \alpha] = [\sigma, \beta] = 1 \rangle$$



$\pi_1^{\mathrm{orb}}(\Sigma_1/\mathbb{D}_6)$

Long answer

$\pi_1^{\mathrm{orb}}(\Sigma_1/\mathbb{D}_6) \cong \pi_1(\Sigma_1/\mathbb{D}_6 \setminus (\{\text{non regular points}\} \cup \text{blue line}))/\langle \mu^2 \rangle$

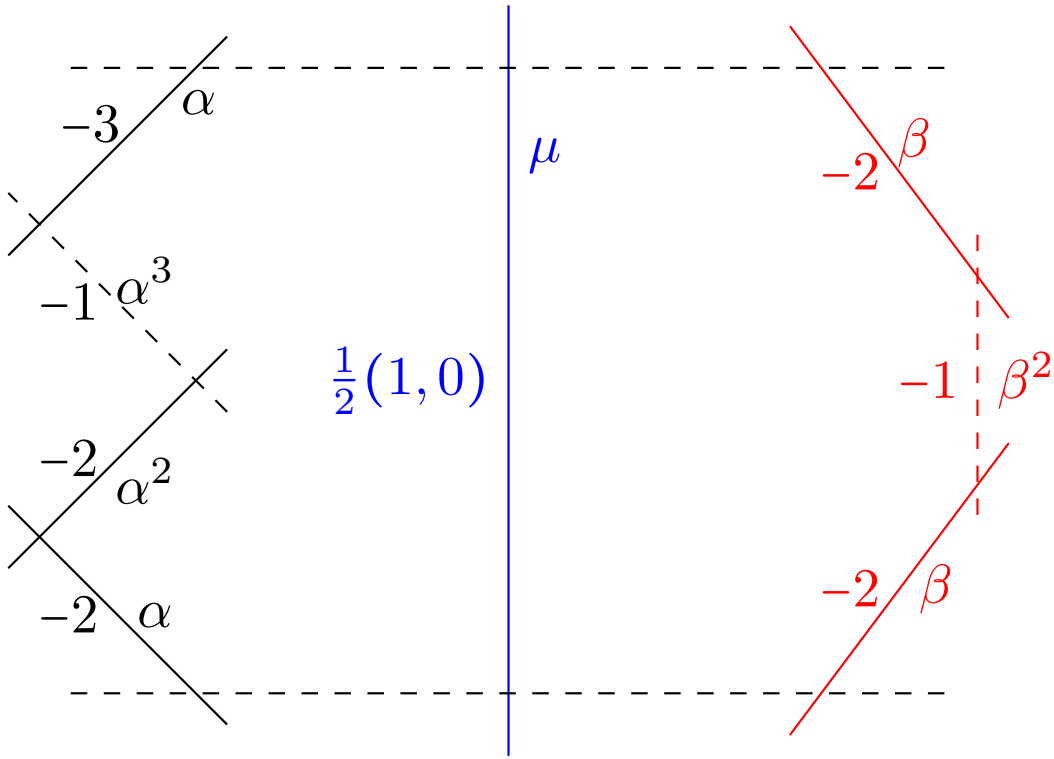


$$\langle \mu, \alpha, \beta \mid \mu^2 = 1, \mu\alpha\beta = 1 \rangle$$

$\pi_1^{\mathrm{orb}}(\Sigma_1/\mathbb{D}_6)$

Long answer

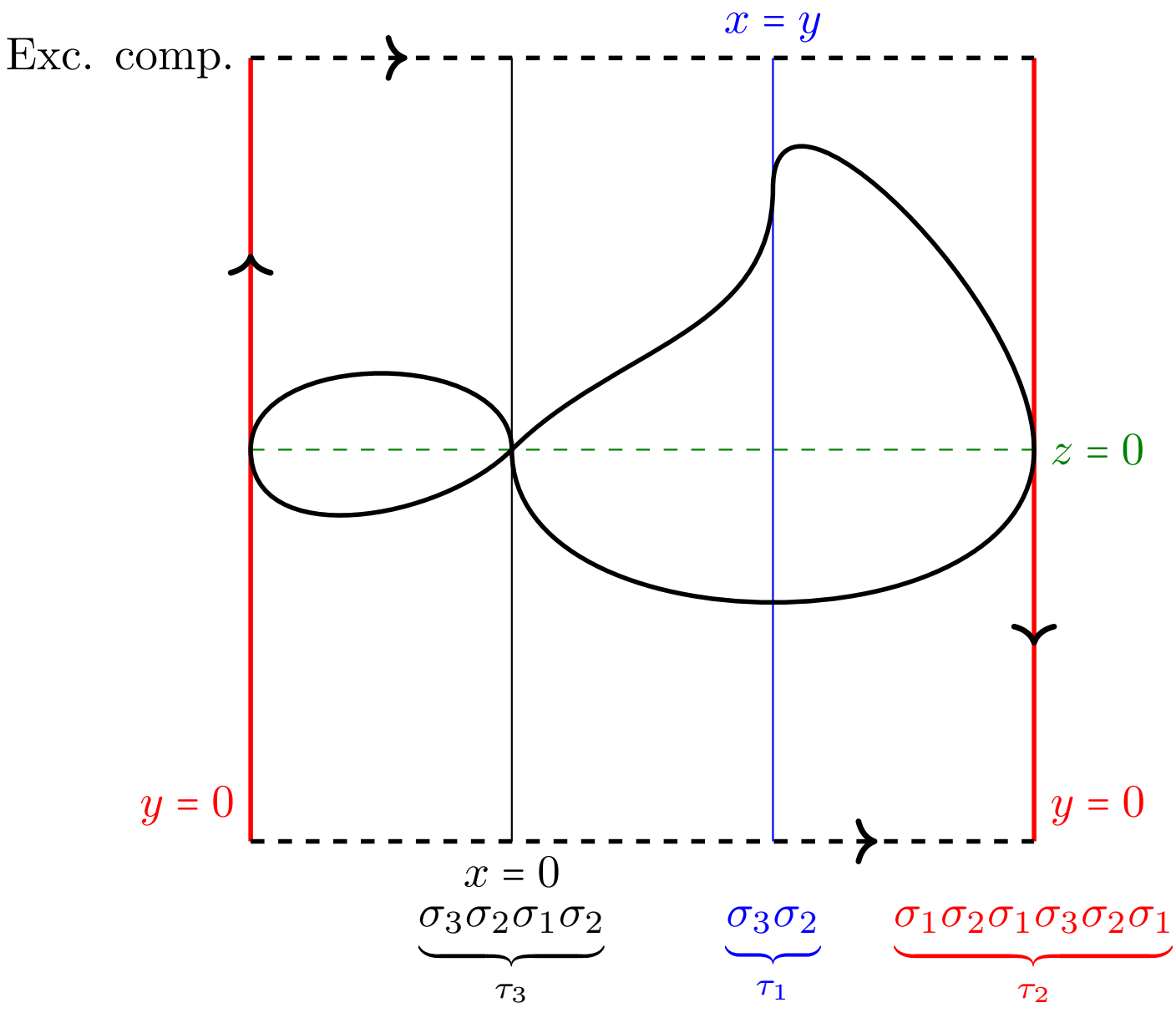
$\pi_1^{\mathrm{orb}}(\Sigma_1/\mathbb{D}_6) \cong \pi_1(\Sigma_1/\mathbb{D}_6 \setminus (\{\text{non regular points}\} \cup \text{blue line}))/\langle \mu^2 \rangle$



$$\langle \mu, \alpha, \beta \mid \mu^2 = 1, \mu\alpha\beta = 1, \alpha^3 = \beta^2 = 1 \rangle \cong \mathbb{D}_6$$



Braid monodromy of $\mathcal{C}_1$



Braid monodromy of $\mathcal{C}_1$

- $\tau_1 := \sigma_3\sigma_2 \in \mathbb{B}_4$
- $\tau_2 := \sigma_1\sigma_2\sigma_1\sigma_3\sigma_2\sigma_1 = \Delta_4$
- $\tau_3 := \sigma_3\sigma_2\sigma_1\sigma_2$
- $\tau_1\tau_2\tau_3 = \Delta_4^2$ generator of the center of $\mathbb{B}_4$
- $\tau_3^3 = \Delta_4^2$.



Braid monodromy of $\mathcal{C}_1$

- $\tau_1 := \sigma_3\sigma_2 \in \mathbb{B}_4$
- $\tau_2 := \sigma_1\sigma_2\sigma_1\sigma_3\sigma_2\sigma_1 = \Delta_4$
- $\tau_3 := \sigma_3\sigma_2\sigma_1\sigma_2$
- $\tau_1\tau_2\tau_3 = \Delta_4^2$ generator of the center of $\mathbb{B}_4$
- $\tau_3^3 = \Delta_4^2$.

Fundamental group of $\mathbb{P}^2 \setminus (\mathcal{C}_1 \cup \{xy(x-y) = 0\})$

- $\langle x_1, x_2, x_3, x_4, y_1, y_2, y_3 \mid x_i^{\tau_j} = y_j^{-1} x_i y_j, x_1x_2x_3x_4y_1y_2y_3 = 1 \rangle$
- $x^{\Delta^2} = (x_1 \dots x_4)x(x_1 \dots x_4)^{-1}$



Braid monodromy of $\mathcal{C}_1$

- $\tau_1 := \sigma_3\sigma_2 \in \mathbb{B}_4$
- $\tau_2 := \sigma_1\sigma_2\sigma_1\sigma_3\sigma_2\sigma_1 = \Delta_4$
- $\tau_3 := \sigma_3\sigma_2\sigma_1\sigma_2$
- $\tau_1\tau_2\tau_3 = \Delta_4^2$ generator of the center of $\mathbb{B}_4$
- $\tau_3^3 = \Delta_4^2$.

Fundamental group of $\mathbb{P}^2 \setminus (\mathcal{C}_1 \cup \{xy(x-y) = 0\})$

- $\langle x_1, x_2, x_3, x_4, y_1, y_2, y_3 \mid x_i^{\tau_j} = y_j^{-1} x_i y_j, x_1x_2x_3x_4y_1y_2y_3 = 1\rangle$
- $x^{\Delta^2} = (x_1 \dots x_4)x(x_1 \dots x_4)^{-1}$

Orbifold fundamental group G_1 of $\Sigma_1/\mathbb{D}_6 \setminus \mathcal{C}_1$

- $y_1 \leftrightarrow \mu \Longrightarrow y_1^2 = 1$
- $y_2 \leftrightarrow \beta \Longrightarrow y_2^2(x_1 \dots x_4) = 1$, compatible with $\tau_2 = \Delta_4$!
- $y_3 \leftrightarrow \alpha \Longrightarrow y_3^3(x_1 \dots x_4) = 1$, compatible with $\tau_3^3 = \Delta_4^2$!



Braid monodromy of $\mathcal{C}_1$

- $\tau_1 := \sigma_3\sigma_2 \in \mathbb{B}_4$
- $\tau_2 := \sigma_1\sigma_2\sigma_1\sigma_3\sigma_2\sigma_1 = \Delta_4$
- $\tau_3 := \sigma_3\sigma_2\sigma_1\sigma_2$
- $\tau_1\tau_2\tau_3 = \Delta_4^2$ generator of the center of $\mathbb{B}_4$
- $\tau_3^3 = \Delta_4^2$.

Fundamental group of $\mathbb{P}^2 \setminus (\mathcal{C}_1 \cup \{xy(x-y) = 0\})$

- $\langle x_1, x_2, x_3, x_4, y_1, y_2, y_3 \mid x_i^{\tau_j} = y_j^{-1} x_i y_j, x_1x_2x_3x_4y_1y_2y_3 = 1 \rangle$
- $x^{\Delta^2} = (x_1 \dots x_4)x(x_1 \dots x_4)^{-1}$

Orbifold fundamental group G_1 of $\Sigma_1/\mathbb{D}_6 \setminus \mathcal{C}_1$

- $x_1x_2x_3x_4 = y_2^2 = y_3^3 = 1 \Longrightarrow H_2 = \langle y_i \rangle = \mathbb{D}_6$
- $H_1 = \langle x_i \rangle = \langle x_2, x_3 \mid x_2x_3x_2 = x_3x_2x_3, x_2^2x_3^2 = 1 \rangle$ of order 12.



Braid monodromy of $\mathcal{C}_1$

- $\tau_1 := \sigma_3\sigma_2 \in \mathbb{B}_4$
- $\tau_2 := \sigma_1\sigma_2\sigma_1\sigma_3\sigma_2\sigma_1 = \Delta_4$
- $\tau_3 := \sigma_3\sigma_2\sigma_1\sigma_2$
- $\tau_1\tau_2\tau_3 = \Delta_4^2$ generator of the center of $\mathbb{B}_4$
- $\tau_3^3 = \Delta_4^2$.

Fundamental group of $\mathbb{P}^2 \setminus (\mathcal{C}_1 \cup \{xy(x-y) = 0\})$

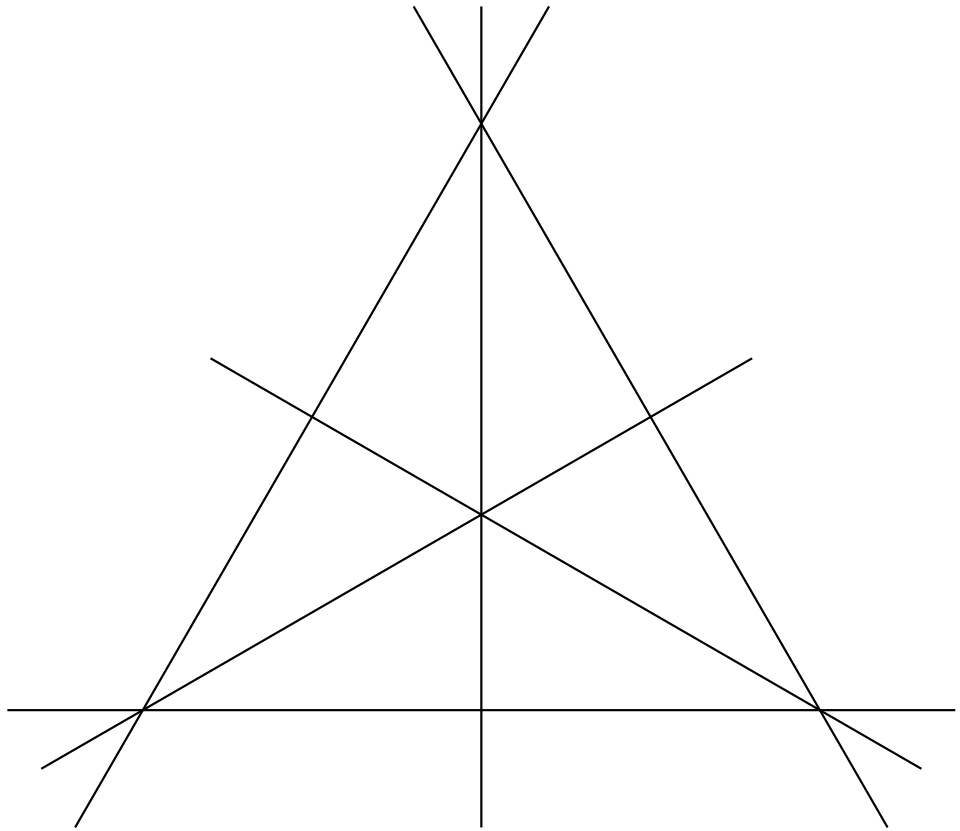
- $\langle x_1, x_2, x_3, x_4, y_1, y_2, y_3 \mid x_i^{\tau_j} = y_j^{-1} x_i y_j, x_1x_2x_3x_4y_1y_2y_3 = 1 \rangle$
- $x^{\Delta^2} = (x_1 \dots x_4)x(x_1 \dots x_4)^{-1}$

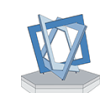
Orbifold fundamental group G_1 of $\Sigma_1/\mathbb{D}_6 \setminus \mathcal{C}_1$

- $x_1x_2x_3x_4 = y_2^2 = y_3^3 = 1 \Longrightarrow H_2 = \langle y_i \rangle = \mathbb{D}_6$
- $H_1 = \langle x_i \rangle = \langle x_2, x_3 \mid x_2x_3x_2 = x_3x_2x_3, x_2^2x_3^2 = 1 \rangle$ of order 12.
- $G_1 = H_1 \rtimes \mathbb{D}_6$, $y_1x_2y_1x_3^{-1} = 1$, $[y_2, x_3] = 1$, $y_2^{-1}x_2y_2 = x_2x_3x_2^{-1}] = 1$
- $H_1 = \pi_1(\mathbb{P}^2 \setminus \mathcal{C})$

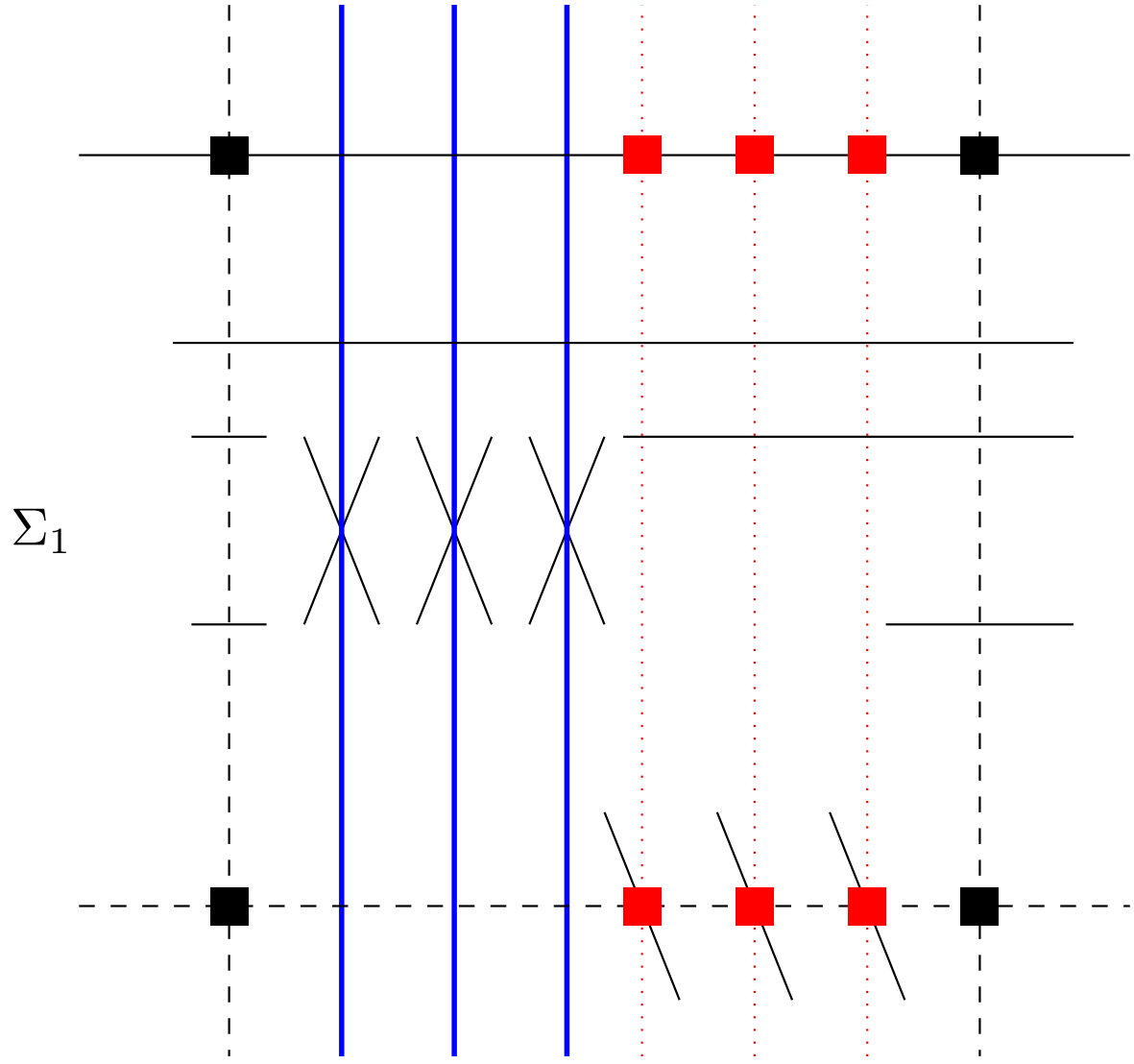


$\mathfrak{S}_3$ action on Ceva arrangement

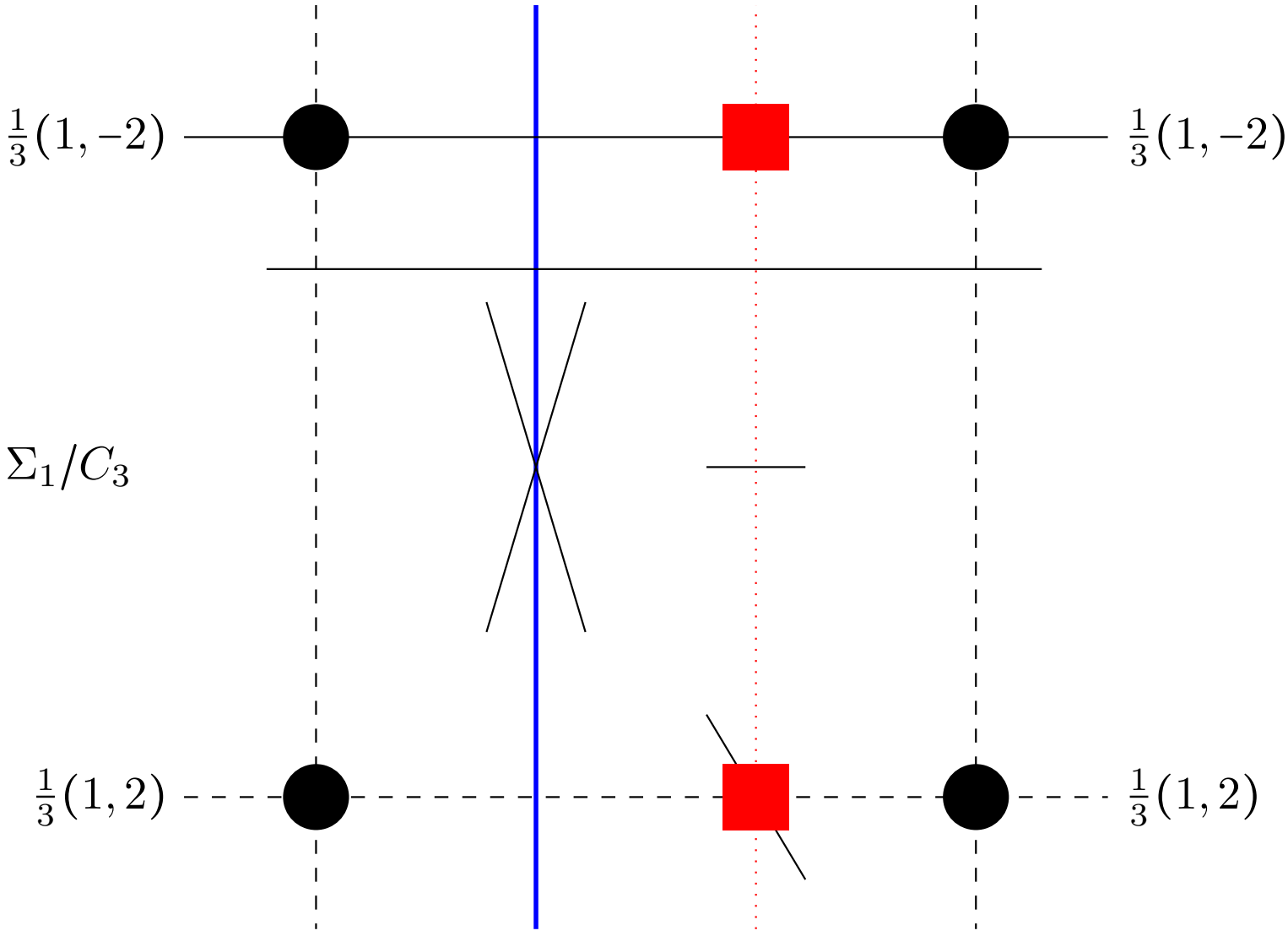




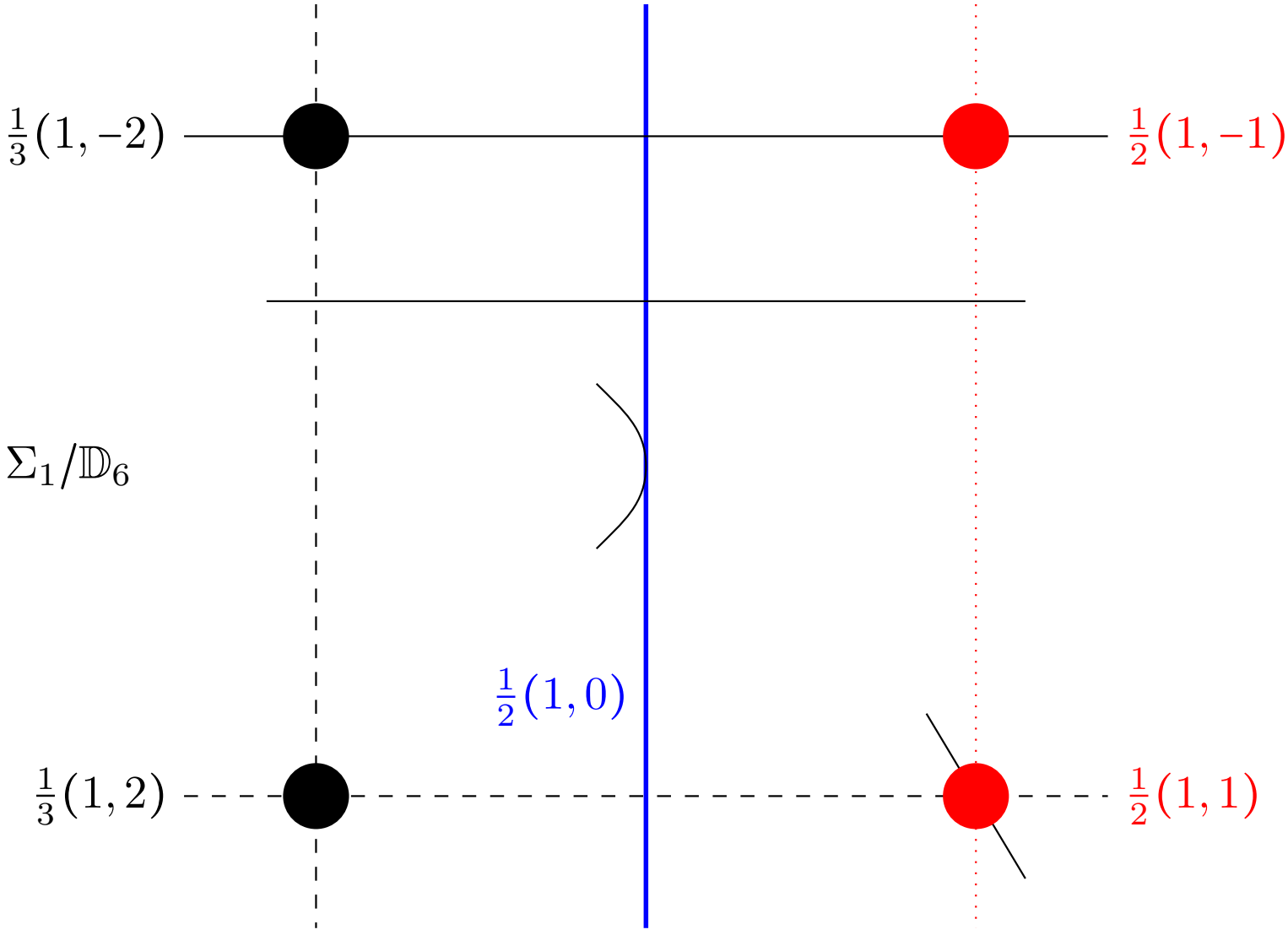
$\mathfrak{S}_3$ action on Ceva arrangement



$\mathfrak{S}_3$ action on Ceva arrangement

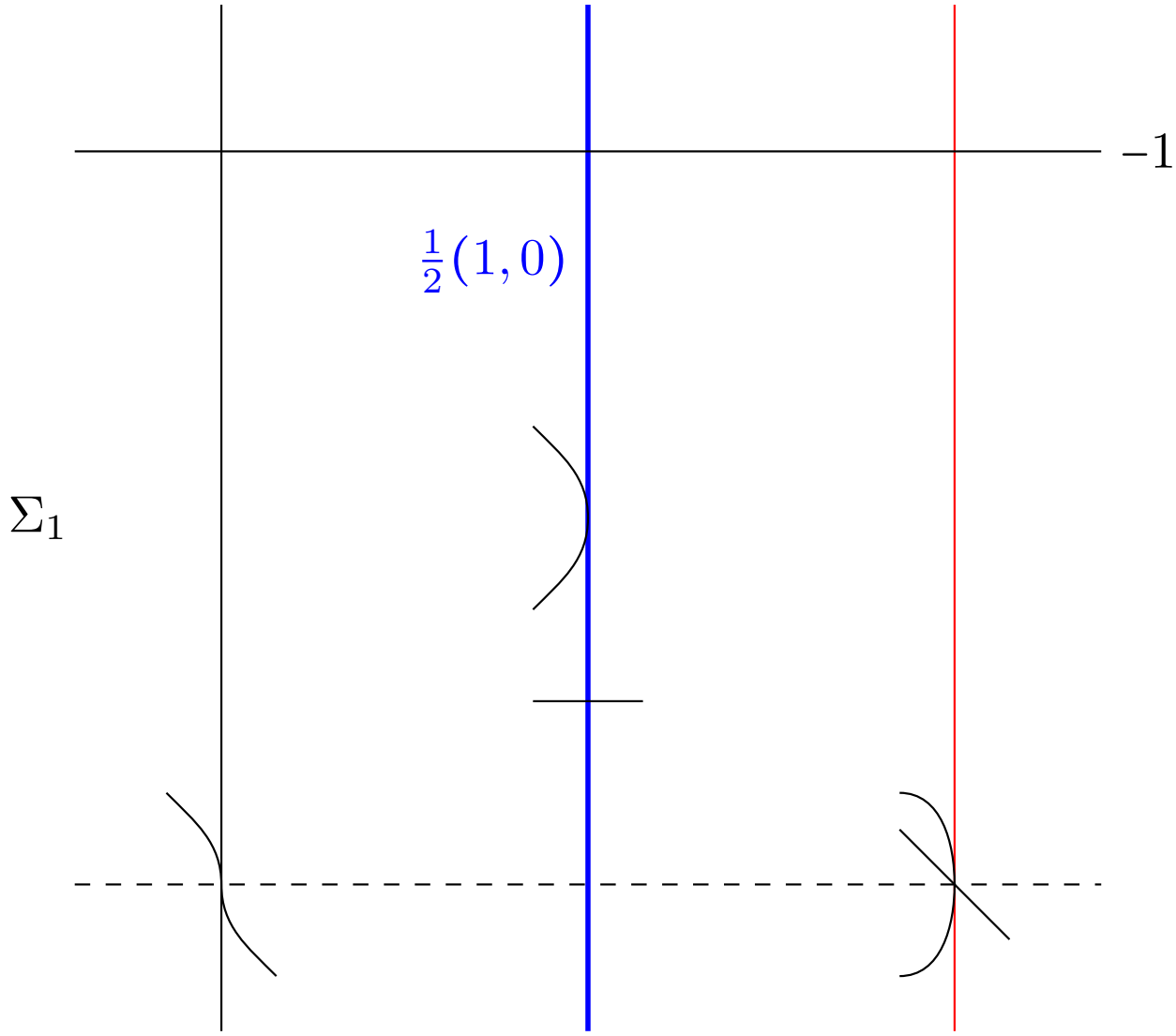


$\mathfrak{S}_3$ action on Ceva arrangement

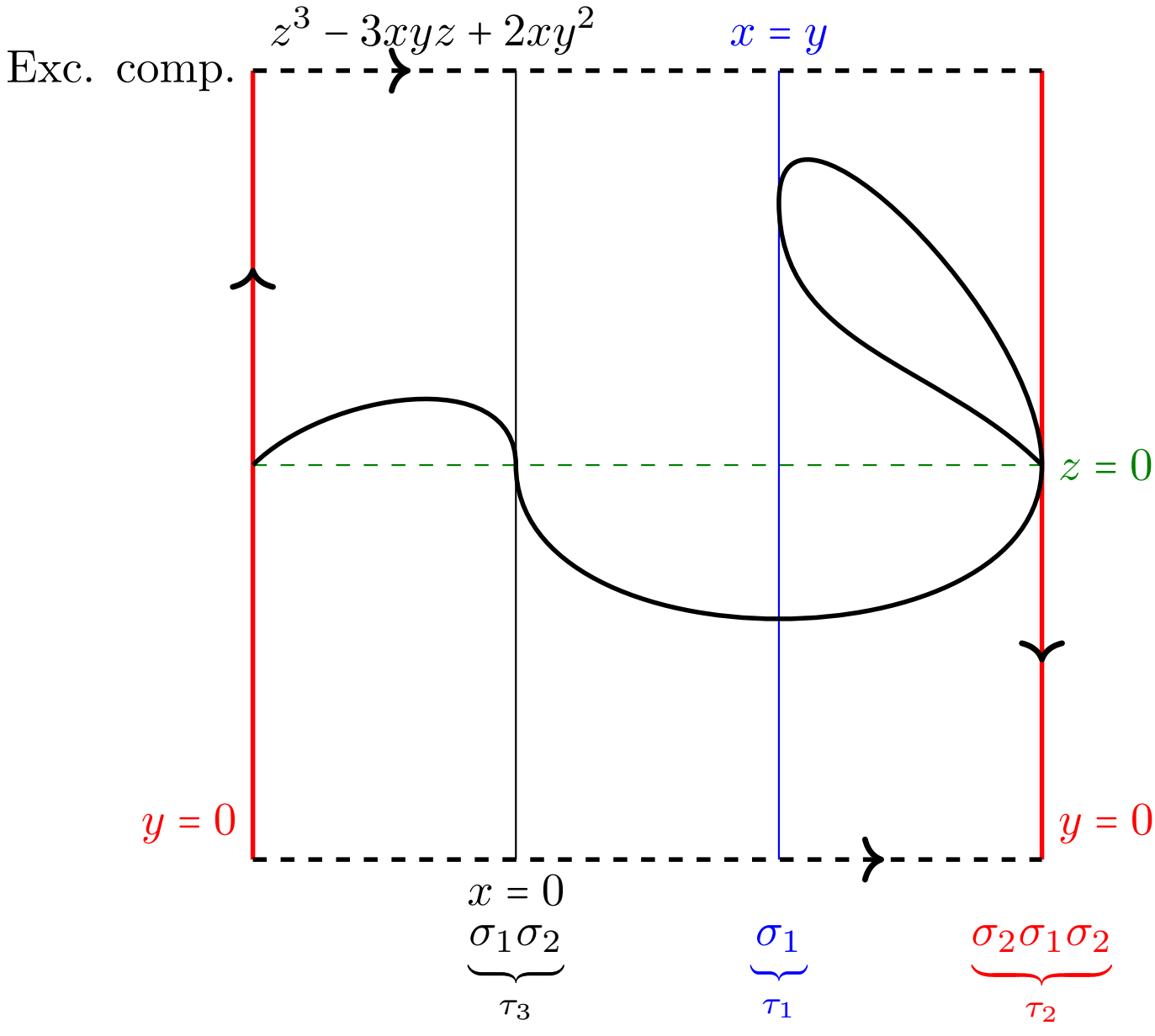




$\mathfrak{S}_3$ action on Ceva arrangement

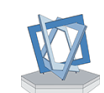


$\mathfrak{S}_3$ action on Ceva arrangement



$\mathfrak{S}_3$ action on Ceva arrangement

- $\tau_1 := \sigma_1, \tau_2 := \Delta_3, \tau_3 := \sigma_1\sigma_2 \in \mathbb{B}_3$
- $\tau_1\tau_2\tau_3 = \Delta_3^2$, $\tau_3^3 = \Delta_3^2$.



$\mathfrak{S}_3$ action on Ceva arrangement

- $\tau_1 := \sigma_1, \tau_2 := \Delta_3, \tau_3 := \sigma_1\sigma_2 \in \mathbb{B}_3$
- $\tau_1\tau_2\tau_3 = \Delta_3^2,\ \tau_3^3 = \Delta_3^2.$

Fundamental group of $\mathbb{P}^2 \setminus (\mathcal{C}_1 \cup \{xy(x-y) = 0\})$

- $\langle x_1, x_2, x_3, y_1, y_2, y_3 \mid x_i^{\tau_j} = y_j^{-1} x_i y_j, x_1x_2x_3y_1y_2y_3 = 1 \rangle$
- $x^{\Delta^2} = (x_1x_2x_3)x(x_1x_2x_3)^{-1}$



$\mathfrak{S}_3$ action on Ceva arrangement

- $\tau_1 := \sigma_1, \tau_2 := \Delta_3, \tau_3 := \sigma_1\sigma_2 \in \mathbb{B}_3$
- $\tau_1\tau_2\tau_3 = \Delta_3^2,\ \tau_3^3 = \Delta_3^2.$

Fundamental group of $\mathbb{P}^2 \setminus (\mathcal{C}_1 \cup \{xy(x-y)=0\})$

- $\langle x_1, x_2, x_3, y_1, y_2, y_3 \mid x_i^{\tau_j} = y_j^{-1} x_i y_j, x_1x_2x_3y_1y_2y_3 = 1\rangle$
- $x^{\Delta^2} = (x_1x_2x_3)x(x_1x_2x_3)^{-1}$

Orbifold fundamental group G_1 of $\Sigma_1/\mathbb{D}_6 \setminus (\mathcal{C}_1 \cup \{x=y\})$

- $y_1 \leftrightarrow \mu,\ y_2 \leftrightarrow \beta \Longrightarrow y_2^2(x_1x_2x_3) = 1,\ y_3 \leftrightarrow \alpha \Longrightarrow y_3^3(x_1x_2x_3) = 1$
- G_1 generators x_3, y_1, y_3
- G_1 relations $[y_1, x_3] = [y_1, y_3y_1y_3] = (y_3x_3)^3 = 1, y_3^2 = y_1y_3y_1\rangle$
- $\Phi : G_1 \twoheadrightarrow \mathfrak{S}_3,\ x_3 \mapsto (),\ y_1 \mapsto (1,2),\ y_3 \mapsto (1,2,3)$
- $\pi_1(\mathbb{P}^2 \setminus \mathcal{C}) = \ker \Phi.$



$\mathfrak{S}_4$ action on Ceva arrangement $\mathscr{C}$

- $\mathscr{C} \leftrightarrow \{(x^2 - y^2)(y^2 - z^2)(z^2 - x^2) = 0\}$, $\operatorname{Aut} \mathcal{C} = \mathfrak{S}_4$
- $[x : y : z] \mapsto [-x : y : z]$, ...
- $1 \to \mu_2 \times \mu_2 \to \mathfrak{S}_4 \to \mathfrak{S}_3 \to 1$



$\mathfrak{S}_4$ action on Ceva arrangement $\mathscr{C}$

- $\mathscr{C} \leftrightarrow \{(x^2 - y^2)(y^2 - z^2)(z^2 - x^2) = 0\}$, $\operatorname{Aut} \mathcal{C} = \mathfrak{S}_4$
- $[x : y : z] \mapsto [-x : y : z]$, ...
- $1 \to \mu_2 \times \mu_2 \to \mathfrak{S}_4 \to \mathfrak{S}_3 \to 1$

Lemma

- $\mathbb{P}^2$ *is the underlying space of the orbifold* $\mathbb{P}^2/\mu_2 \times \mu_2$.



$\mathfrak{S}_4$ action on Ceva arrangement $\mathscr{C}$

- $\mathscr{C} \leftrightarrow \{(x^2 - y^2)(y^2 - z^2)(z^2 - x^2) = 0\}$, $\operatorname{Aut} \mathcal{C} = \mathfrak{S}_4$
- $[x : y : z] \mapsto [-x : y : z]$, ...
- $1 \to \mu_2 \times \mu_2 \to \mathfrak{S}_4 \to \mathfrak{S}_3 \to 1$

Lemma

- $\mathbb{P}^2$ *is the underlying space of the orbifold* $\mathbb{P}^2/\mu_2 \times \mu_2$.
- *The non-regular points are* $xyz = 0$.



$\mathfrak{S}_4$ action on Ceva arrangement $\mathscr{C}$

- $\mathscr{C} \leftrightarrow \{(x^2 - y^2)(y^2 - z^2)(z^2 - x^2) = 0\}$, $\operatorname{Aut} \mathcal{C} = \mathfrak{S}_4$
- $[x : y : z] \mapsto [-x : y : z]$, ...
- $1 \to \mu_2 \times \mu_2 \to \mathfrak{S}_4 \to \mathfrak{S}_3 \to 1$

Lemma

- $\mathbb{P}^2$ *is the underlying space of the orbifold* $\mathbb{P}^2/\mu_2 \times \mu_2$.
- *The non-regular points are* $xyz = 0$.
- *The isotropy groups in the axes outside the vertices are* μ_2
- *The isotropy groups for the vertices are* $\mu_2 \times \mu_2$



$\mathfrak{S}_4$ action on Ceva arrangement $\mathscr{C}$

- $\mathscr{C} \leftrightarrow \{(x^2 - y^2)(y^2 - z^2)(z^2 - x^2) = 0\}$, $\operatorname{Aut} \mathcal{C} = \mathfrak{S}_4$
- $[x : y : z] \mapsto [-x : y : z]$, ...
- $1 \to \mu_2 \times \mu_2 \to \mathfrak{S}_4 \to \mathfrak{S}_3 \to 1$

Lemma

- $\mathbb{P}^2$ *is the underlying space of the orbifold* $\mathbb{P}^2/\mu_2 \times \mu_2$.
- *The non-regular points are* $xyz = 0$.
- *The isotropy groups in the axes outside the vertices are* μ_2
- *The isotropy groups for the vertices are* $\mu_2 \times \mu_2$
- *Kummer covering* $[x : y : z] \to [x^2 : y^2 : z^2]$ *in mind*



$\mathfrak{S}_4$ action on Ceva arrangement $\mathscr{C}$

- $\mathscr{C} \leftrightarrow \{(x^2-y^2)(y^2-z^2)(z^2-x^2)=0\}$, $\operatorname{Aut}\mathcal{C} = \mathfrak{S}_4$
- $[x:y:z] \mapsto [-x:y:z]$, ...
- $1 \to \mu_2 \times \mu_2 \to \mathfrak{S}_4 \to \mathfrak{S}_3 \to 1$

Lemma

- $\mathbb{P}^2$ *is the underlying space of the orbifold* $\mathbb{P}^2/\mu_2 \times \mu_2$.
- *The non-regular points are* $xyz = 0$.
- *The isotropy groups in the axes outside the vertices are* μ_2
- *The isotropy groups for the vertices are* $\mu_2 \times \mu_2$
- *Kummer covering* $[x:y:z] \to [x^2:y^2:z^2]$ *in mind*
- *The image of* $\mathscr{C}$ *is* $(x-y)(y-z)(z-x)=0$.
- *In particular its union with the non regular points is again a Ceva arrangement.*



$\mathfrak{S}_4$ action on Ceva arrangement $\mathscr{C}$

- $\mathscr{C} \leftrightarrow \{(x^2-y^2)(y^2-z^2)(z^2-x^2)=0\}$, $\operatorname{Aut}\mathcal{C} = \mathfrak{S}_4$
- $[x:y:z] \mapsto [-x:y:z]$, ...
- $1 \to \mu_2 \times \mu_2 \to \mathfrak{S}_4 \to \mathfrak{S}_3 \to 1$

Lemma

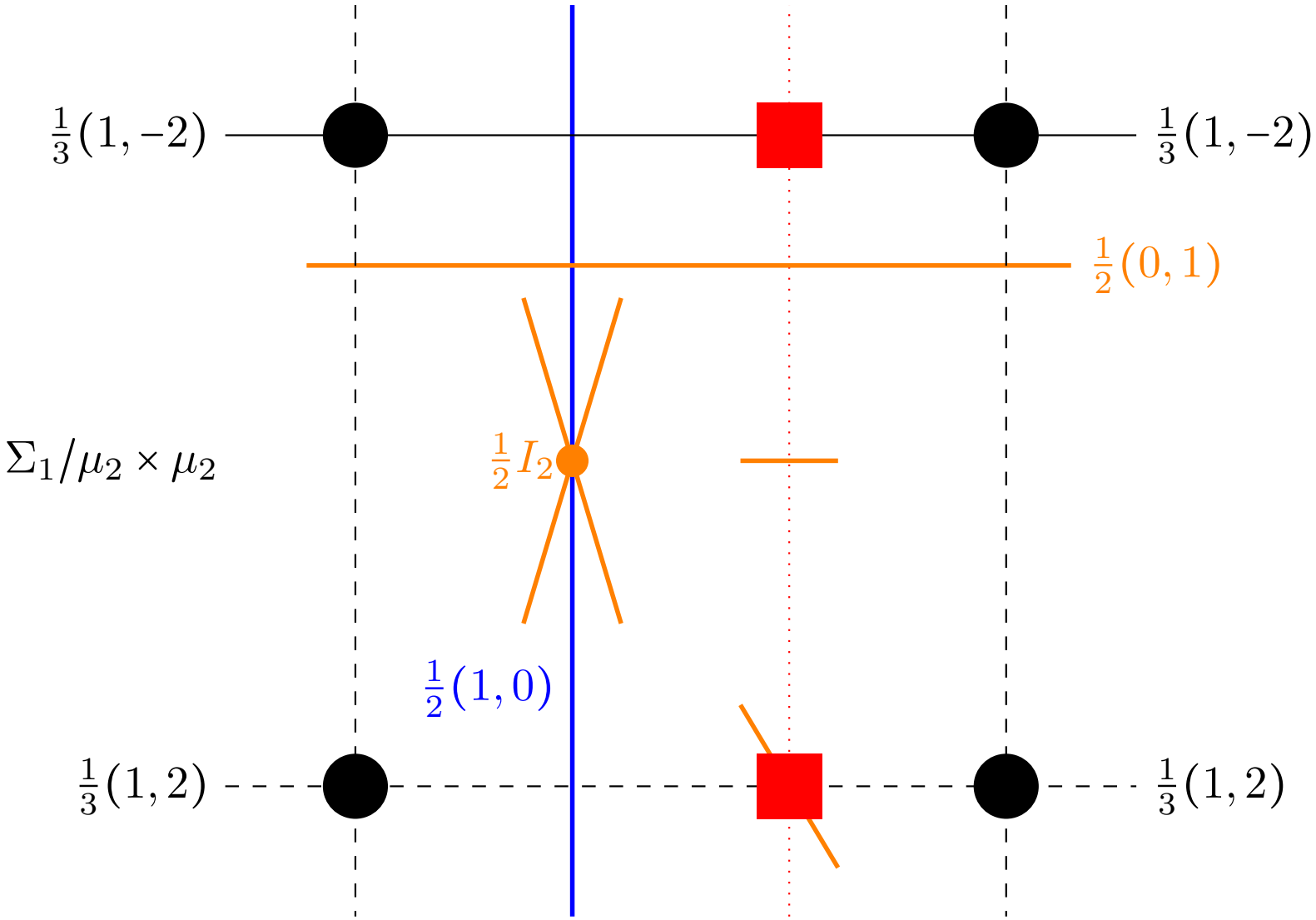
- $\mathbb{P}^2$ *is the underlying space of the orbifold* $\mathbb{P}^2/\mu_2 \times \mu_2$.
- *The non-regular points are* $xyz = 0$.
- *The isotropy groups in the axes outside the vertices are* μ_2
- *The isotropy groups for the vertices are* $\mu_2 \times \mu_2$
- *Kummer covering* $[x:y:z] \to [x^2:y^2:z^2]$ *in mind*
- *The image of* $\mathscr{C}$ *is* $(x-y)(y-z)(z-x)=0$.
- *In particular its union with the non regular points is again a Ceva arrangement.*

Strategy

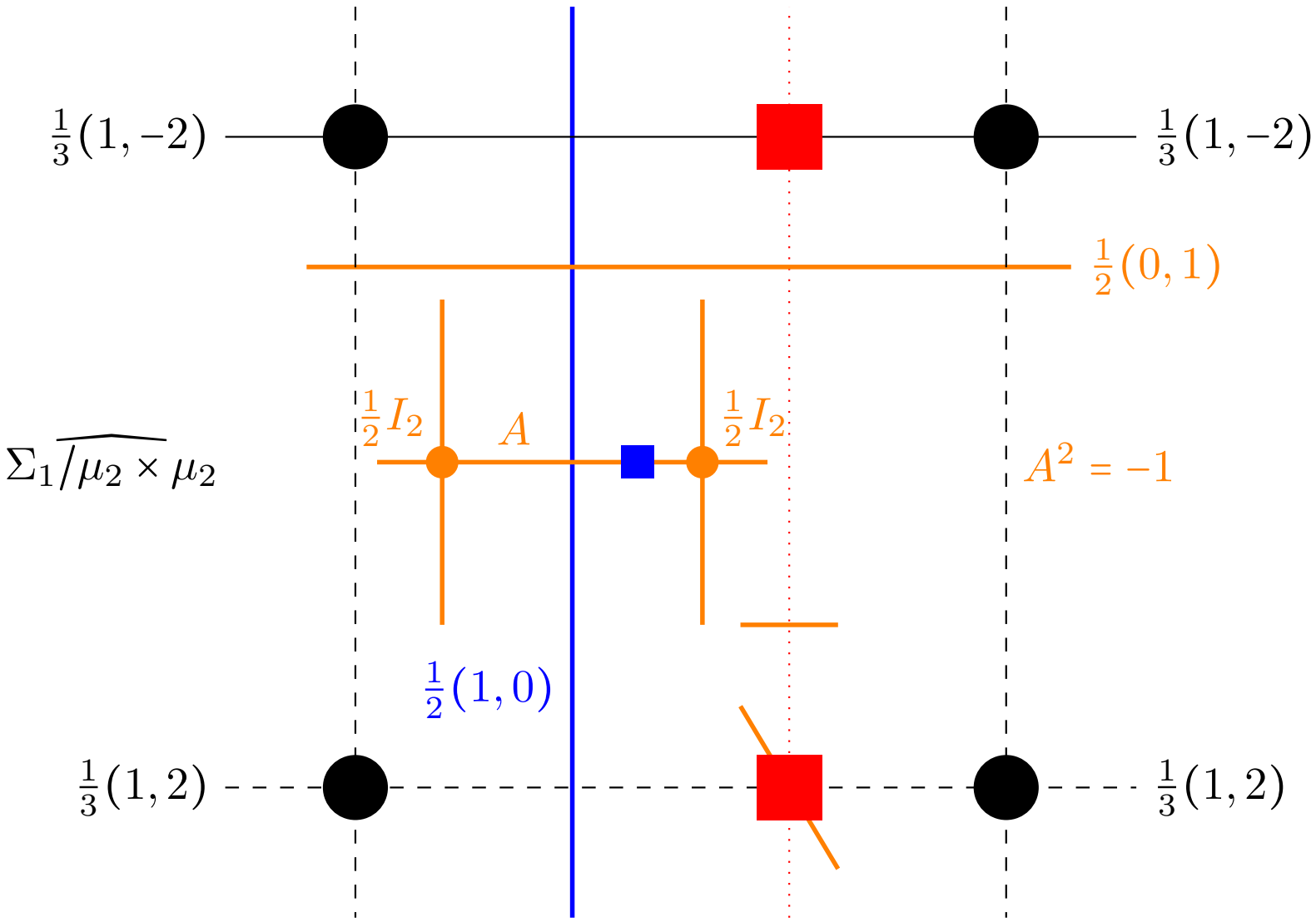
Study the birational model of $(\mathbb{P}^2/\mu_2 \times \mu_2)/\mu_3$.



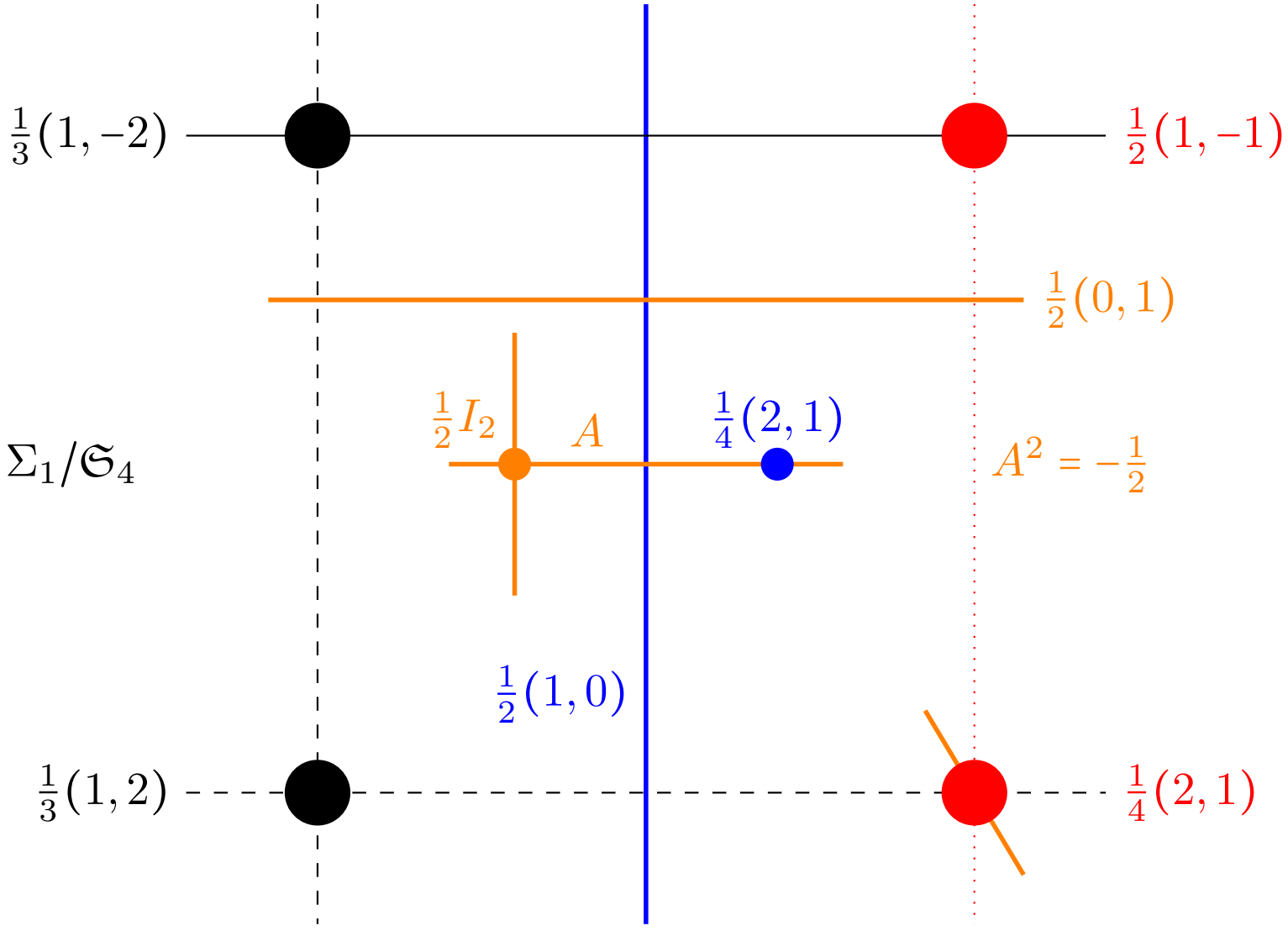
$\mathbb{S}_4$ orbifold



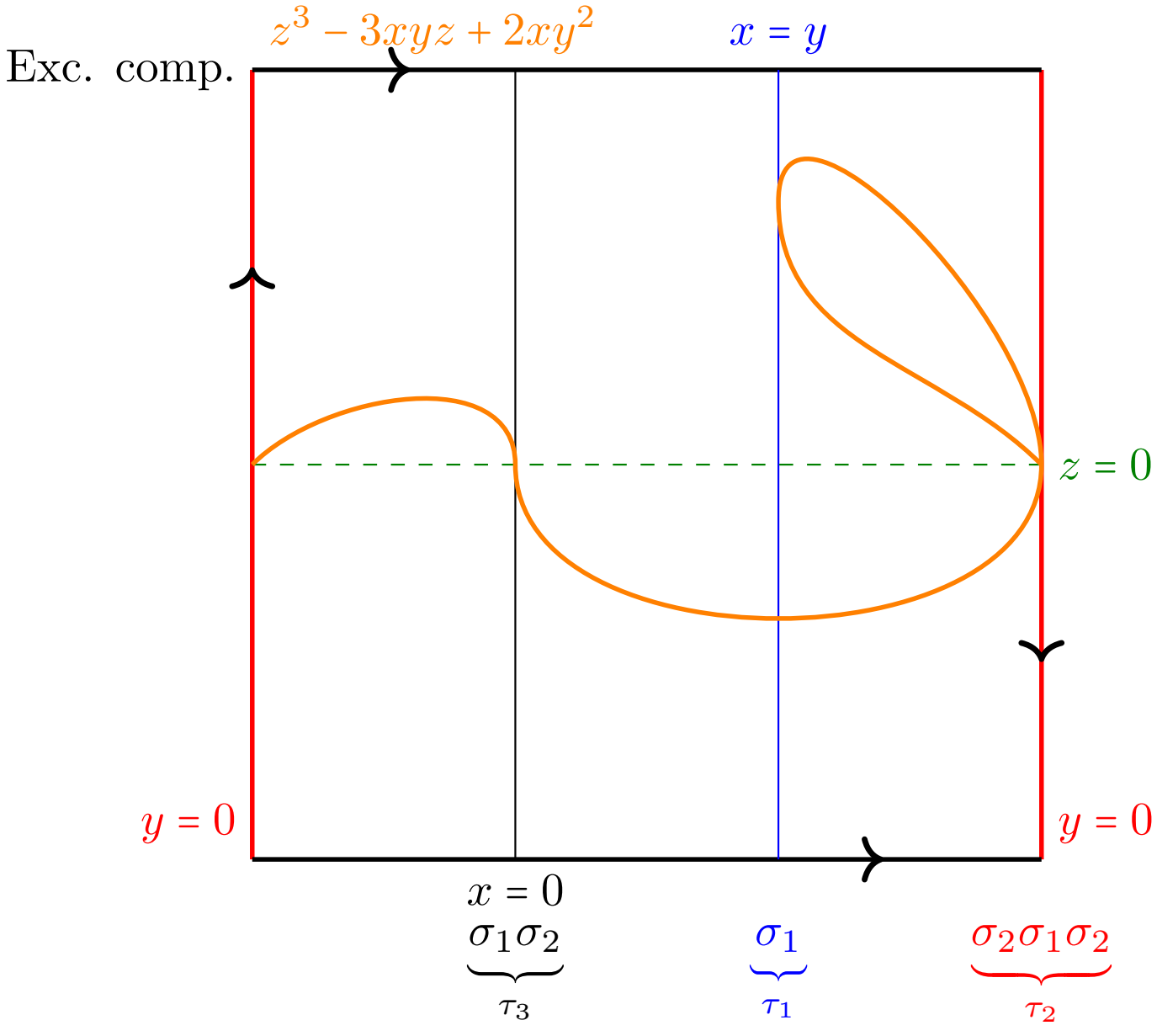
$\mathbb{S}_4$ orbifold



$\mathfrak{S}_4$ orbifold



$\mathbb{S}_4$ orbifold



Generalized Ceva arrangements

Theorem

$$\mathscr{C}_2 = \{(x^2 - y^2)(y^2 - z^2)(z^2 - x^2) = 0\}$$

$$G_1 = \langle x_3, y_1, y_3 \mid [y_1, x_3] = [y_1, (y_3 y_1)^2] = (y_3 x_3)^3 = x_3^2 = 1, y_3^2 = y_1 y_3 y_1 \rangle$$

$$\begin{aligned} G_1 &\xrightarrow{\Phi_2} \mathfrak{S}_4 \\ x_3 &\longmapsto (1,2)(3,4) \\ y_1 &\longmapsto (1,2) \\ y_3 &\longmapsto (1,2,3) \end{aligned}$$

$$\pi_1(\mathbb{P}^2 \smallsetminus \mathscr{C}_2) = \ker \Phi_2$$



Generalized Ceva arrangements

Theorem

$$\mathscr{C}_2 = \{(x^2 - y^2)(y^2 - z^2)(z^2 - x^2) = 0\}$$

$$H_2 = \langle a_1, a_2, a_3 \mid a_1^2 = a_2^2 = a_3^2 = a_1 a_2 a_3 = 1, [a_i, a_j] = 1 \rangle$$

$$\tilde{H}_2 = H_2 \rtimes \mathfrak{S}_3, \quad a_i^\sigma = a_{i\sigma}$$

$$G_1 = \langle x_3, y_1, y_3 \mid [y_1, x_3] = [y_1, (y_3 y_1)^2] = (y_3 x_3)^3 = x_3^2 = 1, y_3^2 = y_1 y_3 y_1 \rangle$$

$$\begin{aligned} G_1 &\xrightarrow{\Phi_2} \tilde{H}_2 \\ x_3 &\longmapsto a_3 \\ y_1 &\longmapsto (1,2) \\ y_3 &\longmapsto (1,2,3) \end{aligned}$$

$$\pi_1(\mathbb{P}^2 \setminus \mathscr{C}_2) = \ker \Phi_2$$



Generalized Ceva arrangements

Theorem

$$\mathscr{C}_n = \{(x^n - y^n)(y^n - z^n)(z^n - x^n) = 0\}$$

$$H_n = \langle a_1, a_2, a_3 \mid a_1^n = a_2^n = a_3^n = a_1 a_2 a_3 = 1, [a_i, a_j] = 1 \rangle$$

$$\tilde{H}_n = H_n \rtimes \mathfrak{S}_3, \quad a_i^\sigma = a_{i\sigma}$$

$$G_1 = \langle x_3, y_1, y_3 \mid [y_1, x_3] = [y_1, (y_3 y_1)^2] = (y_3 x_3)^3 = x_3^n = 1, y_3^2 = y_1 y_3 y_1 \rangle$$

$$\begin{aligned} G_1 &\xrightarrow{\Phi_n} \tilde{H}_n \\ x_3 &\longmapsto a_3 \\ y_1 &\longmapsto (1,2) \\ y_3 &\longmapsto (1,2,3) \end{aligned}$$

$$\pi_1(\mathbb{P}^2 \smallsetminus \mathscr{C}_n) = \ker \Phi_n$$



McLane arrangements $\mathscr{ML}$

Proposition

$\operatorname{Aut} \mathscr{ML} = \mathrm{SL}(2; \mathbb{F}_3)$ *(of order* 24*).*



McLane arrangements $\mathscr{ML}$

Proposition

$\operatorname{Aut} \mathscr{ML} = \mathrm{SL}(2; \mathbb{F}_3)$ *(of order* 24*)*.

Group structure and action



McLane arrangements $\mathscr{ML}$

Proposition

$\operatorname{Aut} \mathscr{ML} = \mathrm{SL}(2;\mathbb{F}_3)$ *(of order* 24*).*

Group structure and action

- $Z(\mathrm{SL}(2;\mathbb{F}_3))$ generated by $-I_2$.



McLane arrangements $\mathscr{ML}$

Proposition

$\operatorname{Aut} \mathscr{ML} = \mathrm{SL}(2;\mathbb{F}_3)$ *(of order 24).*

Group structure and action

- $Z(\mathrm{SL}(2;\mathbb{F}_3))$ generated by $-I_2$.
- $1 \to \mu_2 \times \mu_2 \to \mathrm{PSL}(2;\mathbb{F}_3) \to \mu_3 \to 1$



McLane arrangements $\mathscr{ML}$

Proposition

$\operatorname{Aut} \mathscr{ML} = \mathrm{SL}(2;\mathbb{F}_3)$ *(of order 24).*

Group structure and action

- $Z(\mathrm{SL}(2;\mathbb{F}_3))$ generated by $-I_2$.
- $1 \to \mu_2 \times \mu_2 \to \mathrm{PSL}(2;\mathbb{F}_3) \to \mu_3 \to 1$
- The action on $\mathbb{P}^2$ has a fixed point p.



McLane arrangements $\mathscr{ML}$

Proposition

$\operatorname{Aut} \mathscr{ML} = \mathrm{SL}(2;\mathbb{F}_3)$ *(of order 24).*

Group structure and action

- $Z(\mathrm{SL}(2;\mathbb{F}_3))$ generated by $-I_2$.
- $1 \to \mu_2 \times \mu_2 \to \mathrm{PSL}(2;\mathbb{F}_3) \to \mu_3 \to 1$
- The action on $\mathbb{P}^2$ has a fixed point p.
- The singular points of McLane arrangements are contained in four lines through p, each one containing two triple points.



McLane arrangements $\mathscr{ML}$

Proposition

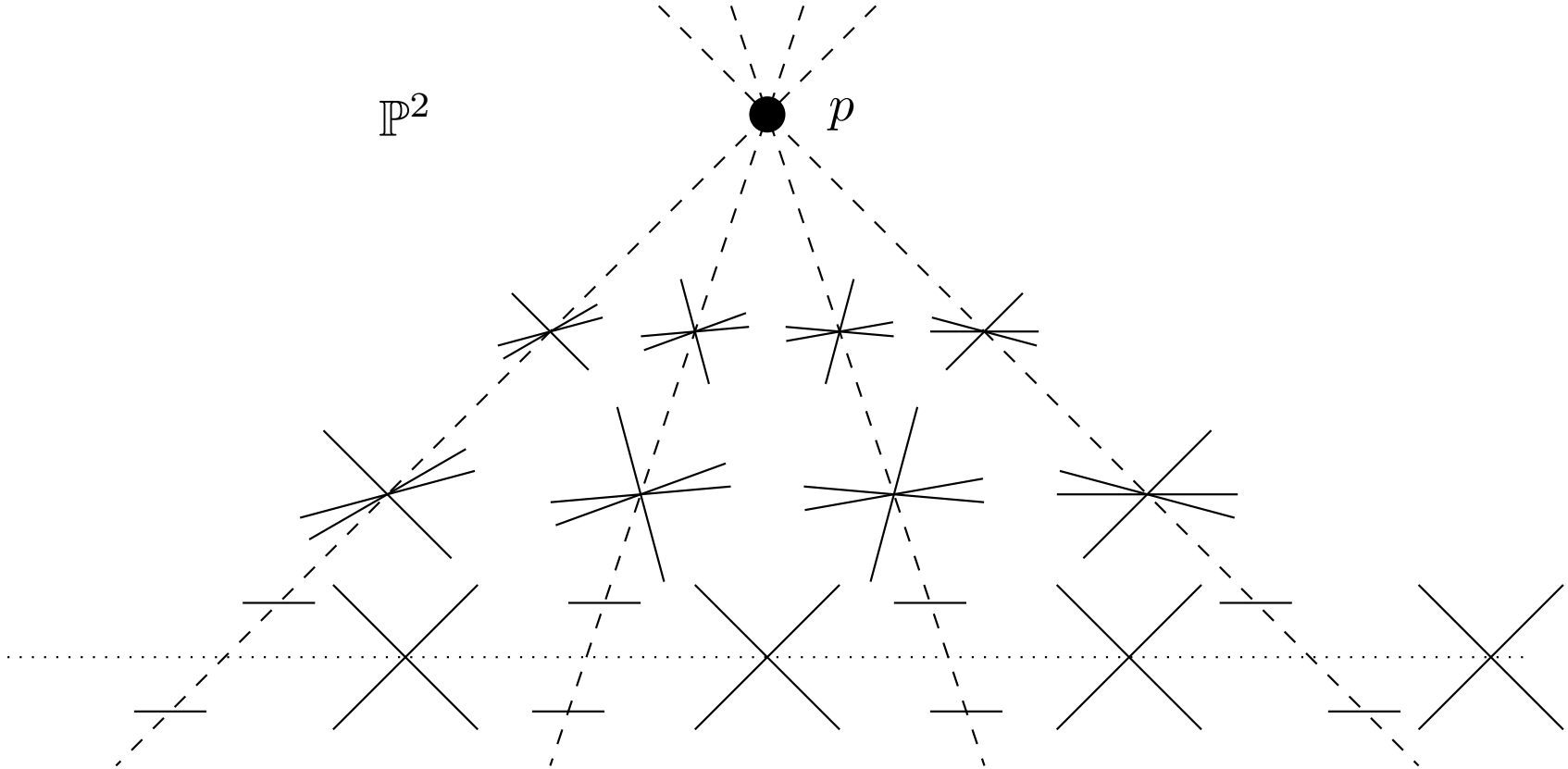
$\operatorname{Aut} \mathscr{ML} = \mathrm{SL}(2;\mathbb{F}_3)$ *(of order 24).*

Group structure and action

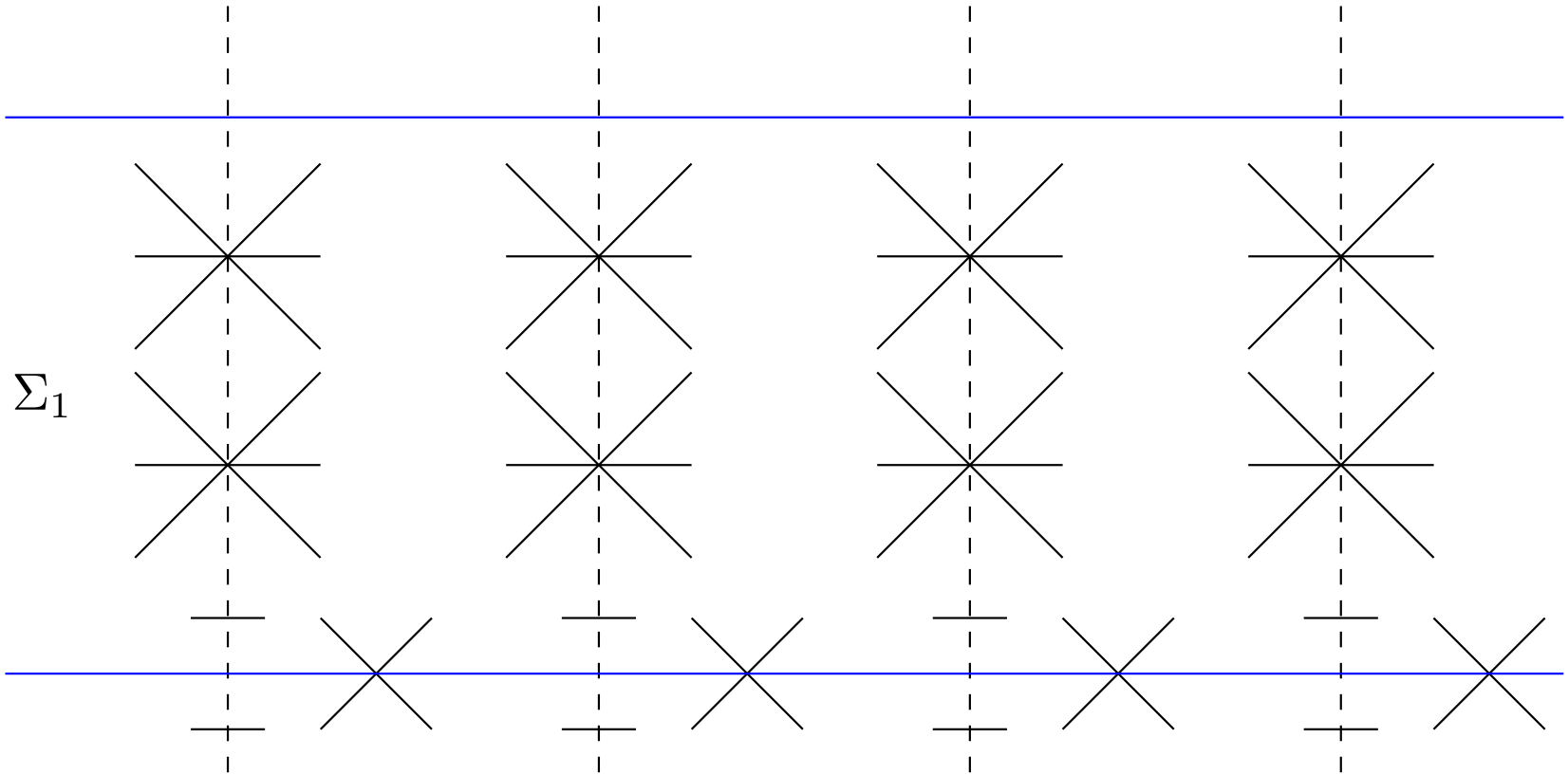
- $Z(\mathrm{SL}(2;\mathbb{F}_3))$ generated by $-I_2$.
- $1 \to \mu_2 \times \mu_2 \to \mathrm{PSL}(2;\mathbb{F}_3) \to \mu_3 \to 1$
- The action on $\mathbb{P}^2$ has a fixed point p.
- The singular points of McLane arrangements are contained in four lines through p, each one containing two triple points.
- The action of the center has also a line of fixed points containing the double points.



McLane quotient

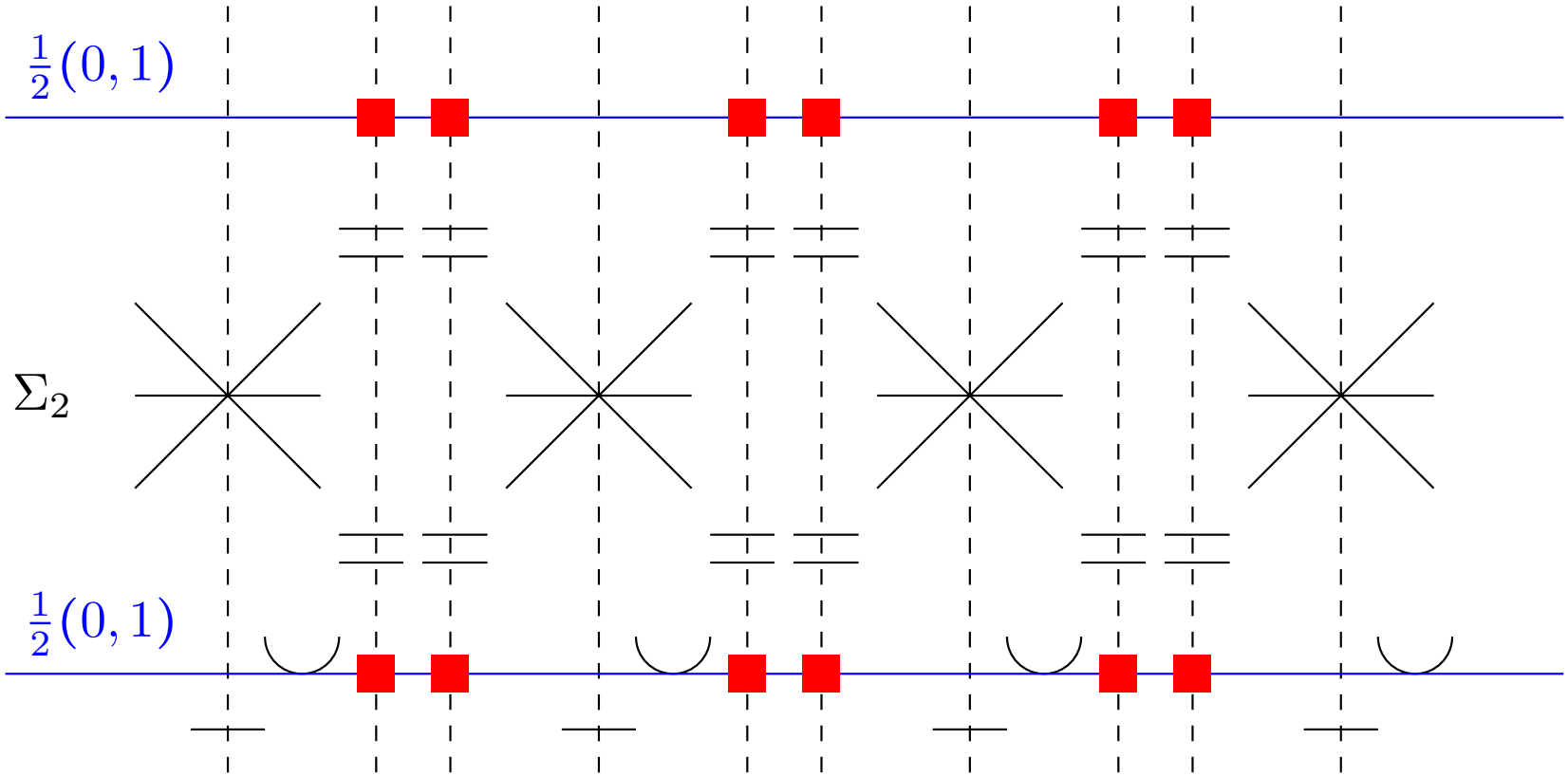


McLane quotient

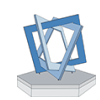


μ_2 action

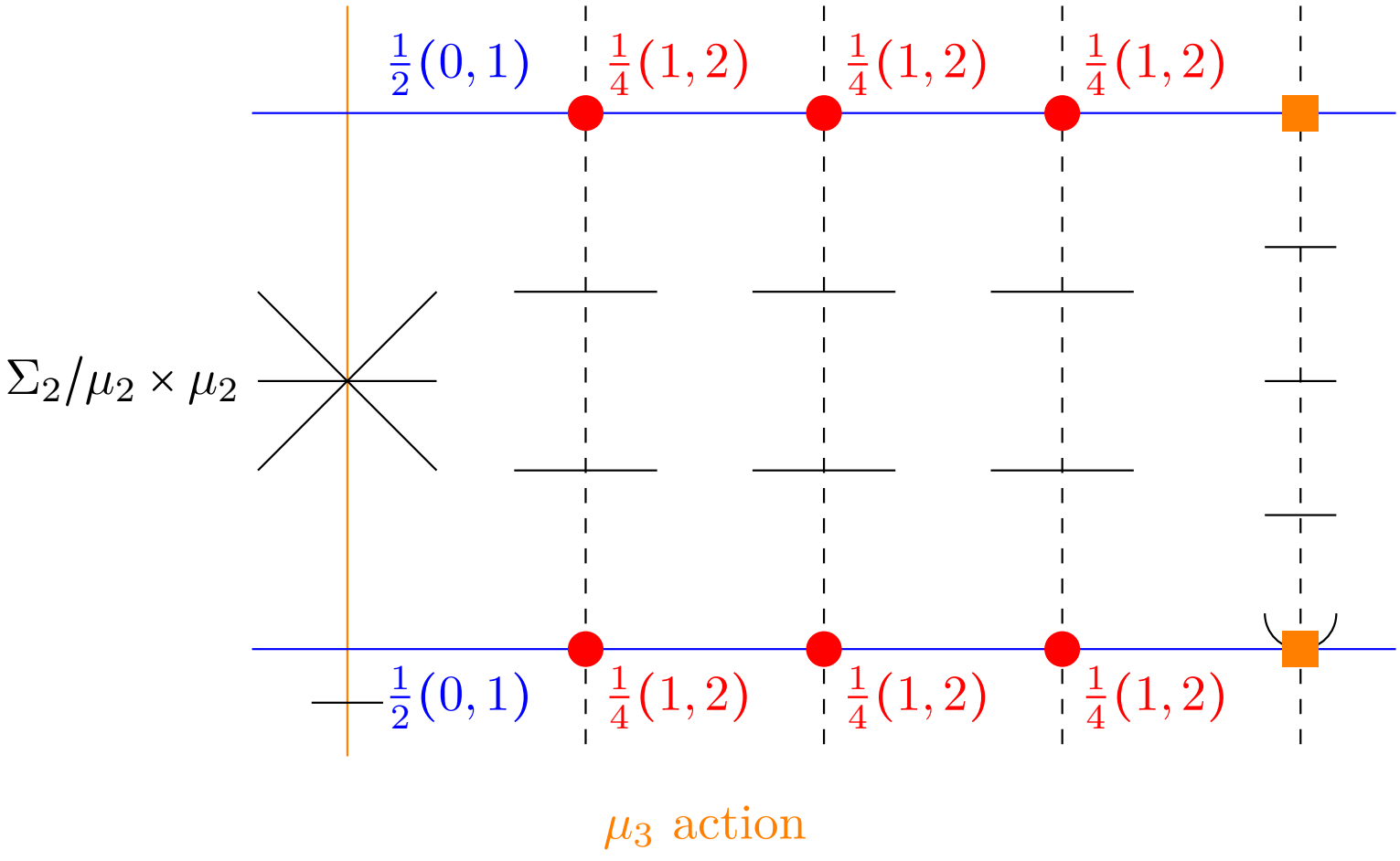
McLane quotient



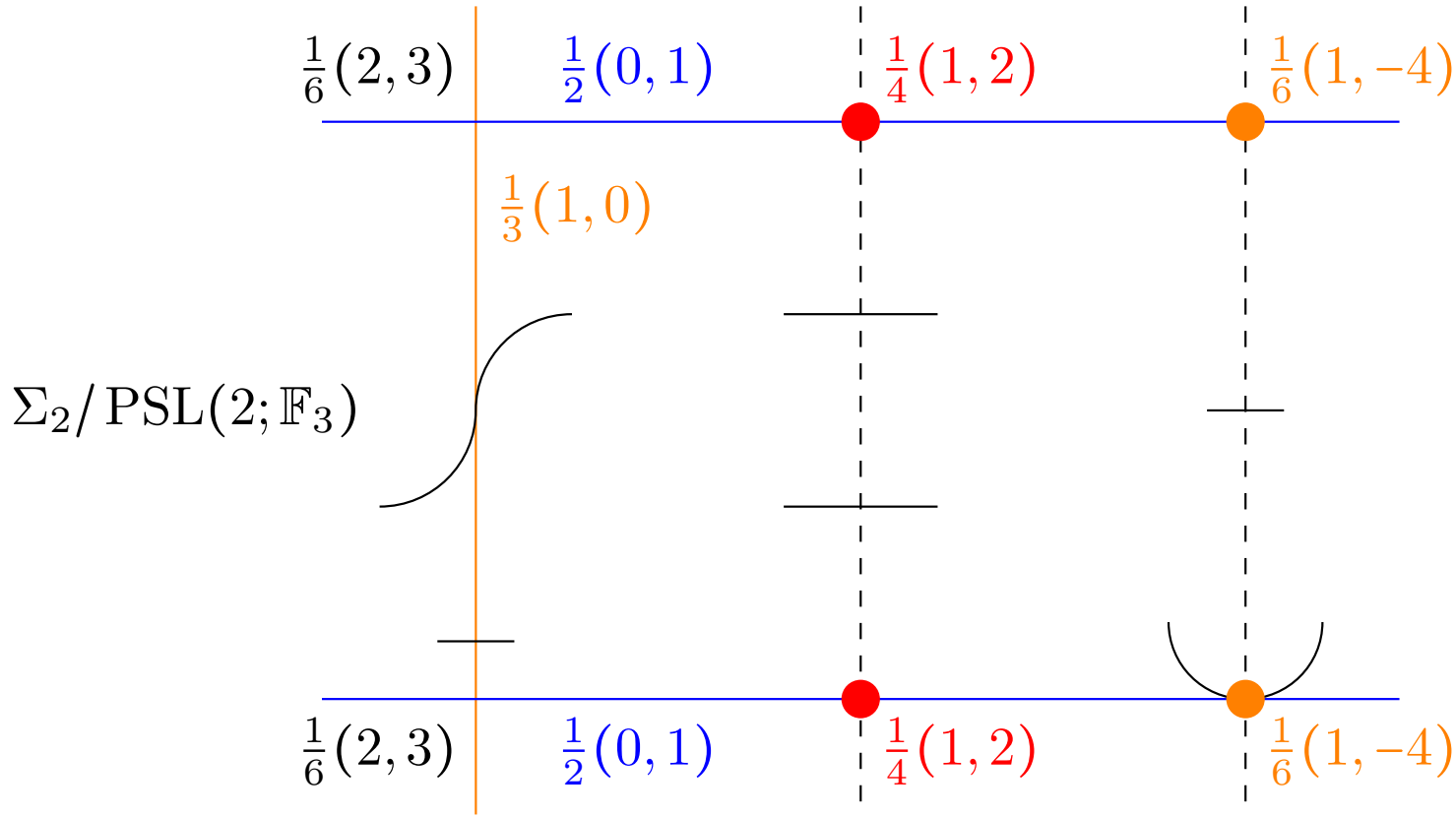
$\mu_2 \times \mu_2$ action



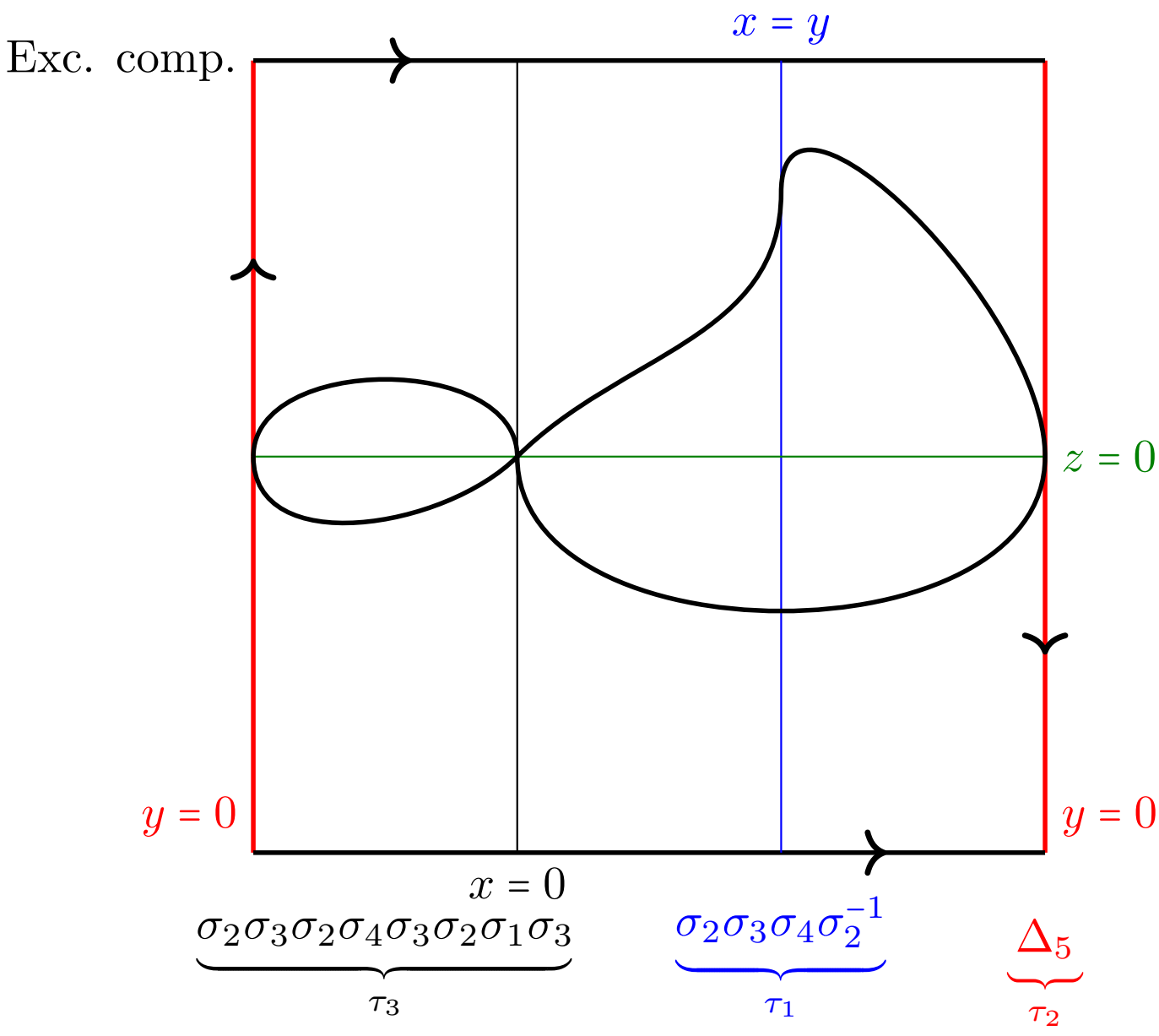
McLane quotient



McLane quotient



McLane quotient



McLane quotient

- $\tau_1\tau_2\tau_3 = \Delta_5^2 \in \mathbb{B}_5$, $\tau_3^3 = \Delta_3^2\sigma_3^4$.

Fundamental group of $\mathbb{P}^2 \setminus (\mathcal{C}_1 \cup \{xy(x-y)=0\})$

- $\langle x_1, \ldots, x_5, y_1, y_2, y_3 \mid x_i^{\tau_j} = y_j^{-1} x_i y_j, x_1 \ldots x_5 y_1 y_2 y_3 = 1 \rangle$
- $x^{\Delta_5^2} = (x_1 \ldots x_5) x (x_1 \ldots x_5)^{-1}$



McLane quotient

- $\tau_1\tau_2\tau_3 = \Delta_5^2 \in \mathbb{B}_5$, $\tau_3^3 = \Delta_3^2\sigma_3^4$.

Fundamental group of $\mathbb{P}^2 \setminus (\mathcal{C}_1 \cup \{xy(x-y)=0\})$

- $\langle x_1, \ldots, x_5, y_1, y_2, y_3 \mid x_i^{\tau_j} = y_j^{-1}x_iy_j, x_1 \ldots x_5y_1y_2y_3 = 1\rangle$
- $x^{\Delta_5^2} = (x_1 \ldots x_5)x(x_1 \ldots x_5)^{-1}$

Orbifold fundamental group G_1 of $\Sigma_2/\mathrm{PSL}(2;\mathbb{F}_3) \setminus \mathcal{C}_1$

- $y_1 \leftrightarrow \mu \Longrightarrow y_1^3 = 1$
- $y_2 \leftrightarrow \beta \Longrightarrow y_2^2(x_1 \ldots x_5) = 1$
- $y_3 \leftrightarrow \alpha \Longrightarrow y_3^3(x_1 \ldots x_5) = 1$
- $x_3^2 = 1$



McLane quotient

- $\tau_1\tau_2\tau_3 = \Delta_5^2 \in \mathbb{B}_5$, $\tau_3^3 = \Delta_3^2\sigma_3^4$.

Fundamental group of $\mathbb{P}^2 \setminus (\mathcal{C}_1 \cup \{xy(x-y)=0\})$

- $\langle x_1, \ldots, x_5, y_1, y_2, y_3 \mid x_i^{\tau_j} = y_j^{-1} x_i y_j, x_1 \ldots x_5 y_1 y_2 y_3 = 1 \rangle$
- $x^{\Delta_5^2} = (x_1 \ldots x_5) x (x_1 \ldots x_5)^{-1}$

Orbifold fundamental group G_1 of $\Sigma_2/\operatorname{PSL}(2;\mathbb{F}_3) \setminus \mathcal{C}_1$

- G_1 generators: x_5, y_1, y_3
- G_1 relations:
  $y_3^2 = y_1 y_3 y_1, [y_1^{-1} y_3, x_5] = [x_5, (x_5 y_1)] = y_1^3 = (x_5 y_1 x_5 y_3)^4 = 1$
- $G = \ker \Phi$

$$
\begin{aligned}
G_1 &\xrightarrow{\Phi} \operatorname{SL}(2;\mathbb{F}_3) \\
x_5 &\longmapsto I_2 \\
y_1 &\longmapsto \left(\begin{smallmatrix} -1 & -1 \\ 1 & 0 \end{smallmatrix}\right) \\
y_3 &\longmapsto \left(\begin{smallmatrix} -1 & -1 \\ 0 & -1 \end{smallmatrix}\right)
\end{aligned}
$$



McLane equations

- $P_4 := 3x^4 + 6x^2y^2 - y^4$
- $Q_4 := 3x^4 - 6x^2y^2 - y^4$
- $P_6 := 6xy(3x^4 + y^4)$

$$\begin{array}{ccc} \mathbb{P}^2 & \overset{\Phi}{\dashrightarrow} & \mathbb{P}^2 \\ [x:y:z] & \longmapsto & [P_4^3 : P_6^2 : P_4P_6z^2] \end{array}$$

- $F = z^8 - 6P_4z^4 - 8P_6z^2 - 3P_4^2$



Hesse $\mathscr{H}$ and dual-Hesse $\check{\mathscr{H}}$ arrangements

Equations and properties

- $\check{\mathscr{H}} : (x^3 - y^3)(y^3 - z^3)(z^3 - x^3) = 0$
- $\mathscr{H}$:
  $xyz(x^3+y^3+z^3-3xyz)((x^3+y^3+z^3)^2+3xyz(x^3+3xyz+y^3+z^3)) = 0$
- Singular fibers of elliptic fibrations



Hesse $\mathscr{H}$ and dual-Hesse $\check{\mathscr{H}}$ arrangements

Equations and properties

- $\check{\mathscr{H}} : (x^3 - y^3)(y^3 - z^3)(z^3 - x^3) = 0$
- $\mathscr{H}$:
  $xyz(x^3+y^3+z^3-3xyz)((x^3+y^3+z^3)^2+3xyz(x^3+3xyz+y^3+z^3)) = 0$
- Singular fibers of elliptic fibrations

Proposition

$\operatorname{Aut}\check{\mathscr{H}} = \operatorname{Aut}\mathscr{H} = \mathbb{F}_3^2 \rtimes \mathrm{SL}(2;\mathbb{F}_3)$



Hesse $\mathscr{H}$ and dual-Hesse $\check{\mathscr{H}}$ arrangements

Equations and properties

- $\check{\mathscr{H}} : (x^3 - y^3)(y^3 - z^3)(z^3 - x^3) = 0$
- $\mathscr{H}$:
  $xyz(x^3+y^3+z^3-3xyz)((x^3+y^3+z^3)^2+3xyz(x^3+3xyz+y^3+z^3)) = 0$
- Singular fibers of elliptic fibrations

Proposition

$\operatorname{Aut}\check{\mathscr{H}} = \operatorname{Aut}\mathscr{H} = \mathbb{F}_3^2 \rtimes \mathrm{SL}(2;\mathbb{F}_3)$

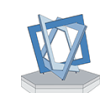
Series of Aut $\check{\mathscr{H}}$

- $H_1 = \mathbb{F}_3^2 \cong \boxed{\mu_3 \times \mu_3}$, $\operatorname{Aut}\check{\mathscr{H}}/H_1 \cong \mathrm{SL}(2;\mathbb{F}_3)$
- $H_2/H_1 \cong Z(\mathrm{SL}(2;\mathbb{F}_3)) \cong \boxed{\mu_2}$, $\operatorname{Aut}\check{\mathscr{H}}/H_2 \cong \mathfrak{A}_4$
- $H_3/H_2 \cong \boxed{\mu_2 \times \mu_2}$, $\operatorname{Aut}\check{\mathscr{H}}/H_3 \cong \boxed{\mu_3}$



Elliptic fibrations and quotient surfaces

- $\pi : X \to \mathbb{P}^2$ blow-up of the base points of $(x^3 + y^3 + z^3, xyz)$



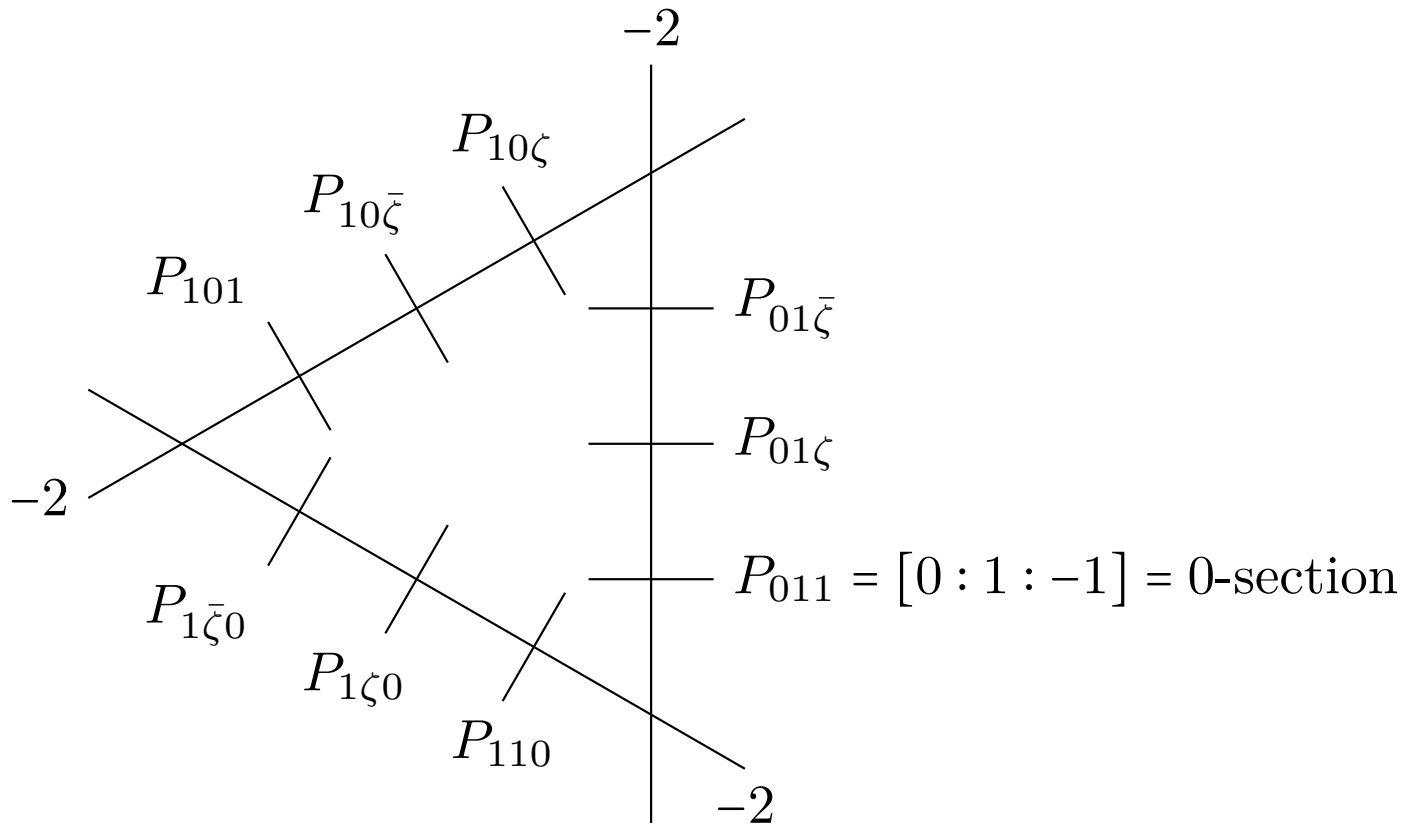
Elliptic fibrations and quotient surfaces

- $\pi : X \to \mathbb{P}^2$ blow-up of the base points of $(x^3 + y^3 + z^3, xyz)$
- $\varphi : X \to \mathbb{P}^1$ with four exceptional fibers (strict transform of $\mathscr{H}$), $P_{ab}, a, b \in \mu_3$, exceptional components.



Elliptic fibrations and quotient surfaces

- $\pi : X \to \mathbb{P}^2$ blow-up of the base points of $(x^3 + y^3 + z^3, xyz)$
- $\varphi : X \to \mathbb{P}^1$ with four exceptional fibers (strict transform of $\mathscr{H}$), $P_{ab}, a, b \in \mu_3$, exceptional components.





Elliptic fibrations and quotient surfaces

- $\pi : X \to \mathbb{P}^2$ blow-up of the base points of $(x^3 + y^3 + z^3, xyz)$
- $\varphi : X \to \mathbb{P}^1$ with four exceptional fibers (strict transform of $\mathscr{H}$), $P_{ab}, a, b \in \mu_3$, exceptional components.
- $\mathrm{MW}(\varphi) \cong \mu_3 \times \mu_3$.

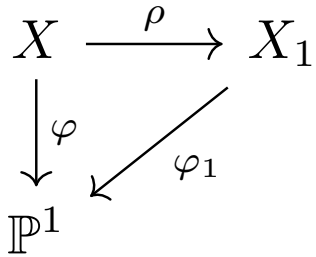


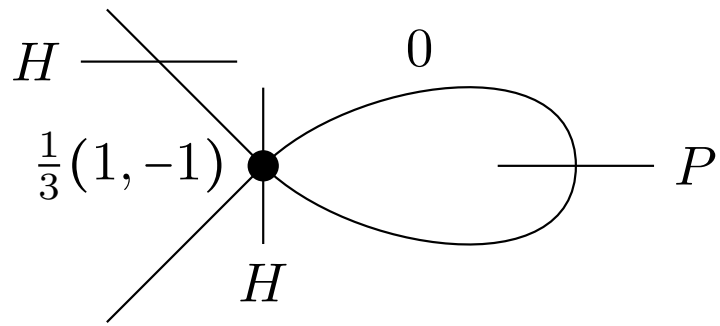
Elliptic fibrations and quotient surfaces

- $\pi : X \to \mathbb{P}^2$ blow-up of the base points of $(x^3 + y^3 + z^3, xyz)$
- $\varphi : X \to \mathbb{P}^1$ with four exceptional fibers (strict transform of $\mathscr{H}$), $P_{ab}, a, b \in \mu_3$, exceptional components.
- $\mathrm{MW}(\varphi) \cong \mu_3 \times \mu_3$.

Proposition

- $X_1 := X/\mu_3 \times \mu_3$
- $\rho(P_{\bullet\bullet\bullet}) = P$ *0-section of* φ_1
- $\rho(\check{\mathscr{H}}) =: H$ *multisection of 2-torsion*
- φ_1 *with four singular fibers*



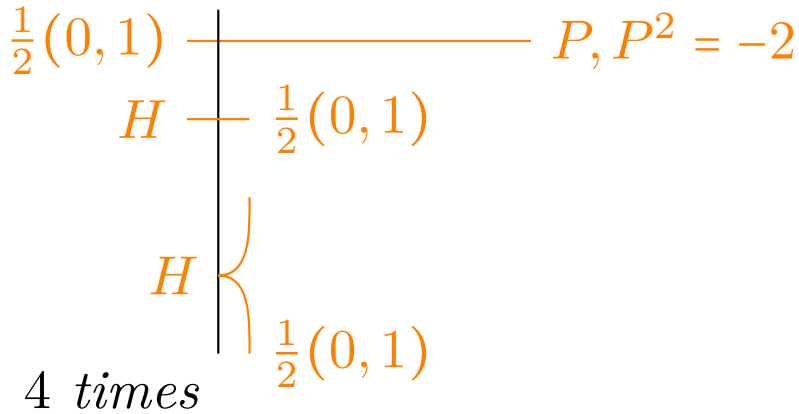


Elliptic fibrations and quotient surfaces

- $\pi : X \to \mathbb{P}^2$ blow-up of the base points of $(x^3 + y^3 + z^3, xyz)$
- $\varphi : X \to \mathbb{P}^1$ with four exceptional fibers (strict transform of $\mathscr{H}$), $P_{ab}, a, b \in \mu_3$, exceptional components.
- $\mathrm{MW}(\varphi) \cong \mu_3 \times \mu_3$.

Proposition

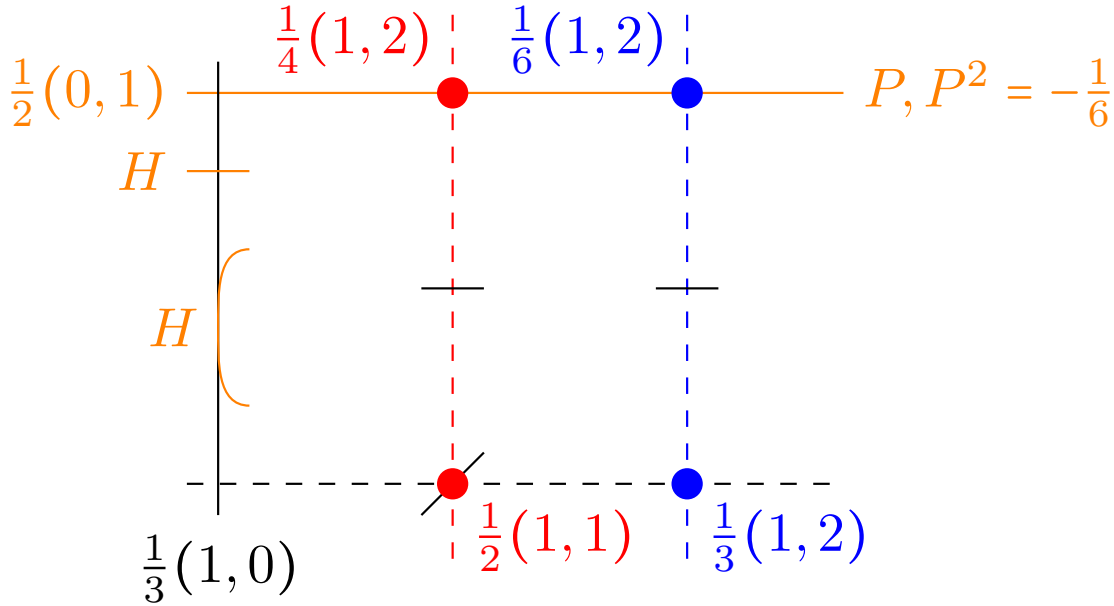
$\Sigma_2 \cong X_1/\mu_2 \equiv X/(x \mapsto -x)$, $\varphi_2 : \Sigma_2 \to \mathbb{P}^1$.



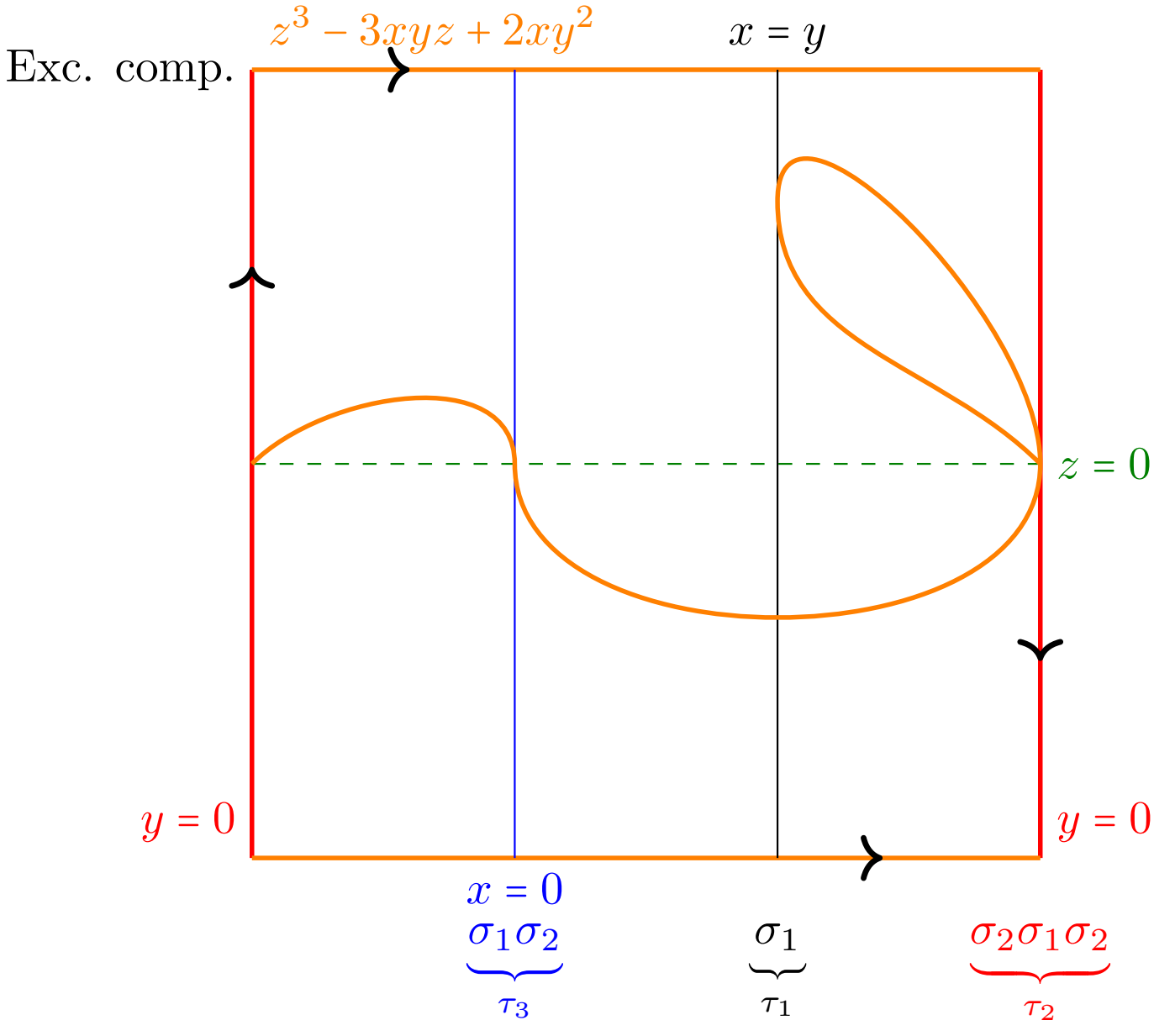
Elliptic fibrations and quotient surfaces

- $\pi : X \to \mathbb{P}^2$ blow-up of the base points of $(x^3 + y^3 + z^3, xyz)$
- $\varphi : X \to \mathbb{P}^1$ with four exceptional fibers (strict transform of $\mathscr{H}$), $P_{ab}, a, b \in \mu_3$, exceptional components.
- $\mathrm{MW}(\varphi) \cong \mu_3 \times \mu_3$.

Proposition



An old friend



A plus of symplectic geometry

Problem

Find all curves $\mathcal{C} \subset \mathbb{P}^2$ such that $\deg \mathcal{C} = 8$ and $\operatorname{Sing} \mathcal{C} = 8\mathbb{E}_6$



A plus of symplectic geometry

Problem

Find all curves $\mathcal{C} \subset \mathbb{P}^2$ such that $\deg \mathcal{C} = 8$ and $\operatorname{Sing} \mathcal{C} = 8\mathbb{E}_6$

Theorem

There are (at least) three curves as above pairwise non-homeomorphic:

1. $\exists \mathcal{C}_1$ *rigid* μ_2*-invariant.*
2. $\exists \mathcal{C}_2$ *rigid* μ_3*-invariant, only isolated fixed points.*
3. $\exists \mathcal{C}_3$ *symplectic* $\mu_2 \times \mu_2$*-invariant.*



A plus of symplectic geometry

Problem

Find all curves $\mathcal{C} \subset \mathbb{P}^2$ such that $\deg \mathcal{C} = 8$ and $\operatorname{Sing} \mathcal{C} = 8\mathbb{E}_6$ Failed!

Theorem

There are (at least) three curves as above pairwise non-homeomorphic:

1. $\exists \mathcal{C}_1$ *rigid* μ_2*-invariant.*
2. $\exists \mathcal{C}_2$ *rigid* μ_3*-invariant, only isolated fixed points.*
3. $\exists \mathcal{C}_3$ *symplectic* $\mu_2 \times \mu_2$*-invariant.*

Proof.

1, 2 Computations thanks to orbifolds.



A plus of symplectic geometry

Problem

Find all curves $\mathcal{C} \subset \mathbb{P}^2$ such that $\deg \mathcal{C} = 8$ and $\operatorname{Sing} \mathcal{C} = 8\mathbb{E}_6$ Failed!

Theorem

There are (at least) three curves as above pairwise non-homeomorphic:

1. $\exists \mathcal{C}_1$ *rigid* μ_2*-invariant.*
2. $\exists \mathcal{C}_2$ *rigid* μ_3*-invariant, only isolated fixed points.*
3. $\exists \mathcal{C}_3$ *symplectic* $\mu_2 \times \mu_2$*-invariant.*

Proof.

1, 2 Computations thanks to orbifolds.

3 Wrong pictures with the tricuspidal quartic. □



A plus of symplectic geometry

Problem

Find all curves $\mathcal{C} \subset \mathbb{P}^2$ such that $\deg \mathcal{C} = 8$ and $\operatorname{Sing} \mathcal{C} = 8\mathbb{E}_6$ Failed!

Theorem

There are (at least) three curves as above pairwise non-homeomorphic:

1. $\exists \mathcal{C}_1$ *rigid* μ_2*-invariant.*
2. $\exists \mathcal{C}_2$ *rigid* μ_3*-invariant, only isolated fixed points.*
3. $\exists \mathcal{C}_3$ *symplectic* $\mu_2 \times \mu_2$*-invariant.*

Proof.

1, 2 Computations thanks to orbifolds.

3 Wrong pictures with the tricuspidal quartic. □

Question

Is $\mathcal{C}_3$ isotopic to an algebraic curve?



A plus of symplectic geometry

Problem

Find all curves $\mathcal{C} \subset \mathbb{P}^2$ such that $\deg \mathcal{C} = 8$ and $\operatorname{Sing} \mathcal{C} = 8\mathbb{E}_6$ Failed!

Theorem

There are (at least) three curves as above pairwise non-homeomorphic:

1. $\exists \mathcal{C}_1$ *rigid* μ_2*-invariant.*
2. $\exists \mathcal{C}_2$ *rigid* μ_3*-invariant, only isolated fixed points.*
3. $\exists \mathcal{C}_3$ *symplectic* $\mu_2 \times \mu_2$*-invariant.*

Proof.

1, 2 Computations thanks to orbifolds.

3 Wrong pictures with the tricuspidal quartic. □

Question

Is $\mathcal{C}_3$ isotopic to an algebraic curve? Not to $\mathcal{C}_1, \mathcal{C}_2$!



Happy birthday,
Alex and Laurențiu!

