Orbifolds for understanding symmetric curves and line arrangements

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Geometry, Topology and Algebra of Singular Spaces
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Questions on McLane arrangements

McLane arrangements $\mathscr{ML}$

- $\mathscr{ML}_\zeta := \{xyz(x-y)(x-z)(\zeta y+z)(\bar{\zeta}x+\zeta y+z)(x+\bar{\zeta}y-z)=0\}$, $\zeta = \exp\frac{\pm 2i\pi}{3}$

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Can symmetry provide more understanding to the topology of $(\mathbb{P}^2, \mathscr{ML})$?

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Strategy

Work with $\mathrm{Aut}\,\mathscr{ML}$ and study $\mathbb{P}^2/\,\mathrm{Aut}\,\mathscr{ML}$.

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- $\mu_n := \{\zeta \in \mathbb{C} \mid \zeta^n = 1\}$
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- $\prod_{j=1}^{r} \mu_{m_j} \times \mathbb{C}^n$, $\zeta_j \cdot (z_1, \ldots, z_n) := (\zeta_j^{a_{j1}} \cdot z_1, \ldots, \zeta_j^{a_{jn}} \cdot z_n)$.

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- $\begin{pmatrix} \frac{1}{m_1} \\ \vdots \\ \frac{1}{m_r} \end{pmatrix} \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{r1} & \dots & a_{rn} \end{pmatrix}$

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A symmetric curve

Tricuspidal quartic

- $\mathcal{C} : x^2y^2 + x^2z^2 + y^2z^2 - 2xyz(x+y+z) = 0$
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- $\mathcal{C} : z^4 - 24xyz^2 + 32(x^3 + y^3)z - 48x^2y^2 = 0.$
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  - $[x : y : z] \mapsto [\zeta x : \overline{\zeta} y : z].$
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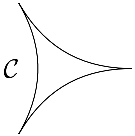
- Birational model of $\mathbb{P}^2 \to \mathbb{P}^2/\mathbb{D}_6$:
  $[x : y : z] \mapsto [4x^3y^3 : (x^3 + y^3)^2 : xyz(x^3 + y^3)]$
- $\mathcal{C}$ comes from $\mathcal{C}_1 : z^4 - 6xyz^2 + 8xy^2z - 3x^2y^2 = 0.$

Geometry



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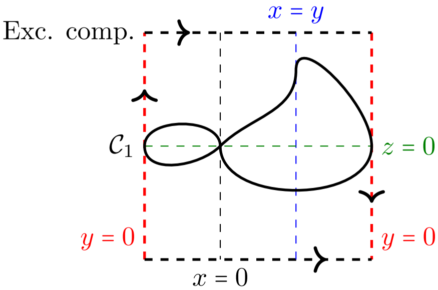

- $y = 0$: $\mathcal{C}_1$ is like $u^2 = v^4$.
- $x = 0$: ordinary node with flex vertical branch.
- $x = y$: vertical tangent at flex.

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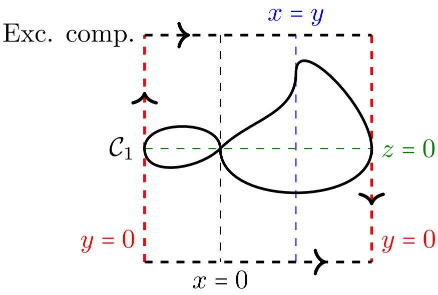

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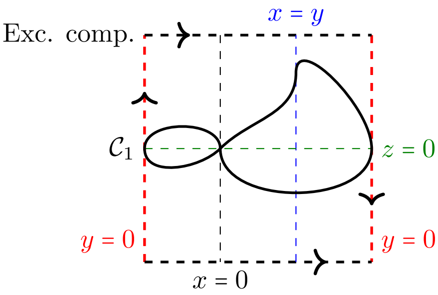

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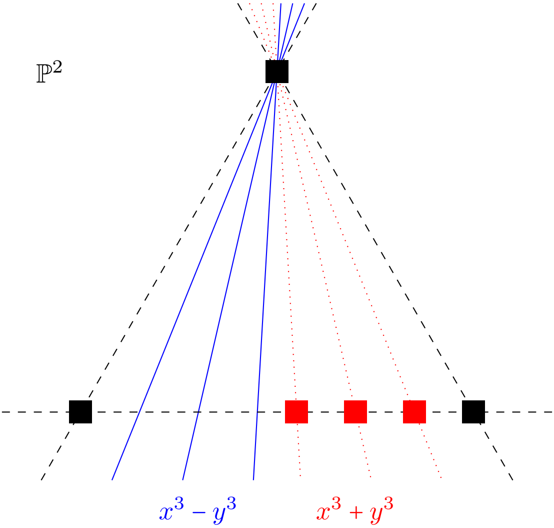
- Klein bottle from the real blow-up of $\mathbb{P}^2$ at $[0:0:1]$.

The movie of the action



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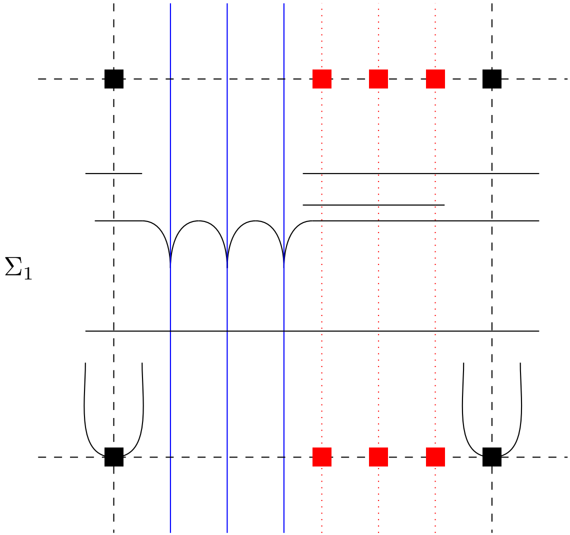

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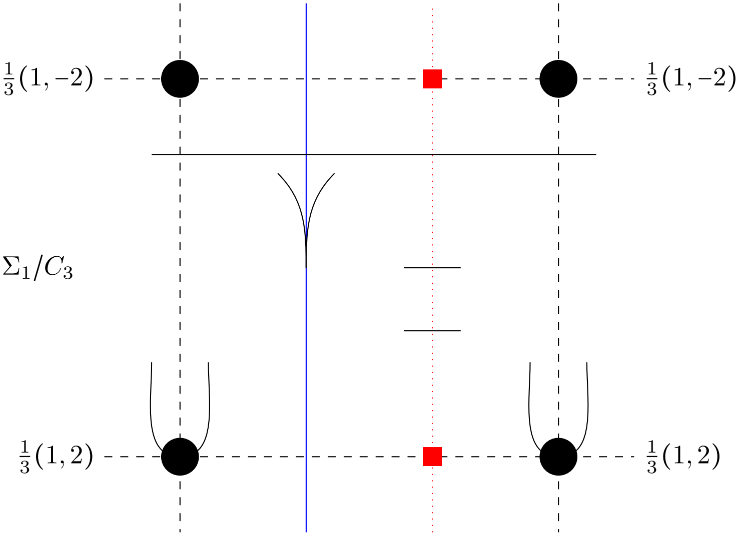

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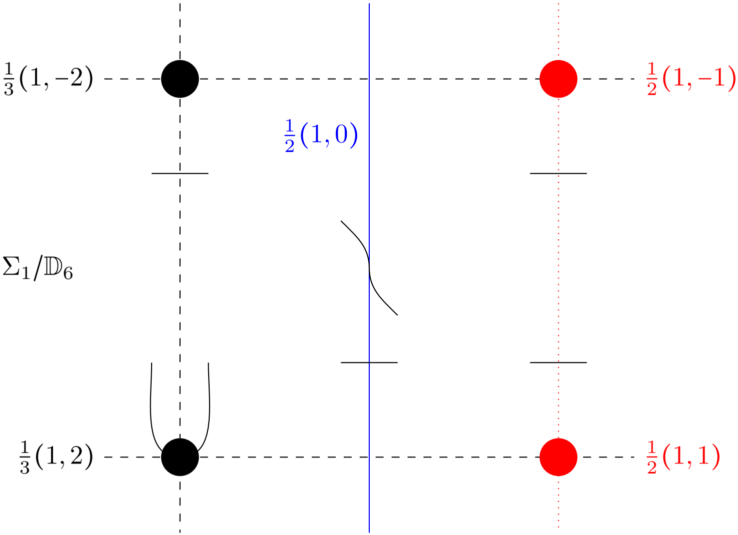

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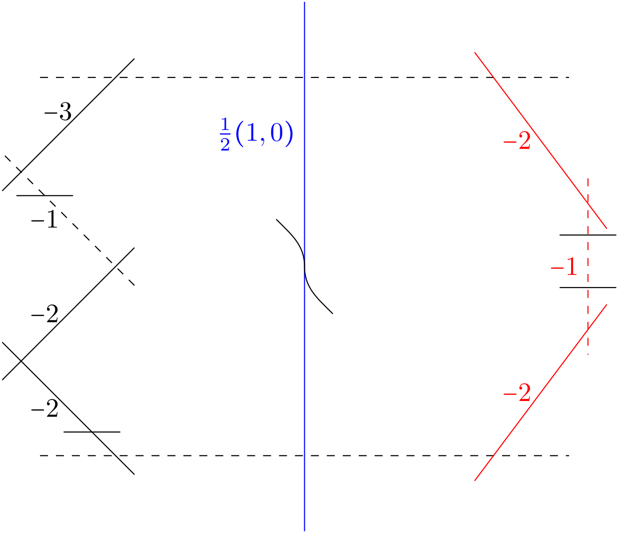

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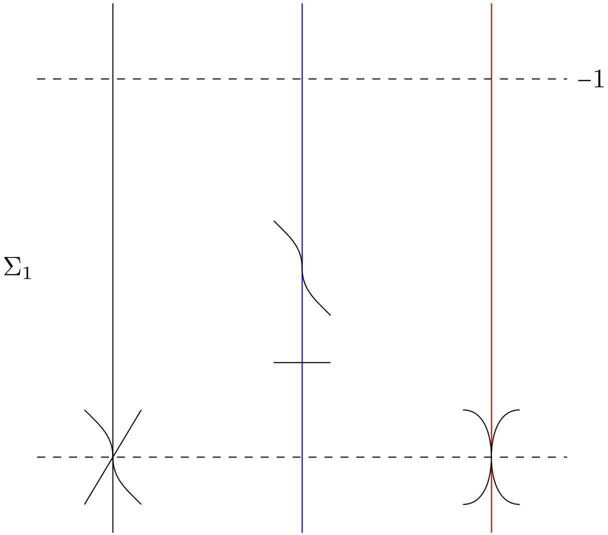

Orbifolds

Definition

An orbifold is a pair $(X, \mathcal{O})$, X topological space and $\mathcal{O}$ is the following data: $\forall p \in X$ there exist

- G finite group
- $V \subset \mathbb{C}^n$ open set (or $\mathbb{R}^n$)
- $U \subset X$ open neighbourhood of p
- $\tilde{\mathbf{x}} : V/G \to U$ homeomorphism $(\mathbf{x} : V \to U)$.
- Analytic (or continuous, differentiable) compatibility of the change of charts.

$p \in X \rightsquigarrow$ *isotropy group* G_p, p regular if $G_p = \{1\}$.

Examples

$\mathbb{C}^n/G$ (G finite), weighted projective spaces, ...

Maps and orbifold fundamental groups

Definition

$(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$, $(f, f_{\mathcal{O}}) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ orbifold map composed by an underlying holomorphic (continuous, ...) map $f : X \to Y$ and a law of local liftings.

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Examples

- $\pi_1(\mathbb{C}/\mu_n)$ generated by the class of γ.

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\begin{array}{ccc}
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 & \overset{\tilde{\gamma}}{\nearrow} & \downarrow \pi \\
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- Orbifold covers and orbifold fundamental groups mimic covers and fundamental groups.

$\pi_1^{\text{orb}}(\Sigma_1/\mathbb{D}_6)$

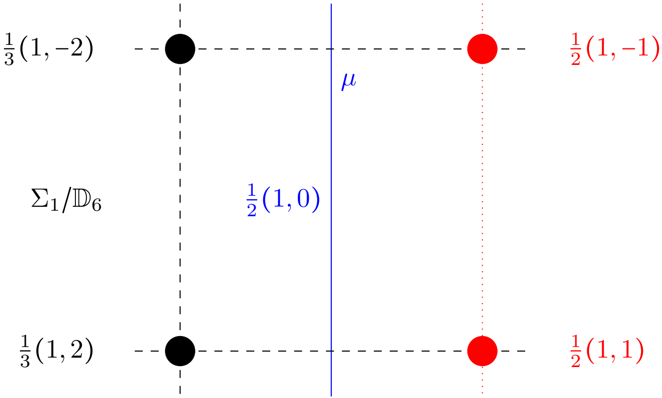
Short answer

$\mathbb{D}_6$

$\pi_1^{\text{orb}}(\Sigma_1/\mathbb{D}_6)$

Long answer

$\pi_1^{\text{orb}}(\Sigma_1/\mathbb{D}_6) \cong \pi_1(\Sigma_1/\mathbb{D}_6 \setminus (\{\text{non regular points}\} \cup \text{blue line}))/\langle \mu^2 \rangle$



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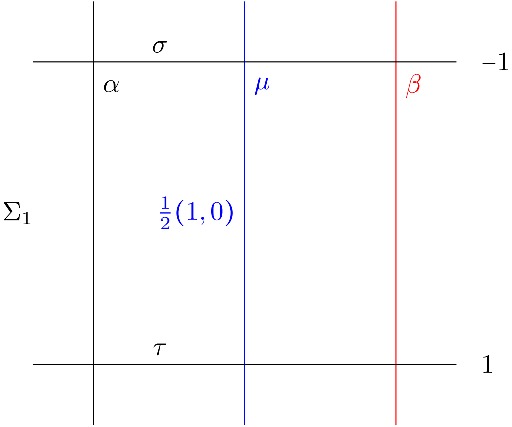

$$\langle \mu, \alpha, \beta, \sigma, \tau \mid \mu^2 = 1, \mu\alpha\beta = \sigma, \sigma\tau = 1, [\sigma, \alpha] = [\sigma, \beta] = 1 \rangle$$

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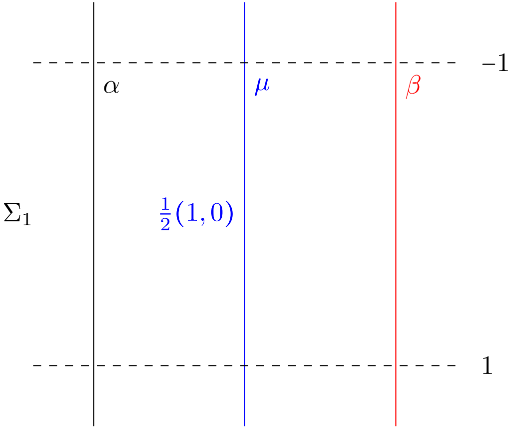

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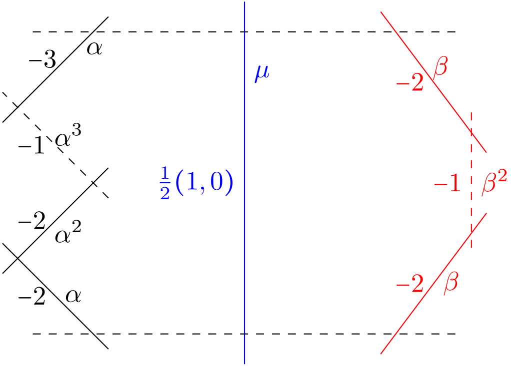
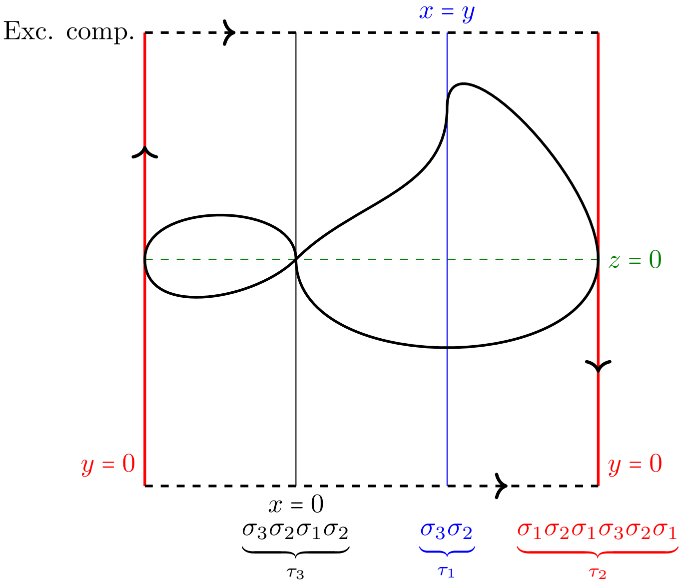


$\langle \mu, \alpha, \beta \mid \mu^2 = 1, \mu\alpha\beta = 1, \alpha^3 = \beta^2 = 1\rangle \cong \mathbb{D}_6$

Braid monodromy of $\mathcal{C}_1$



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- $\tau_1 := \sigma_3\sigma_2 \in \mathbb{B}_4$
- $\tau_2 := \sigma_1\sigma_2\sigma_1\sigma_3\sigma_2\sigma_1 = \Delta_4$
- $\tau_3 := \sigma_3\sigma_2\sigma_1\sigma_2$
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Fundamental group of $\mathbb{P}^2 \setminus (\mathcal{C}_1 \cup \{xy(x-y)=0\})$

- $\langle x_1, x_2, x_3, x_4, y_1, y_2, y_3 \mid x_i^{\tau_j} = y_j^{-1} x_i y_j, x_1x_2x_3x_4y_1y_2y_3 = 1\rangle$
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Orbifold fundamental group G_1 of $\Sigma_1/\mathbb{D}_6 \setminus \mathcal{C}_1$

- $y_1 \leftrightarrow \mu \Longrightarrow y_1^2 = 1$
- $y_2 \leftrightarrow \beta \Longrightarrow y_2^2(x_1\ldots x_4) = 1$, compatible with $\tau_2 = \Delta_4$!
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- $x_1x_2x_3x_4 = y_2^2 = y_3^3 = 1 \Longrightarrow H_2 = \langle y_i \rangle = \mathbb{D}_6$
- $H_1 = \langle x_i \rangle = \langle x_2, x_3 \mid x_2x_3x_2 = x_3x_2x_3, x_2^2x_3^2 = 1\rangle$ of order 12.

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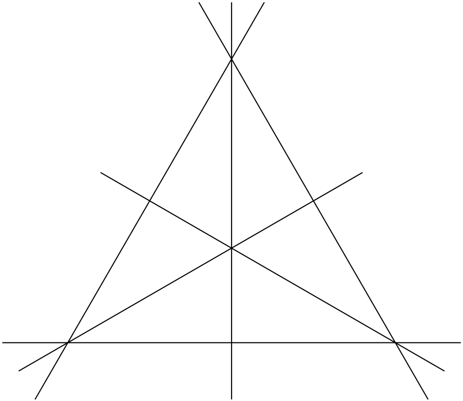
Orbifold fundamental group G_1 of $\Sigma_1/\mathbb{D}_6 \setminus \mathcal{C}_1$

- $x_1x_2x_3x_4 = y_2^2 = y_3^3 = 1 \Longrightarrow H_2 = \langle y_i\rangle = \mathbb{D}_6$
- $H_1 = \langle x_i\rangle = \langle x_2, x_3 \mid x_2x_3x_2 = x_3x_2x_3, x_2^2x_3^2 = 1\rangle$ of order 12.
- $G_1 = H_1 \rtimes \mathbb{D}_6$, $y_1x_2y_1x_3^{-1} = 1$, $[y_2, x_3] = 1$, $y_2^{-1}x_2y_2 = x_2x_3x_2^{-1}] = 1$
- $H_1 = \pi_1(\mathbb{P}^2 \setminus \mathcal{C})$

$\mathfrak{S}_3$ action on Ceva arrangement



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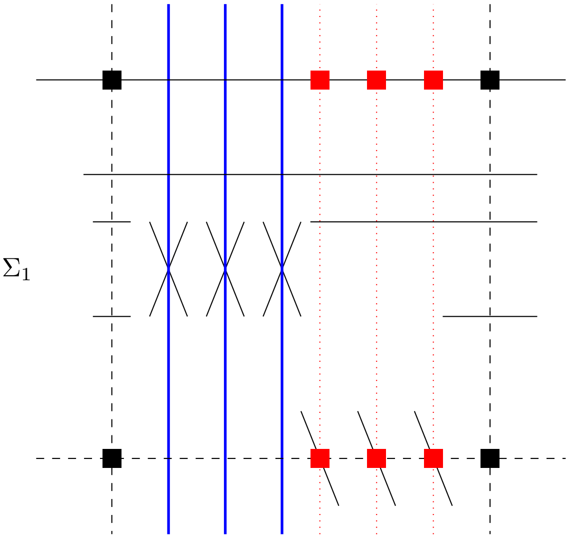

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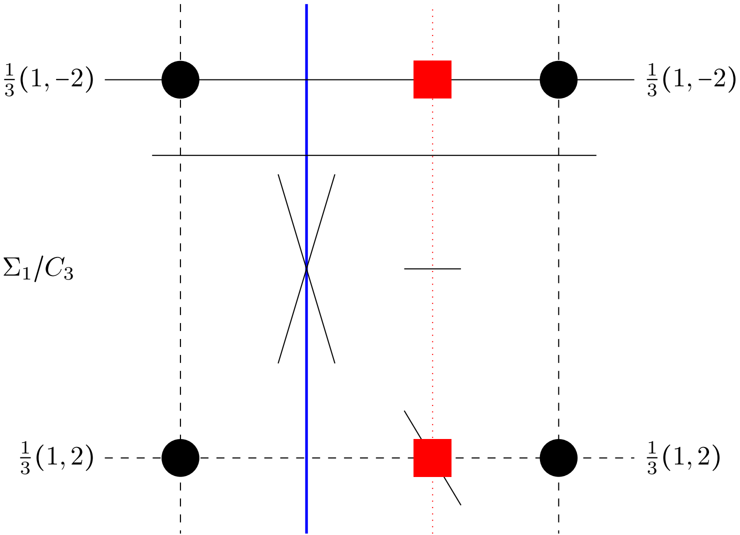

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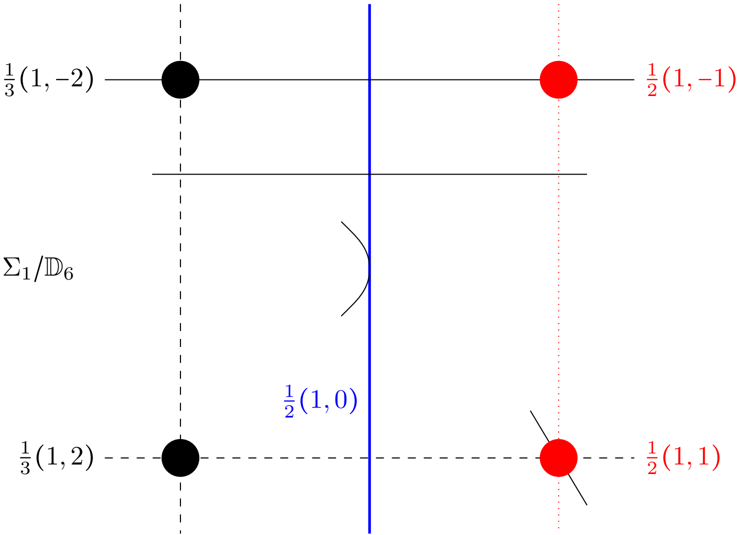

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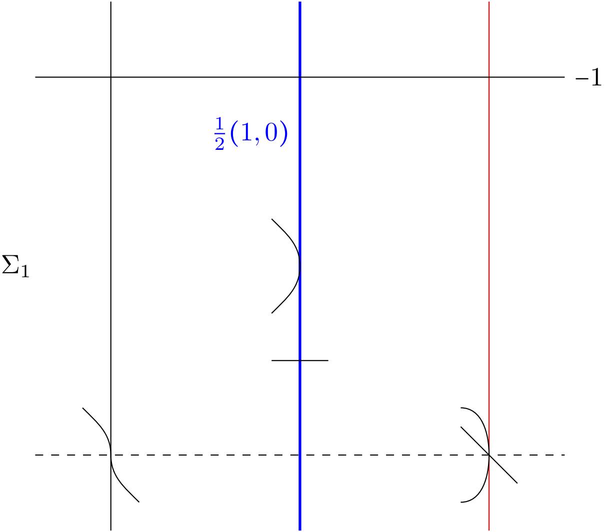

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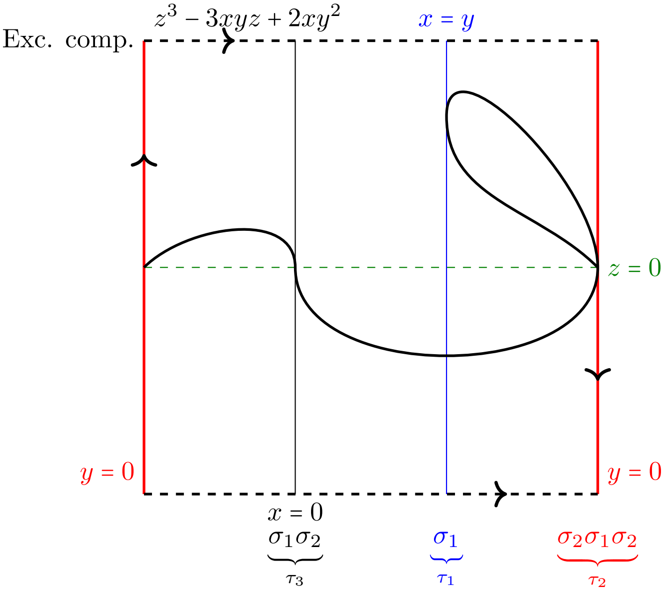

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- $\tau_1 := \sigma_1, \tau_2 := \Delta_3, \tau_3 := \sigma_1\sigma_2 \in \mathbb{B}_3$
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Fundamental group of $\mathbb{P}^2 \setminus (\mathcal{C}_1 \cup \{xy(x-y)=0\})$

- $\langle x_1, x_2, x_3, y_1, y_2, y_3 \mid x_i^{\tau_j} = y_j^{-1} x_i y_j, x_1x_2x_3y_1y_2y_3 = 1\rangle$
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Orbifold fundamental group G_1 of $\Sigma_1/\mathbb{D}_6 \setminus (\mathcal{C}_1 \cup \{x=y\})$

- $y_1 \leftrightarrow \mu,\ y_2 \leftrightarrow \beta \Longrightarrow y_2^2(x_1x_2x_3) = 1,\ y_3 \leftrightarrow \alpha \Longrightarrow y_3^3(x_1x_2x_3) = 1$
- G_1 generators x_3, y_1, y_3
- G_1 relations $[y_1, x_3] = [y_1, y_3y_1y_3] = (y_3x_3)^3 = 1, y_3^2 = y_1y_3y_1\rangle$
- $\Phi : G_1 \twoheadrightarrow \mathfrak{S}_3,\ x_3 \mapsto (),\ y_1 \mapsto (1,2),\ y_3 \mapsto (1,2,3)$
- $\pi_1(\mathbb{P}^2 \setminus \mathcal{C}) = \ker\Phi.$

$\mathfrak{S}_4$ action on Ceva arrangement $\mathscr{C}$

- $\mathscr{C} \leftrightarrow \{(x^2 - y^2)(y^2 - z^2)(z^2 - x^2) = 0\}$, $\operatorname{Aut} \mathcal{C} = \mathfrak{S}_4$
- $[x : y : z] \mapsto [-x : y : z]$, ...
- $1 \to \mu_2 \times \mu_2 \to \mathfrak{S}_4 \to \mathfrak{S}_3 \to 1$

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Lemma

- *$\mathbb{P}^2$ is the underlying space of the orbifold $\mathbb{P}^2 / \mu_2 \times \mu_2$.*

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- *The isotropy groups in the axes outside the vertices are μ_2*
- *The isotropy groups for the vertices are $\mu_2 \times \mu_2$*

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- *Kummer covering* $[x : y : z] \to [x^2 : y^2 : z^2]$ *in mind*
- *The image of* $\mathscr{C}$ *is* $(x - y)(y - z)(z - x) = 0$.
- *In particular its union with the non regular points is again a Ceva arrangement.*

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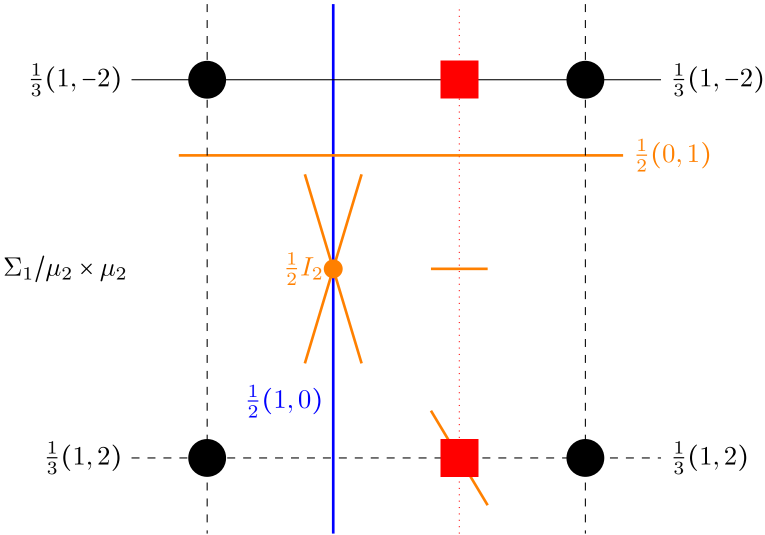
Strategy

Study the birational model of $(\mathbb{P}^2/\mu_2 \times \mu_2)/\mu_3$.

$\mathfrak{S}_4$ orbifold



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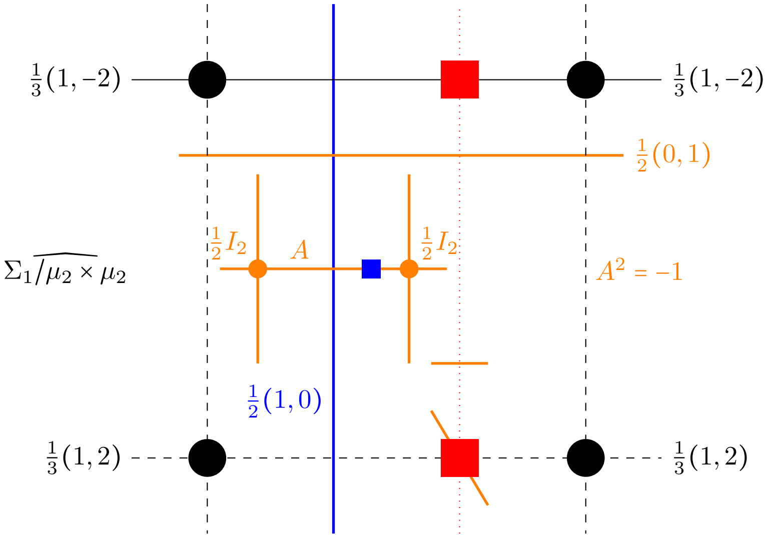

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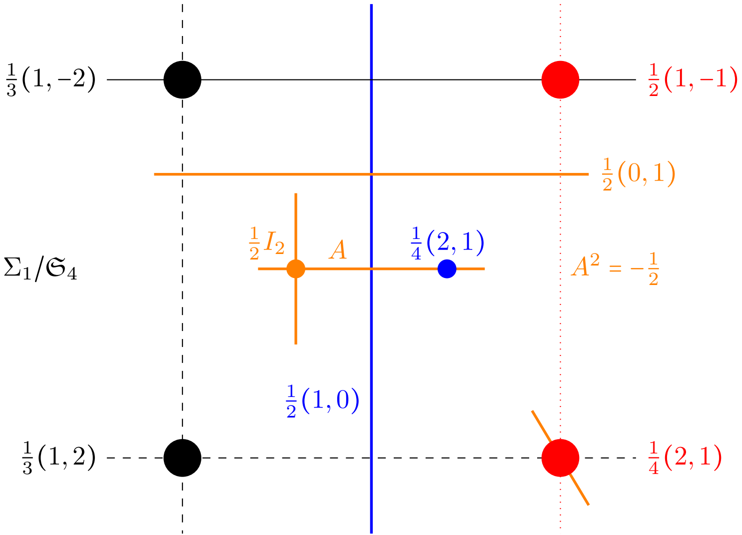
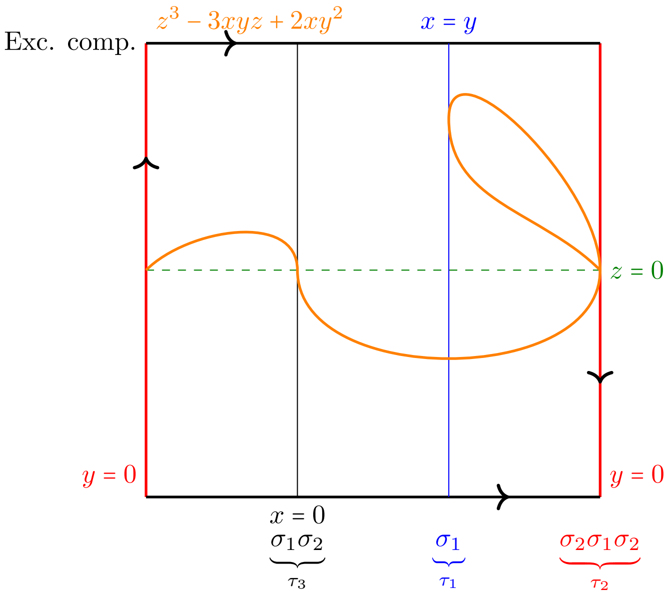


$\mathfrak{S}_4$ orbifold



Generalized Ceva arrangements

Theorem

$$\mathscr{C}_2 = \{(x^2 - y^2)(y^2 - z^2)(z^2 - x^2) = 0\}$$

$$G_1 = \langle x_3, y_1, y_3 \mid [y_1, x_3] = [y_1, (y_3 y_1)^2] = (y_3 x_3)^3 = x_3^2 = 1, y_3^2 = y_1 y_3 y_1 \rangle$$

$$\begin{aligned} G_1 &\xrightarrow{\Phi_2} \mathfrak{S}_4 \\ x_3 &\longmapsto (1,2)(3,4) \\ y_1 &\longmapsto (1,2) \\ y_3 &\longmapsto (1,2,3) \end{aligned}$$

$$\pi_1(\mathbb{P}^2 \setminus \mathscr{C}_2) = \ker \Phi_2$$

Generalized Ceva arrangements

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$$\mathscr{C}_2 = \{(x^2 - y^2)(y^2 - z^2)(z^2 - x^2) = 0\}$$

$$H_2 = \langle a_1, a_2, a_3 \mid a_1^2 = a_2^2 = a_3^2 = a_1 a_2 a_3 = 1, [a_i, a_j] = 1\rangle$$

$$\tilde{H}_2 = H_2 \rtimes \mathfrak{S}_3, \quad a_i^\sigma = a_{i^\sigma}$$

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Generalized Ceva arrangements

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McLane arrangements $\mathscr{ML}$

Proposition

$\operatorname{Aut}\mathscr{ML} = \mathrm{SL}(2;\mathbb{F}_3)$ *(of order* 24*).*

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Group structure and action

McLane arrangements $\mathscr{ML}$

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$\operatorname{Aut}\mathscr{ML} = \mathrm{SL}(2;\mathbb{F}_3)$ *(of order* 24*).*

Group structure and action

- $Z(\mathrm{SL}(2;\mathbb{F}_3))$ generated by $-I_2$.

McLane arrangements $\mathscr{ML}$

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$\operatorname{Aut}\mathscr{ML} = \mathrm{SL}(2;\mathbb{F}_3)$ *(of order 24).*

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- $Z(\mathrm{SL}(2;\mathbb{F}_3))$ generated by $-I_2$.
- $1 \to \mu_2 \times \mu_2 \to \mathrm{PSL}(2;\mathbb{F}_3) \to \mu_3 \to 1$

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- The action on $\mathbb{P}^2$ has a fixed point p.

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- The singular points of McLane arrangements are contained in four lines through p, each one containing two triple points.

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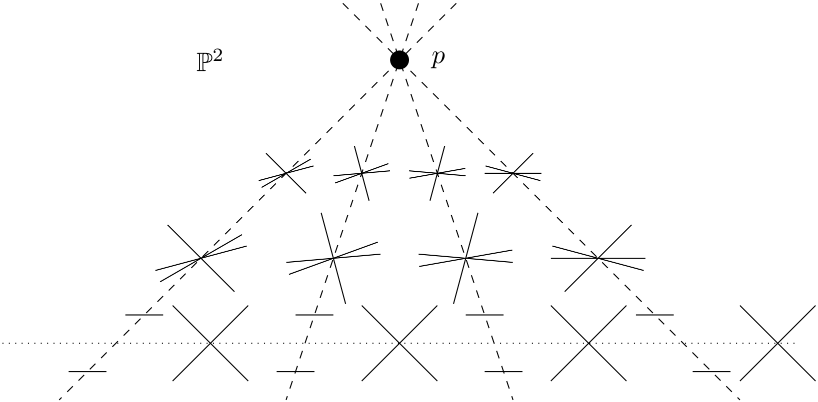
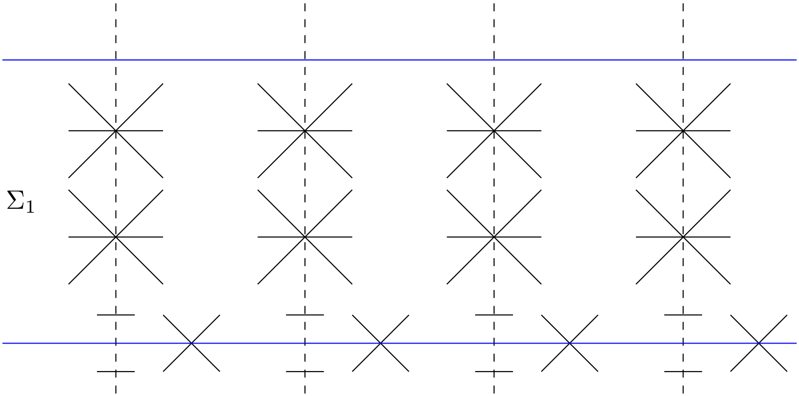
- The action on $\mathbb{P}^2$ has a fixed point p.
- The singular points of McLane arrangements are contained in four lines through p, each one containing two triple points.
- The action of the center has also a line of fixed points containing the double points.

McLane quotient

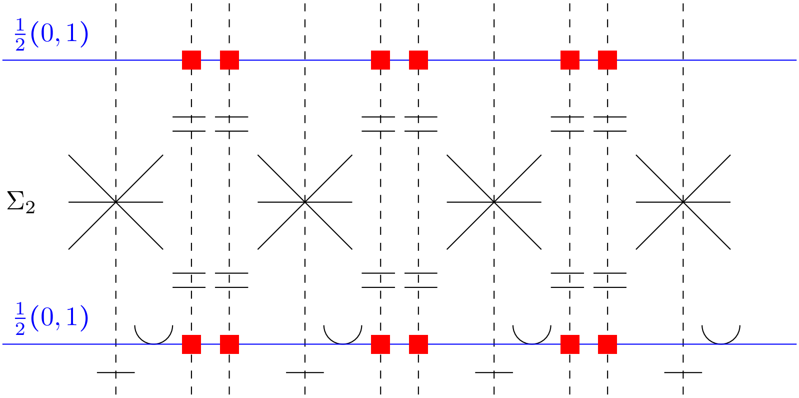


McLane quotient



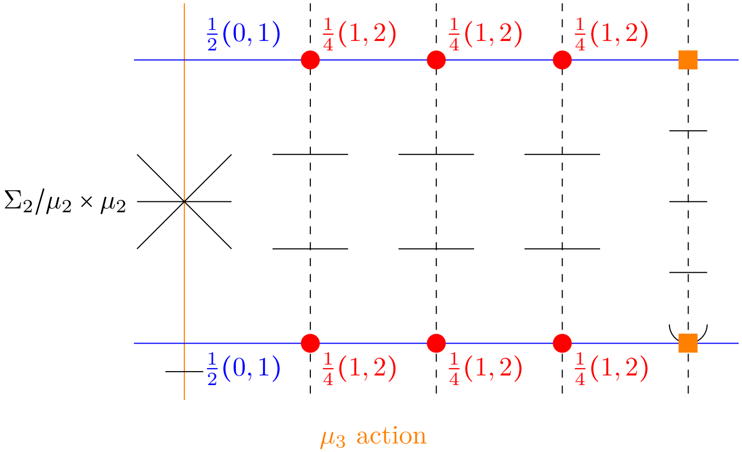
μ_2 action

McLane quotient



$\mu_2 \times \mu_2$ action

McLane quotient



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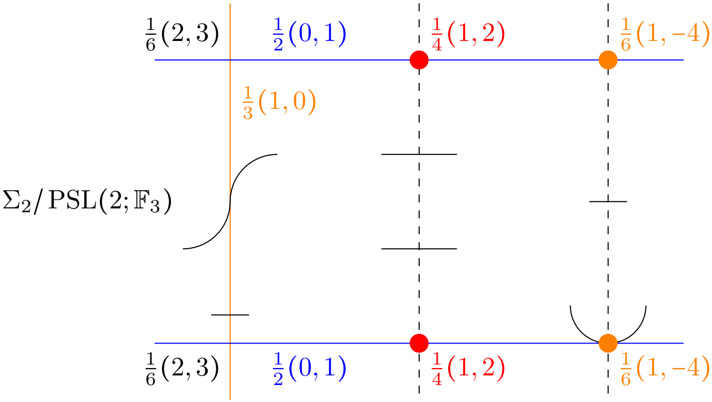

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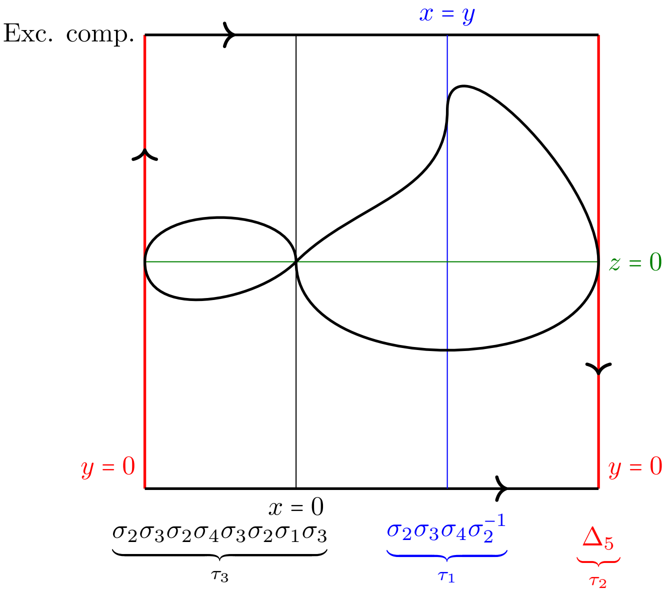

McLane quotient

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Fundamental group of $\mathbb{P}^2 \setminus (\mathcal{C}_1 \cup \{xy(x-y)=0\})$

- $\langle x_1,\ldots,x_5,y_1,y_2,y_3 \mid x_i^{\tau_j} = y_j^{-1}x_iy_j, x_1\ldots x_5y_1y_2y_3 = 1\rangle$
- $x^{\Delta_5^2} = (x_1\ldots x_5)x(x_1\ldots x_5)^{-1}$

McLane quotient

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Orbifold fundamental group G_1 of $\Sigma_2/\operatorname{PSL}(2;\mathbb{F}_3) \setminus \mathcal{C}_1$

- $y_1 \leftrightarrow \mu \Longrightarrow y_1^3 = 1$
- $y_2 \leftrightarrow \beta \Longrightarrow y_2^2(x_1\dots x_5) = 1$
- $y_3 \leftrightarrow \alpha \Longrightarrow y_3^3(x_1\dots x_5) = 1$
- $x_3^2 = 1$

McLane quotient

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Fundamental group of $\mathbb{P}^2 \setminus (\mathcal{C}_1 \cup \{xy(x-y)=0\})$

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Orbifold fundamental group G_1 of $\Sigma_2/\operatorname{PSL}(2;\mathbb{F}_3) \setminus \mathcal{C}_1$

- G_1 generators: x_5, y_1, y_3
- G_1 relations:
  $y_3^2 = y_1y_3y_1, [y_1^{-1}y_3, x_5] = [x_5,(x_5y_1)] = y_1^3 = (x_5y_1x_5y_3)^4 = 1$
- $G = \ker\Phi$

$$
\begin{aligned}
G_1 &\xrightarrow{\ \Phi\ } \operatorname{SL}(2;\mathbb{F}_3)\\
x_5 &\longmapsto I_2\\
y_1 &\longmapsto \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}\\
y_3 &\longmapsto \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}
\end{aligned}
$$

McLane equations

- $P_4 := 3x^4 + 6x^2y^2 - y^4$
- $Q_4 := 3x^4 - 6x^2y^2 - y^4$
- $P_6 := 6xy(3x^4 + y^4)$

$$\begin{array}{ccc} \mathbb{P}^2 & \overset{\Phi}{\dashrightarrow} & \mathbb{P}^2 \\ [x : y : z] & \longmapsto & [P_4^3 : P_6^2 : P_4 P_6 z^2] \end{array}$$

- $F = z^8 - 6P_4z^4 - 8P_6z^2 - 3P_4^2$

Hesse $\mathscr{H}$ and dual-Hesse $\check{\mathscr{H}}$ arrangements

Equations and properties

- $\check{\mathscr{H}} : (x^3 - y^3)(y^3 - z^3)(z^3 - x^3) = 0$
- $\mathscr{H}$:
  $xyz(x^3+y^3+z^3-3xyz)((x^3+y^3+z^3)^2+3xyz(x^3+3xyz+y^3+z^3)) = 0$
- Singular fibers of elliptic fibrations

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Series of Aut $\check{\mathscr{H}}$

- $H_1 = \mathbb{F}_3^2 \cong \boxed{\mu_3 \times \mu_3}$, $\mathrm{Aut}\,\check{\mathscr{H}}/H_1 \cong \mathrm{SL}(2;\mathbb{F}_3)$
- $H_2/H_1 \cong Z(\mathrm{SL}(2;\mathbb{F}_3)) \cong \boxed{\mu_2}$, $\mathrm{Aut}\,\check{\mathscr{H}}/H_2 \cong \mathfrak{A}_4$
- $H_3/H_2 \cong \boxed{\mu_2 \times \mu_2}$, $\mathrm{Aut}\,\check{\mathscr{H}}/H_3 \cong \boxed{\mu_3}$

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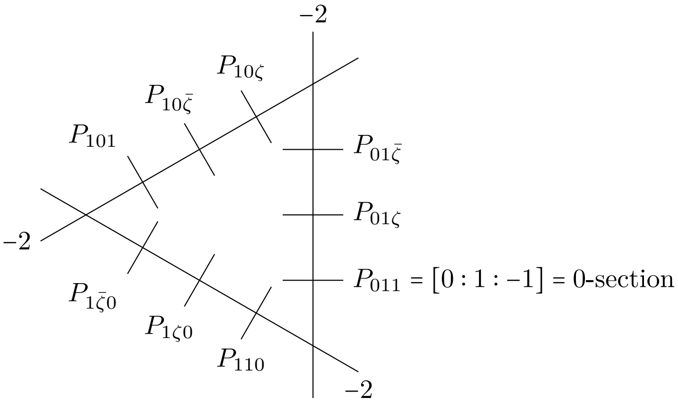

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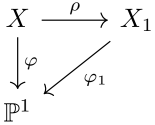
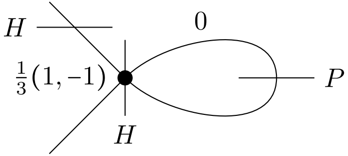
Proposition

- $X_1 := X/\mu_3 \times \mu_3$
- $\rho(P_{\bullet\bullet\bullet}) = P$ *0-section of* φ_1
- $\rho(\check{\mathscr{H}}) =: H$ *multisection of 2-torsion*
- φ_1 *with four singular fibers*





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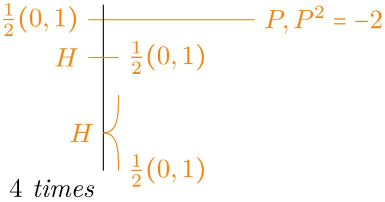
Proposition

$\Sigma_2 \cong X_1/\mu_2 \equiv X/(x \mapsto -x)$, $\varphi_2 : \Sigma_2 \to \mathbb{P}^1$.



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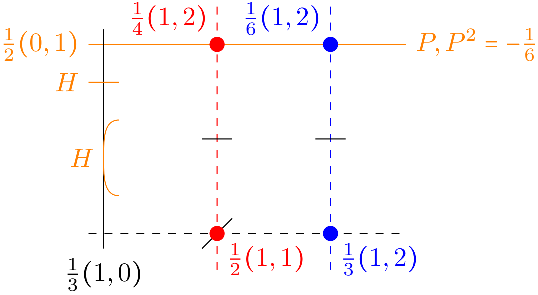
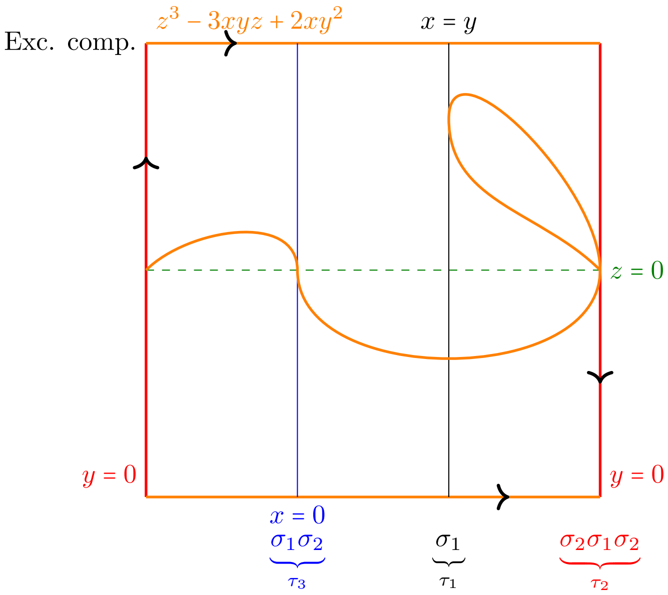


An old friend



A plus of symplectic geometry

Problem

Find all curves $\mathcal{C} \subset \mathbb{P}^2$ such that $\deg \mathcal{C} = 8$ and $\operatorname{Sing} \mathcal{C} = 8\mathbb{E}_6$

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There are (at least) three curves as above pairwise non-homeomorphic:

1. $\exists \mathcal{C}_1$ *rigid* μ_2*-invariant.*
2. $\exists \mathcal{C}_2$ *rigid* μ_3*-invariant, only isolated fixed points.*
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Is $\mathcal{C}_3$ isotopic to an algebraic curve?

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Is $\mathcal{C}_3$ isotopic to an algebraic curve? Not to $\mathcal{C}_1, \mathcal{C}_2$!

Happy birthday,
Alex and Laurențiu!